# Robot in a Maze
## Math 314—006
## Application Mini-Project #2
### *Due: Thursday, October 16*

**Goal & Background:** To investigate the role of a *Markov Chain* when studying an evolving process which consists of a finite number of states. To do this project, you will need to read (and understand!) the portion of Section 3.7 on *Markov Chains*, pp. 228–233.

**Instructions:** *This project is to be done in groups of two or three students, as assigned by the instructor..* Groups of size four or more are not permitted. Each group will turn in one solution to the tasks described below, and each member of the group will receive the same grade on the project. You should be careful to understand each part of the project – related questions may appear on exams.

**Submission Guidelines:** Please use the following guidelines when preparing your project for submission:

1. Include a cover page on which each member of the group *signs* their full name. Also, in the top right hand corner of each page submitted, write the names (first and last, written legibly) of each group member. You should not include student ID or social security numbers.

2. Only use one side of each sheet of paper and staple the pages together.

3. Presentation will be considered when grading this project. You should take care to write legibly and hand in full sheets of paper with no fringe, tears, etc. You should also clearly label each problem and submit these in order.

4. Give justification (in complete sentences!) for your answers.

5. Be academically honest. This means, for example, providing a list of sources other than the textbook (if any) that you used to do the assignment; stating clearly that you're copying or mimicking an example from the book in order to do the assignment (if appropriate).

6. The project is due at the beginning of class. Under certain circumstances, late submissions may be accepted, but they will be penalized.

**Setup:** A robot is placed in the maze below and is programmed to move at random. Each minute, the robot either stays in the room it is currently located or chooses a hallway at random and moves into the adjoining room, and each possibility is equally likely. For example, if the robot happens to be in room 11, then there is a $\frac{1}{3}$ chance it will move next to room 10, a $\frac{1}{3}$ chance it will move next to room 7, and a $\frac{1}{3}$ chance it will stay in room 11.
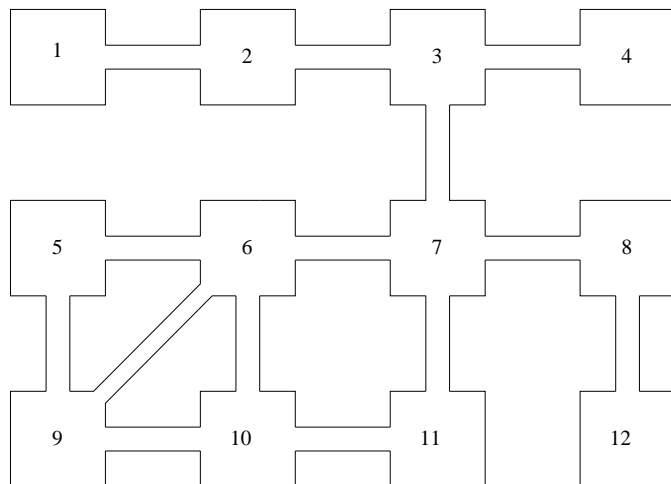


**Project Tasks:**

1. Find the appropriate transition matrix $P$ for the Markov chain describing the movement of the robot.

2. Recall that a *probability vector* is a vector all of whose entries are between 0 and 1 and such that the sum of these entries is 1. Suppose $\mathbf{v}$ is probability vector in $\mathbb{R}^{12}$ whose $i$-th entry gives the probability that the robot is currently located in room $i$. What does the vector $P\mathbf{v}$ tell you and why?

3. Carefully explain the real-world significance of the matrix $P^2$. Be sure to justify, in your own words, the meaning of the entries of the matrix $P^2$. It's helpful to give explicit examples, of the sort "The $(3,2)$ entry of $P^2$ tells me blah, blah, blah." Do the same thing for $P^3$, $P^4$, etc.

4. An important concept in studying stochastic matrices is that of "regularity". A stochastic matrix is called *regular* if some power of it has no zero entries. Show that $P$ is regular. Also, find the smallest integer $k$ such that $P^k$ has no zero entries. What is the practical meaning (in terms of properties of the maze) of this value of $k$?

5. Calculate $P^k$ for a very large value of $k$. (One fast way to do this is to compute $P^2$, $P^4$, $P^8$, etc. by repeatedly squaring.) What do you notice about the entries of $P^k$ for $k$ very large?

6. What is the approximate probability that the robot will wind up in each of the twelve rooms after a very long time has passed? To answer this, it might help to use the answer to the previous part, and to remember that $P^k \mathbf{e}_j$ is the $j$-th column of $P^k$. Does your answer depend on where the robot begins his journey?

7. Find a probability vector that solves the equation $P\mathbf{x} = \mathbf{x}$. This is called a *steady state* vector for the system. What's the connection between this vector and your answer to the previous part?

8. What connection is there between the entries of the steady-state vector and the nature of the maze? Can you explain heuristically why some of the entries of the steady-state vector are equal to others?

9. Suppose the maze is now modified so that the hallway joining rooms 3 and 7 is permanently blocked off. This leads to a new transition matrix — call it $Q$. Is $Q$ regular? How can you be sure of your answer?

**Computers**: You can conceivably do this project using just a fancy calculator. I encourage you, however, to use the software package *Maple* (or an equivalent one). For one thing, it is far easier to enter, manipulate, and view large amounts of data in *Maple* than it is on a calculator, and this project will require the use of rather large matrices. Also, I believe it is a valuable experience to learn how to use a computer algebra system such as *Maple*.

*Maple* is available on the Mathlab computer system and everyone in class should have access to Mathlab. The Mathlab is located in room 18 of Avery Hall and you can see what its hours are at the website:

http://www.math.unl.edu/pi/studentResources/labs.

Your login and password for the computers in the Mathlab should be the same as for your usual university "active directory" account.

Here are some helpful tips concerning *Maple*. These are based on Version 10 of *Maple*.

- You will need to "import" the linear algebra package into *Maple*. Enter the command with(LinearAlgebra): at the very start of your *Maple* session. (The trailing colon suppresses a long list of commands that are imported with the LinearAlgebra package — to see this list, replace the colon with a semi-colon.)

- By default, *Maple* only displays in full matrices of size at most $10 \times 10$. To increase this to 12, use interface(rtablesize = 12);.

- To learn more about a *Maple* command or topic, type in the command name or topic preceded by a question mark. For example, try entering ?Matrix; into *Maple*. (You can also use the menu-driven help documentation.)

- Here are a couple of useful commands:

  - The command B := Matrix([[a,b,c],[d,e,f]]); sets $B$ to be the evident $2 \times 3$ matrix. Note that in Maple, all commands should be ended with a ";" (or with a ":" if you don't want to see the output of the command for some reason).

  - You can use plus, minus, and exponents in the usual manner for matrix operations. Instead of using an asterisk for matrix multiplication, however, use the period key. For example, enter $A$ and $B$ to be your favorite $3 \times 3$ and $2 \times 3$ matrices, and try B.(A+3A^2);.

  - The command ReducedRowEchelonForm(A); puts the matrix $A$ in to reduced row echelon form.

  - Suppose $A$ is a matrix with rational numbers as entries and you would rather have them as decimals. Then you could use the command map(evalf,A);. Suppose you want to round off the entries of $A$ to 3 decimal places of accuracy. (This is useful for printing a large matrix so that it fits on one page.) Then use map(x -> evalf(x,3),A);.