MATH 721, Algebra II
Exercises 4
Due Wed 11 Feb

Exercise 1. Fix integers $n \geq m \geq 1$ and consider a free R-module F with basis $e_1, \ldots, e_n$. Given $r_1, \ldots, r_m \in R$, prove that there is an R-module isomorphism

$$F/\langle r_1 e_1, \ldots, r_m e_m \rangle \cong (R/\langle r_1 \rangle) \oplus \cdots \oplus (R/\langle r_m \rangle) \oplus R^{n-m}.$$

Exercise 2. Let k be a field, and consider a non-zero polynomial $f \in k[x]$. Prove that the k-vector space $k[x]/\langle f \rangle$ has $\dim_k(k[x]/\langle f \rangle) = \deg(f)$. (Hint: division algorithm)

Exercise 3. Let R be a PID, and let M be a finitely generated R-module. Prove that there are integers $n \geq m \geq 1$ and elements $d_1, \ldots, d_m \in R$ such that

$$M \cong (R/\langle d_1 \rangle) \oplus \cdots \oplus (R/\langle d_m \rangle) \oplus R^{n-m}$$

and $d_1 | d_2 | \cdots | d_m$.