MATH 721, Algebra II
Exercises 1
Due Wed 21 Jan

Exercise 1 (Hom-tensor adjointness)**.** Let R be a commutative ring with identity, and let L, M, and N be R-modules.

(a) Let $f\colon L \otimes_R M \to N$ be an R-module homomorphism, and let $x \in L$. Let $f_x\colon M \to N$ be given by $f_x(m) := f(x \otimes m)$. Prove that f_x is a well-defined R-module homomorphism, that is, that $f_x \in \operatorname{Hom}_R(M, N)$.

(b) Let $f\colon L \otimes_R M \to N$ be an R-module homomorphism. Let $f'\colon L \to \operatorname{Hom}_R(M, N)$ be given by $f'(x) := f_x$. Prove that f' is a well-defined R-module homomorphism, that is, that $f' \in \operatorname{Hom}_R(L, \operatorname{Hom}_R(M, N))$.

(c) Let $\Phi\colon \operatorname{Hom}_R(L \otimes_R M, N) \to \operatorname{Hom}_R(L, \operatorname{Hom}_R(M, N))$ be given by $\Phi(f) := f'$. Prove that Φ is a well-defined R-module homomorphism.

(d) Let $g\colon L \to \operatorname{Hom}_R(M, N)$ be an R-module homomorphism. Let $\widetilde{g}\colon L \otimes_R M \to N$ be given by $\widetilde{g}(x \otimes m) := g(x)(m)$. Prove that $\widetilde{g}$ is a well-defined R-module homomorphism, that is, that $\widetilde{g} \in \operatorname{Hom}_R(L \otimes_R M, N)$.

(e) Let $\Psi\colon \operatorname{Hom}_R(L, \operatorname{Hom}_R(M, N)) \to \operatorname{Hom}_R(L \otimes_R M, N)$ be given by $\Psi(g) := \widetilde{g}$. Prove that Ψ is a well-defined R-module homomorphism.

(f) Prove that $\Phi \circ \Psi$ and $\Psi \circ \Phi$ are the identities on $\operatorname{Hom}_R(L, \operatorname{Hom}_R(M, N))$ and $\operatorname{Hom}_R(L \otimes_R M, N)$, respectively. Conclude that we have $\operatorname{Hom}_R(L \otimes_R M, N) \cong \operatorname{Hom}_R(L, \operatorname{Hom}_R(M, N))$.