MATH 721, Algebra II
Exercises 5
Due Wed 18 Feb

Exercise 1. Let R be a commutative ring with identity, and let M be a unital R-module with a fixed R-module homomorphism $\phi\colon M \to M$. Prove that M has a well-defined $R[x]$-module structure where $xm := \phi(m)$ for all $m \in M$.

Exercise 2. Let k be a field, and let A, P be $n \times n$ matrices with entries in k. Assume that P is invertible, and let $\lambda \in k$. Set $A' := PAP^{-1}$.

Recall that λ is an *eigenvalue* for A if there is a non-zero vector $v \in k^n$ such that $Av = \lambda v$; in this event, v is an *eigenvector* of A associated to λ. The *eigenspace* of A associated to λ is

$$E_\lambda(A) := \{\text{eigenvectors of } A \text{ associated to } \lambda\} \cup \{0\}.$$

(a) Prove that v is an eigenvector of A associated to λ if and only if Pv is an eigenvector of A' associated to λ.
(b) Prove that λ is an eigenvalue for A if and only if λ is an eigenvalue for A'.
(c) Prove that for all $\lambda \in k$, the set $E_\lambda(A)$ is a subspace of k^n.
(d) Prove that for all $\lambda \in k$, there is a k-linear isomorphism $E_\lambda(A) \cong E_\lambda(A')$.

Exercise 3. Find the rational canonical form of the following matrix in $\mathrm{Mat}_{3\times 3}(\mathbb{R})$.

$$\begin{pmatrix} 2 & 0 & 0 \\ 1 & 2 & 0 \\ 2 & 5 & 3 \end{pmatrix}$$