

MATH 721, Algebra II
Exercises 6
Due Wed 25 Feb

Throughout this homework set, let k be a field, and let A, B be $n \times n$ matrices with entries in k .

- Exercise 1.** (a) Prove that if A and B are similar, then A and B have the same characteristic polynomial and the same minimal polynomial.
(b) Show that if A and B have the same characteristic polynomial, then they need not be similar. Justify your answer.
(c) (Bonus) Prove or give a counterexample to the converse of part (a). Justify your answer.

Exercise 2. Prove that the following conditions are equivalent.

- (i) k^n has a basis consisting of eigenvectors of A .
- (ii) A is diagonalizable, that is, there exists an invertible matrix P with entries in k such that PAP^{-1} is diagonal.

Exercise 3. We say that A is *nilpotent* if there is a natural number $q \geq 1$ such that $A^q = 0$.

- (a) Prove that the following conditions are equivalent.
 - (i) A is nilpotent.
 - (ii) the minimal polynomial of A is of the form x^m .
 - (iii) the characteristic polynomial of A is x^n .
- (b) Prove that A is nilpotent if and only if $A^n = 0$.