MATH 721, Algebra II
Exercises 3
Due Wed 04 Feb

Throughout this homework set, let $R$ be a commutative ring with identity, and let $M$ be an $R$-module.

**Exercise 1.** Let $\varphi\colon R \to S$ be a homomorphism of commutative rings with identity. Prove that the $R$-module $S \otimes_R M$ has a well-defined $S$-module structure given by $s(s' \otimes m) := (ss') \otimes m$. In particular, if $U \subseteq R$ is multiplicatively closed, then $(U^{-1}R) \otimes_R M$ has a well-defined $U^{-1}R$-module structure given by

$$\frac{r}{u}\left(\frac{r'}{u'} \otimes m\right) := \frac{rr'}{uu'} \otimes m.$$

**Exercise 2.** Let $U \subseteq R$ be a multiplicatively closed subset. Prove that the $R$-module isomorphism $g\colon (U^{-1}R) \otimes_R M \to U^{-1}M$ given by $(r/u) \otimes m \mapsto (rm)/u$ is a $U^{-1}R$-module isomorphism.

**Exercise 3.** Prove that the following conditions are equivalent.

(i) $M = 0$.
(ii) For every multiplicatively closed subset $U \subseteq R$, we have $U^{-1}M = 0$.
(iii) For every prime ideal $\mathfrak{p} \subset R$, we have $M_{\mathfrak{p}} = 0$.
(iv) For every maximal ideal $\mathfrak{m} \subset R$, we have $M_{\mathfrak{m}} = 0$.

Hint: For the implication (iv) $\implies$ (i), suppose that $0 \neq x \in M$. Then there is an ideal $I \subset R$ such that $0 \neq R/I \cong Rx \subseteq M$. Let $\mathfrak{m} \subset R$ be a maximal ideal containing $I$, and show that there is a monomorphism $(Rx)_{\mathfrak{m}} \to M_{\mathfrak{m}}$ with $(Rx)_{\mathfrak{m}} \neq 0$.