Solutions to MiniProject 2 -- Robots in a Maze

As instructed, we begin by importing the Linear Algebra package and then set the display matrix size to 12x12:

```
with(LinearAlgebra) :
interface(rtablesize = 12);
```

$$10 \tag{1}$$

(1) We wish to find the transition matrix P for the Markov chain describing the robot's movement. This matrix is below. Note that tt's easier to think about what the columns of P should be than it is to think about the rows, and so we enter P as the transpose of a matrix. To see how we got this matrix, consider (for example) the 7th column. A robot in room 7 can stay put or go to room 3, room 6, room 8 or room 10. Each of these 5 possibilities happens with equal likelihood, which means that the 7th column has a 1/5 in the 3rd, 6th, 7th, 8th and 10th positions and 0's elsewhere.

```
P := Transpose(Matrix([[1 / 2, 1 / 2, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0], [1 / 3, 1 / 3, 1 / 3, 0, 0, 0, 0, 0,
   0, 0, 0, 0], [0, 1 / 4, 1 / 4, 1 / 4, 0, 0, 1 / 4, 0, 0, 0, 0, 0], [0, 0, 1 / 2, 1 / 2, 0, 0, 0, 0, 0, 0, 0,
   0], [0, 0, 0, 0, 1 / 3, 1 / 3, 0, 0, 1 / 3, 0, 0, 0], [0, 0, 0, 0, 1 / 5, 1 / 5, 1 / 5, 0, 1 / 5, 1 / 5, 0,
   0], [0, 0, 1 / 5, 0, 0, 1 / 5, 1 / 5, 1 / 5, 0, 0, 1 / 5, 0], [0, 0, 0, 0, 0, 0, 1 / 3, 1 / 3, 0, 0, 0, 1
   / 3], [0, 0, 0, 0, 1 / 4, 1 / 4, 0, 0, 1 / 4, 1 / 4, 0, 0], [0, 0, 0, 0, 0, 1 / 4, 0, 0, 1 / 4, 1 / 4, 1 / 4,
   0], [0, 0, 0, 0, 0, 0, 1 / 3, 0, 0, 1 / 3, 1 / 3, 0], [0, 0, 0, 0, 0, 0, 0, 1 / 2, 0, 0, 0, 1 / 2]]));
```

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & \frac{1}{4} & \frac{1}{2} & 0 & 0 & \frac{1}{5} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{4} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{3} & \frac{1}{5} & 0 & 0 & \frac{1}{4} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{3} & \frac{1}{5} & \frac{1}{5} & 0 & \frac{1}{4} & \frac{1}{4} & 0 & 0 \\ 0 & 0 & \frac{1}{4} & 0 & 0 & \frac{1}{5} & \frac{1}{5} & \frac{1}{3} & 0 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{5} & \frac{1}{3} & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & \frac{1}{3} & \frac{1}{5} & 0 & 0 & \frac{1}{4} & \frac{1}{4} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{5} & 0 & 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{5} & 0 & 0 & \frac{1}{4} & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{3} & 0 & 0 & 0 & \frac{1}{2} \end{bmatrix} \tag{2}$$

(2) If $\mathbf{v}$ is a probability vector in R^{12} whose i–th entry gives the probability that the robot is currently located in room i, then $P\mathbf{v}$ is a probability vector whose i–th entry gives the probability that the robot will be located in room i after one move. To see this, note that if we write $P = (p_{ij})$ and $\mathbf{v} = (v_1, ..., v_{12})$, then the i–th entry of $P\mathbf{v}$ is $p_{i1}\, v_1 + ... + p_{i,12}\, v_{12}$. Since p_{ij} is the probability that a robot who starts in room j moves to room i and v_j is the probability that the robot starts in room j, we have that $p_{ij}\, v_j$ is the probability that a robot starts in room j and moves to room i. Therefore, the sum is the probability that the robot starts somewhere and moves to room i, i.e., the probability that the robot ends up in room i.

(3) The matrix P^2 is the transition matrix for the Markov chain describing the movement of the robot, taken two moves at a time. For example, the (3,2) entry of P^2 tells me the probability that a robot who starts in room 2 ends up in room 3 after two moves. The matrix P^3 is the transition matrix for the Markov chain describing the movement of the robot, taken three moves at a time. For example, the (3,2) entry of P^3 tells me the probability that a robot who starts in room 2 ends up in room 3

after three moves. The matrix P^4 is the transition matrix for the Markov chain describing the movement of the robot, taken four moves at a time. For example, the (3,2) entry of P^4 tells me the probability that a robot who starts in room 2 ends up in room 3 after four moves. In general, the matrix P^k is the transition matrix for the Markov chain describing the movement of the robot, taken k moves at a time. For example, the (3,2) entry of P^k tells me the probability that a robot who starts in room 2 ends up in room 3 after k moves.

(4) To see that P is regular, we can just start computing powers of P.

P^2;

$$
\left[\begin{array}{cccccccccccc}
\frac{5}{12} & \frac{5}{18} & \frac{1}{12} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{5}{12} & \frac{13}{36} & \frac{7}{48} & \frac{1}{8} & 0 & 0 & \frac{1}{20} & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{6} & \frac{7}{36} & \frac{77}{240} & \frac{3}{8} & 0 & \frac{1}{25} & \frac{9}{100} & \frac{1}{15} & 0 & 0 & \frac{1}{15} & 0 \\
0 & \frac{1}{12} & \frac{3}{16} & \frac{3}{8} & 0 & 0 & \frac{1}{20} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{47}{180} & \frac{47}{300} & \frac{1}{25} & 0 & \frac{47}{240} & \frac{9}{80} & 0 & 0 \\
0 & 0 & \frac{1}{20} & 0 & \frac{47}{180} & \frac{37}{150} & \frac{2}{25} & \frac{1}{15} & \frac{31}{120} & \frac{7}{40} & \frac{3}{20} & 0 \\
0 & \frac{1}{12} & \frac{9}{80} & \frac{1}{8} & \frac{1}{15} & \frac{2}{25} & \frac{79}{300} & \frac{8}{45} & \frac{1}{20} & \frac{2}{15} & \frac{8}{45} & \frac{1}{6} \\
0 & 0 & \frac{1}{20} & 0 & 0 & \frac{1}{25} & \frac{8}{75} & \frac{31}{90} & 0 & 0 & \frac{1}{15} & \frac{5}{12} \\
0 & 0 & 0 & 0 & \frac{47}{180} & \frac{31}{150} & \frac{1}{25} & 0 & \frac{31}{120} & \frac{7}{40} & \frac{1}{12} & 0 \\
0 & 0 & 0 & 0 & \frac{3}{20} & \frac{7}{50} & \frac{8}{75} & 0 & \frac{7}{40} & \frac{31}{120} & \frac{7}{36} & 0 \\
0 & 0 & \frac{1}{20} & 0 & 0 & \frac{9}{100} & \frac{8}{75} & \frac{1}{15} & \frac{1}{16} & \frac{7}{48} & \frac{47}{180} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{15} & \frac{5}{18} & 0 & 0 & 0 & \frac{5}{12}
\end{array}\right] \quad \textbf{(3)}
$$

P^3;

$$
\left[\left[\frac{25}{72}, \frac{7}{27}, \frac{13}{144}, \frac{1}{24}, 0, 0, \frac{1}{60}, 0, 0, 0, 0, 0\right],\right. \quad \textbf{(4)}
$$

$$\left[\frac{7}{18}, \frac{133}{432}, \frac{491}{2880}, \frac{13}{96}, 0, \frac{1}{100}, \frac{47}{1200}, \frac{1}{60}, 0, 0, \frac{1}{60}, 0\right],$$

$$\left[\frac{13}{72}, \frac{491}{2160}, \frac{3529}{14400}, \frac{167}{480}, \frac{1}{75}, \frac{13}{500}, \frac{701}{6000}, \frac{47}{900}, \frac{1}{100}, \frac{2}{75}, \frac{47}{900}, \frac{1}{30}\right],$$

$$\left[\frac{1}{24}, \frac{13}{144}, \frac{167}{960}, \frac{9}{32}, 0, \frac{1}{100}, \frac{19}{400}, \frac{1}{60}, 0, 0, \frac{1}{60}, 0\right],$$

$$\left[0, 0, \frac{1}{100}, 0, \frac{2209}{10800}, \frac{1379}{9000}, \frac{59}{1500}, \frac{1}{75}, \frac{1307}{7200}, \frac{93}{800}, \frac{61}{1200}, 0\right],$$

$$\left[0, \frac{1}{60}, \frac{13}{400}, \frac{1}{40}, \frac{1379}{5400}, \frac{919}{4500}, \frac{89}{750}, \frac{11}{225}, \frac{847}{3600}, \frac{83}{400}, \frac{27}{200}, \frac{1}{30}\right],$$

$$\left[\frac{1}{24}, \frac{47}{720}, \frac{701}{4800}, \frac{19}{160}, \frac{59}{900}, \frac{89}{750}, \frac{2921}{18000}, \frac{547}{2700}, \frac{33}{400}, \frac{397}{3600}, \frac{517}{2700}, \frac{31}{180}\right],$$

$$\left[0, \frac{1}{60}, \frac{47}{1200}, \frac{1}{40}, \frac{1}{75}, \frac{11}{375}, \frac{547}{4500}, \frac{781}{2700}, \frac{1}{100}, \frac{2}{75}, \frac{13}{225}, \frac{137}{360}\right],$$

$$\left[0, 0, \frac{1}{100}, 0, \frac{1307}{5400}, \frac{847}{4500}, \frac{33}{500}, \frac{1}{75}, \frac{811}{3600}, \frac{217}{1200}, \frac{179}{1800}, 0\right],$$

$$\left[0, 0, \frac{2}{75}, 0, \frac{31}{200}, \frac{83}{500}, \frac{397}{4500}, \frac{8}{225}, \frac{217}{1200}, \frac{691}{3600}, \frac{1007}{5400}, 0\right],$$

$$\left[0, \frac{1}{60}, \frac{47}{1200}, \frac{1}{40}, \frac{61}{1200}, \frac{81}{1000}, \frac{517}{4500}, \frac{13}{225}, \frac{179}{2400}, \frac{1007}{7200}, \frac{1849}{10800}, \frac{1}{30}\right],$$

$$\left[0, 0, \frac{1}{60}, 0, 0, \frac{1}{75}, \frac{31}{450}, \frac{137}{540}, 0, 0, \frac{1}{45}, \frac{25}{72}\right]\Bigg]$$

P^4;

$$\Bigg[\left[\frac{131}{432}, \frac{301}{1296}, \frac{881}{8640}, \frac{19}{288}, 0, \frac{1}{300}, \frac{77}{3600}, \frac{1}{180}, 0, 0, \frac{1}{180}, 0\right], \tag{5}$$

$$\left[\frac{301}{864}, \frac{7493}{25920}, \frac{28207}{172800}, \frac{881}{5760}, \frac{1}{300}, \frac{59}{6000}, \frac{3643}{72000}, \frac{67}{3600}, \frac{1}{400}, \frac{1}{150}, \frac{67}{3600}, \frac{1}{120}\right],$$

$$\left[\frac{881}{4320}, \frac{28207}{129600}, \frac{202421}{864000}, \frac{8539}{28800}, \frac{37}{2250}, \frac{1157}{30000}, \frac{35449}{360000}, \frac{3643}{54000}, \frac{19}{1000}, \frac{517}{18000}, \frac{3523}{54000}, \frac{77}{1800}\right],$$

$$\left[\frac{19}{288}, \frac{881}{8640}, \frac{8539}{57600}, \frac{437}{1920}, \frac{1}{300}, \frac{23}{2000}, \frac{1271}{24000}, \frac{77}{3600}, \frac{1}{400}, \frac{1}{150}, \frac{77}{3600}, \frac{1}{120}\right],$$

$$\left[0, \frac{1}{300}, \frac{37}{3000}, \frac{1}{200}, \frac{58243}{324000}, \frac{37523}{270000}, \frac{4801}{90000}, \frac{79}{4500}, \frac{35399}{216000}, \frac{3011}{24000}, \frac{2477}{36000}, \frac{1}{150}\right],$$

$$\left[\frac{1}{120}, \frac{59}{3600}, \frac{1157}{24000}, \frac{23}{800}, \frac{37523}{162000}, \frac{3446}{16875}, \frac{9707}{90000}, \frac{226}{3375}, \frac{6091}{27000}, \frac{391}{2000}, \frac{2767}{18000}, \frac{37}{900}\right],$$

$$\left[\frac{77}{1440}, \frac{3643}{43200}, \frac{35449}{288000}, \frac{1271}{9600}, \frac{4801}{54000}, \frac{9707}{90000}, \frac{177349}{1080000}, \frac{29003}{162000}, \frac{377}{4000}, \frac{13579}{108000}, \frac{12529}{81000}, \frac{253}{1350}\right],$$

$$\left[\frac{1}{120}, \frac{67}{3600}, \frac{3643}{72000}, \frac{77}{2400}, \frac{79}{4500}, \frac{226}{5625}, \frac{29003}{270000}, \frac{21367}{81000}, \frac{119}{6000}, \frac{557}{18000}, \frac{103}{1500}, \frac{3617}{10800}\right],$$

$$\left[0, \frac{1}{300}, \frac{19}{1000}, \frac{1}{200}, \frac{35399}{162000}, \frac{6091}{33750}, \frac{377}{5000}, \frac{119}{4500}, \frac{11291}{54000}, \frac{1561}{9000}, \frac{6233}{54000}, \frac{1}{150}\right],$$

$$\left[0, \frac{2}{225}, \frac{517}{18000}, \frac{1}{75}, \frac{3011}{18000}, \frac{391}{2500}, \frac{13579}{135000}, \frac{557}{13500}, \frac{1561}{9000}, \frac{9791}{54000}, \frac{25199}{162000}, \frac{4}{225}\right],$$

$$\left[\frac{1}{120}, \frac{67}{3600}, \frac{3523}{72000}, \frac{77}{2400}, \frac{2477}{36000}, \frac{2767}{30000}, \frac{12529}{135000}, \frac{103}{1500}, \frac{6233}{72000}, \frac{25199}{216000}, \frac{46003}{324000}, \frac{41}{900}\right],$$

$$\left[0, \frac{1}{180}, \frac{77}{3600}, \frac{1}{120}, \frac{1}{225}, \frac{37}{2250}, \frac{253}{3375}, \frac{3617}{16200}, \frac{1}{300}, \frac{2}{225}, \frac{41}{1350}, \frac{649}{2160}\right]\right]$$

P^5;

$$\left[\left[\frac{347}{1296}, \frac{16523}{77760}, \frac{54637}{518400}, \frac{1451}{17280}, \frac{1}{900}, \frac{89}{18000}, \frac{5953}{216000}, \frac{97}{10800}, \frac{1}{1200}, \frac{1}{450}, \frac{97}{10800}, \frac{1}{360}\right],\right. \qquad \textbf{(6)}$$

$$\left[\frac{16523}{51840}, \frac{415081}{1555200}, \frac{1700003}{10368000}, \frac{54637}{345600}, \frac{47}{9000}, \frac{5251}{360000}, \frac{225407}{4320000}, \frac{1861}{72000}, \frac{67}{12000}, \frac{677}{72000}, \frac{607}{24000}, \frac{97}{7200}\right],$$

$$\left[\frac{54637}{259200}, \frac{1700003}{7776000}, \frac{10975729}{51840000}, \frac{458591}{1728000}, \frac{6661}{270000}, \frac{72433}{1800000}, \frac{2177381}{21600000}, \frac{225407}{3240000}, \frac{1541}{60000}, \frac{40913}{1080000}, \frac{207827}{3240000}, \frac{5953}{108000}\right],$$

$$\left[\frac{1451}{17280}, \frac{54637}{518400}, \frac{458591}{3456000}, \frac{21649}{115200}, \frac{13}{2250}, \frac{1847}{120000}, \frac{73579}{1440000}, \frac{5953}{216000}, \frac{3}{500}, \frac{757}{72000}, \frac{5833}{216000}, \frac{107}{7200}\right],$$

$$\left[\frac{1}{600}, \frac{47}{9000}, \frac{6661}{360000}, \frac{13}{1500}, \frac{1563691}{9720000}, \frac{535753}{4050000}, \frac{157147}{2700000}, \frac{2327}{90000}, \frac{7696}{50625}, \frac{44741}{360000}, \frac{29713}{360000}, \frac{109}{9000}\right],$$

$$\left[\frac{89}{7200}, \frac{5251}{216000}, \frac{72433}{1440000}, \frac{1847}{48000}, \frac{535753}{2430000}, \frac{781471}{4050000}, \frac{627433}{5400000}, \frac{58301}{810000}, \frac{173527}{810000}, \frac{2191}{11250}, \frac{41137}{270000}, \frac{1459}{27000}\right],$$

$$\left[\frac{5953}{86400}, \frac{225407}{2592000}, \frac{2177381}{17280000}, \frac{73579}{576000}, \frac{157147}{1620000}, \frac{627433}{5400000}, \frac{9446081}{64800000}, \frac{1719307}{9720000}, \frac{37507}{360000}, \frac{195419}{1620000}, \frac{1440577}{9720000}, \frac{59363}{324000}\right],$$

$$\left[\frac{97}{7200}, \frac{1861}{72000}, \frac{225407}{4320000}, \frac{5953}{144000}, \frac{2327}{90000}, \frac{58301}{1350000}, \frac{1719307}{16200000}, \frac{285977}{1215000}, \frac{4883}{180000}, \frac{7183}{180000}, \frac{27949}{405000}, \frac{96989}{324000}\right],$$

$$\left[\frac{1}{600}, \frac{67}{9000}, \frac{1541}{60000}, \frac{3}{250}, \frac{30784}{151875}, \frac{173527}{1012500}, \frac{37507}{450000}, \frac{4883}{135000}, \frac{316517}{1620000}, \frac{91589}{540000}, \frac{98353}{810000}, \frac{149}{9000}\right],$$

$$\left[\frac{1}{225}, \frac{677}{54000}, \frac{40913}{1080000}, \frac{757}{36000}, \frac{44741}{270000}, \frac{4382}{28125}, \frac{195419}{2025000}, \frac{7183}{135000}, \frac{91589}{540000}, \frac{270017}{1620000}, \frac{177167}{1215000}, \frac{797}{27000}\right],$$

$$\left[\frac{97}{7200}, \frac{607}{24000}, \frac{207827}{4320000}, \frac{5833}{144000}, \frac{29713}{360000}, \frac{41137}{450000}, \frac{1440577}{16200000}, \frac{27949}{405000}, \frac{98353}{1080000}, \frac{177167}{1620000}, \frac{1138711}{9720000}, \frac{257}{4500}\right],$$

$$\left[\frac{1}{360}, \frac{97}{10800}, \frac{5953}{216000}, \frac{107}{7200}, \frac{109}{13500}, \frac{1459}{67500}, \frac{59363}{810000}, \frac{96989}{486000}, \frac{149}{18000}, \frac{797}{54000}, \frac{257}{6750}, \frac{16969}{64800}\right]\right]$$

That's hard to read, so we'll do the trick from the handout to get it into decimal form with 3 significant digits.$map(x \rightarrow evalf(x, 3), P^5)$;

[[0.268, 0.212, 0.105, 0.0840, 0.00111, 0.00494, 0.0276, 0.00898, 0.000833, 0.00222, 0.00898, 0.00278], **(7)**
[0.319, 0.267, 0.164, 0.158, 0.00522, 0.0146, 0.0522, 0.0258, 0.00558, 0.00940, 0.0253, 0.0135],
[0.211, 0.219, 0.212, 0.265, 0.0247, 0.0402, 0.101, 0.0696, 0.0257, 0.0379, 0.0641, 0.0551],
[0.0840, 0.105, 0.133, 0.188, 0.00578, 0.0154, 0.0511, 0.0276, 0.00600, 0.0105, 0.0270, 0.0149],
[0.00167, 0.00522, 0.0185, 0.00867, 0.161, 0.132, 0.0582, 0.0259, 0.152, 0.124, 0.0825, 0.0121],
[0.0124, 0.0243, 0.0503, 0.0385, 0.220, 0.193, 0.116, 0.0720, 0.214, 0.195, 0.152, 0.0540],
[0.0689, 0.0870, 0.126, 0.128, 0.0970, 0.116, 0.146, 0.177, 0.104, 0.121, 0.148, 0.183],
[0.0135, 0.0258, 0.0522, 0.0413, 0.0259, 0.0432, 0.106, 0.235, 0.0271, 0.0399, 0.0690, 0.299],
[0.00167, 0.00744, 0.0257, 0.0120, 0.203, 0.171, 0.0833, 0.0362, 0.195, 0.170, 0.121, 0.0166],
[0.00444, 0.0125, 0.0379, 0.0210, 0.166, 0.156, 0.0965, 0.0532, 0.170, 0.167, 0.146, 0.0295],
[0.0135, 0.0253, 0.0481, 0.0405, 0.0825, 0.0914, 0.0889, 0.0690, 0.0911, 0.109, 0.117, 0.0571],
[0.00278, 0.00898, 0.0276, 0.0149, 0.00807, 0.0216, 0.0733, 0.200, 0.00828, 0.0148, 0.0381, 0.262]]

Since we still saw some 0's in *P*^4 but there aren't any in *P*^5, we see that 5 is the smallest integer *k* such that *P*^*k* has no zero entries. In practical terms, this means that if we pick any two rooms *a* and *b* of the maze, then there is a nonzero probability that the robot will move from room *a* to room *b* in 5 steps. In other words, there is a way of getting from any given room to any other using at most 5 steps. The fact that the (1,5) and (5,1) entries of P^4 are 0 means that it's impossible to move from room 5 to room 1 or from room 1 to room 5 in 4 steps.

(5) Here is a computation of *P*^256:

$$map\left(x \rightarrow evalf(x, 3), \left(\left(\left(\left(\left(\left(\left((P^2)^2\right)^2\right)^2\right)^2\right)^2\right)^2\right)^2\right)\right);$$

[[0.0500, 0.0500, 0.0500, 0.0500, 0.0500, 0.0500, 0.0500, 0.0500, 0.0500, 0.0500, 0.0500, 0.0500], **(8)**

[0.0750, 0.0750, 0.0750, 0.0750, 0.0750, 0.0750, 0.0750, 0.0750, 0.0750, 0.0750, 0.0750, 0.0750],
[0.100, 0.100, 0.100, 0.100, 0.100, 0.100, 0.100, 0.100, 0.100, 0.100, 0.100, 0.100],
[0.0500, 0.0500, 0.0500, 0.0500, 0.0500, 0.0500, 0.0500, 0.0500, 0.0500, 0.0500, 0.0500, 0.0500],
[0.0750, 0.0750, 0.0750, 0.0750, 0.0750, 0.0750, 0.0750, 0.0750, 0.0750, 0.0750, 0.0750, 0.0750],
[0.125, 0.125, 0.125, 0.125, 0.125, 0.125, 0.125, 0.125, 0.125, 0.125, 0.125, 0.125],
[0.125, 0.125, 0.125, 0.125, 0.125, 0.125, 0.125, 0.125, 0.125, 0.125, 0.125, 0.125],
[0.0750, 0.0750, 0.0750, 0.0750, 0.0750, 0.0750, 0.0750, 0.0750, 0.0750, 0.0750, 0.0750, 0.0750],
[0.100, 0.100, 0.100, 0.100, 0.100, 0.100, 0.100, 0.100, 0.100, 0.100, 0.100, 0.100],
[0.100, 0.100, 0.100, 0.100, 0.100, 0.100, 0.100, 0.100, 0.100, 0.100, 0.100, 0.100],
[0.0750, 0.0750, 0.0750, 0.0750, 0.0750, 0.0750, 0.0750, 0.0750, 0.0750, 0.0750, 0.0750, 0.0750],
[0.0500, 0.0500, 0.0500, 0.0500, 0.0500, 0.0500, 0.0500, 0.0500, 0.0500, 0.0500, 0.0500, 0.0500]]

We note that every column of this matrix is the same, i.e., for a given row, every entry of that row is the same.

(6) Since P^k **e**_j is the j^th column of P^k and all the columns of P^k are the same if k is big enough, we see that the probability that a robot will eventually end up in room i is independent of where the robot starts. In particular, the probability that the robot will end up in each room is as follows:

room 1 -- .05
room 2 -- .075
room 3 -- .1
room 4 -- .05
room 5 -- .075
room 6 -- .125
room 7 -- .125
room 8 -- .075
room 9 -- .1
room 10 -- .1
room 11 -- .075
room 12 -- .05

(7) To say $P\mathbf{x} = \mathbf{x}$ is the same as $P\mathbf{x} - \mathbf{x} = 0$, which is the same as $P\mathbf{x} - I\mathbf{x} = \mathbf{0}$, which is the same as $(P-I)\mathbf{x} = \mathbf{0}$. So we set $A = P-I$ and find solutions to the equation $A\mathbf{x} = \mathbf{0}$ (i.e., we find the nullspace of A).

$A := P\text{-}IdentityMatrix(12);$

$$\begin{bmatrix} -\frac{1}{2} & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & -\frac{2}{3} & \frac{1}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & -\frac{3}{4} & \frac{1}{2} & 0 & 0 & \frac{1}{5} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{4} & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{2}{3} & \frac{1}{5} & 0 & 0 & \frac{1}{4} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{3} & -\frac{4}{5} & \frac{1}{5} & 0 & \frac{1}{4} & \frac{1}{4} & 0 & 0 \\ 0 & 0 & \frac{1}{4} & 0 & 0 & \frac{1}{5} & -\frac{4}{5} & \frac{1}{3} & 0 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{5} & -\frac{2}{3} & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & \frac{1}{3} & \frac{1}{5} & 0 & 0 & -\frac{3}{4} & \frac{1}{4} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{5} & 0 & 0 & \frac{1}{4} & -\frac{3}{4} & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{5} & 0 & 0 & \frac{1}{4} & -\frac{2}{3} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{3} & 0 & 0 & 0 & -\frac{1}{2} \end{bmatrix} \quad \textbf{(9)}$$

$ReducedRowEchelonForm(A);$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{3}{2} \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{3}{2} \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -\frac{5}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -\frac{5}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -\frac{3}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -\frac{3}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad \textbf{(10)}$$

We see that the nullspace is 1–dimensional, spanned by the transpose of the vector **y**:

$$y := Vector[column]\left(\left[1, \frac{3}{2}, 2, 1, \frac{3}{2}, \frac{5}{2}, \frac{5}{2}, \frac{3}{2}, 2, 2, \frac{3}{2}, 1\right]\right);$$

$$\begin{bmatrix} 1 \\ \frac{3}{2} \\ 2 \\ 1 \\ \frac{3}{2} \\ \frac{5}{2} \\ \frac{5}{2} \\ \frac{3}{2} \\ 2 \\ 2 \\ \frac{3}{2} \\ 1 \end{bmatrix} \tag{11}$$

We want a probability vector, i.e., a vector whose entries sum to 1. So we divide **y** by the sum of its entries and set that new vector to be **x**.

$$x := \frac{1}{\left(1 + \frac{3}{2} + 2 + 1 + \frac{3}{2} + \frac{5}{2} + \frac{5}{2} + \frac{3}{2} + 2 + 2 + \frac{3}{2} + 1\right)} y;$$

$$\begin{bmatrix} \frac{1}{20} \\ \frac{3}{40} \\ \frac{1}{10} \\ \frac{1}{20} \\ \frac{3}{40} \\ \frac{1}{8} \\ \frac{1}{8} \\ \frac{3}{40} \\ \frac{1}{10} \\ \frac{1}{10} \\ \frac{3}{40} \\ \frac{1}{20} \end{bmatrix} \tag{12}$$

Now let's double check; it'll be easier to compare if we again convert to decimals:

$map(t \rightarrow evalf(t, 3), x);$

$$\begin{bmatrix} 0.0500 \\ 0.0750 \\ 0.100 \\ 0.0500 \\ 0.0750 \\ 0.125 \\ 0.125 \\ 0.0750 \\ 0.100 \\ 0.100 \\ 0.0750 \\ 0.0500 \end{bmatrix} \quad (13)$$

$map(t \rightarrow evalf(t, 3), P.x)$

$$\begin{bmatrix} 0.0500 \\ 0.0750 \\ 0.100 \\ 0.0500 \\ 0.0750 \\ 0.125 \\ 0.125 \\ 0.0750 \\ 0.100 \\ 0.100 \\ 0.0750 \\ 0.0500 \end{bmatrix} \quad (14)$$

So our vector **x** is indeed the probability vector we seek. We notice that it is equal to each column of *P*^256.

(8) The rooms with the fewest connections correspond to the smallest entries of x, and the more connections, the larger the corresponding entry. For example, both room 1 and room 12 have only one hallway connecting to them, and 1st and 12th entries of x are both only .05. We notice also that the 2nd, 5th, 8th and 11th entries are all .075, and the corresponding rooms each have 2 hallways. Similarly, the 3rd, 9th and 10th entries are .1 and those rooms have 3 hallways each. And the 6th and

7th entries are .125, with the corresponding rooms having 4 hallways each.

(9) If we form a new maze by blocking off the hallway joining rooms 3 and 7, we do not expect the corresponding transition matrix Q to be regular. This is because it will now be impossible to ever get from rooms 1, 2, 3 or 4 to any of rooms 5--12 and conversely. So we will always have a 0 in both the (i,j) and (j,i) entries of Q^k for 1 <= i <= 4 and 5 <= j <= 12. For fun, let's do it:

Q := *Transpose*(*Matrix*([[1 / 2, 1 / 2, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0], [1 / 3, 1 / 3, 1 / 3, 0, 0, 0, 0, 0, 0, 0, 0, 0], [0, 1 / 3, 1 / 3, 1 / 3, 0, 0, 0, 0, 0, 0, 0, 0], [0, 0, 1 / 2, 1 / 2, 0, 0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 1 / 3, 1 / 3, 0, 0, 1 / 3, 0, 0, 0], [0, 0, 0, 0, 1 / 5, 1 / 5, 1 / 5, 0, 1 / 5, 1 / 5, 0, 0], [0, 0, 0, 0, 0, 1 / 4, 1 / 4, 1 / 4, 0, 0, 1 / 4, 0], [0, 0, 0, 0, 0, 0, 1 / 3, 1 / 3, 0, 0, 0, 1 / 3], [0, 0, 0, 0, 1 / 4, 1 / 4, 0, 0, 1 / 4, 1 / 4, 0, 0], [0, 0, 0, 0, 0, 1 / 4, 0, 0, 1 / 4, 1 / 4, 1 / 4, 0], [0, 0, 0, 0, 0, 0, 1 / 3, 0, 0, 1 / 3, 1 / 3, 0], [0, 0, 0, 0, 0, 0, 0, 1 / 2, 0, 0, 0, 1 / 2]]));

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{3} & \frac{1}{5} & 0 & 0 & \frac{1}{4} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{3} & \frac{1}{5} & \frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{4} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{5} & \frac{1}{4} & \frac{1}{3} & 0 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{4} & \frac{1}{3} & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & \frac{1}{3} & \frac{1}{5} & 0 & 0 & \frac{1}{4} & \frac{1}{4} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{5} & 0 & 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{4} & 0 & 0 & \frac{1}{4} & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{3} & 0 & 0 & 0 & \frac{1}{2} \end{bmatrix} \quad \textbf{(15)}$$

$$map\left(x \to evalf(x, 3), \left(\left(\left(\left(\left(\left(\left(\left(Q^2\right)^2\right)^2\right)^2\right)^2\right)^2\right)^2\right)^2\right)\right);$$

$$[[0.200, 0.200, 0.200, 0.200, 0., 0., 0., 0., 0., 0., 0., 0.],$$
$$[0.300, 0.300, 0.300, 0.300, 0., 0., 0., 0., 0., 0., 0., 0.],$$
$$[0.300, 0.300, 0.300, 0.300, 0., 0., 0., 0., 0., 0., 0., 0.],$$
$$[0.200, 0.200, 0.200, 0.200, 0., 0., 0., 0., 0., 0., 0., 0.],$$
$$[0., 0., 0., 0., 0.107, 0.107, 0.107, 0.107, 0.107, 0.107, 0.107, 0.107],$$
$$[0., 0., 0., 0., 0.179, 0.179, 0.179, 0.179, 0.179, 0.179, 0.179, 0.179],$$
$$[0., 0., 0., 0., 0.143, 0.143, 0.143, 0.143, 0.143, 0.143, 0.143, 0.143],$$
$$[0., 0., 0., 0., 0.107, 0.107, 0.107, 0.107, 0.107, 0.107, 0.107, 0.107],$$
$$[0., 0., 0., 0., 0.143, 0.143, 0.143, 0.143, 0.143, 0.143, 0.143, 0.143],$$
$$[0., 0., 0., 0., 0.143, 0.143, 0.143, 0.143, 0.143, 0.143, 0.143, 0.143],$$
$$[0., 0., 0., 0., 0.107, 0.107, 0.107, 0.107, 0.107, 0.107, 0.107, 0.107],$$
$$[0., 0., 0., 0., 0.0714, 0.0714, 0.0714, 0.0714, 0.0714, 0.0714, 0.0714, 0.0714]] \quad (16)$$

To be sure, we can compute the steady-state vector. We have:

$B := Q - IdentityMatrix(12);$

$$\begin{bmatrix} -\frac{1}{2} & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & -\frac{2}{3} & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & -\frac{2}{3} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{2}{3} & \frac{1}{5} & 0 & 0 & \frac{1}{4} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{3} & -\frac{4}{5} & \frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{4} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{5} & -\frac{3}{4} & \frac{1}{3} & 0 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{4} & -\frac{2}{3} & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & \frac{1}{3} & \frac{1}{5} & 0 & 0 & -\frac{3}{4} & \frac{1}{4} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{5} & 0 & 0 & \frac{1}{4} & -\frac{3}{4} & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{4} & 0 & 0 & \frac{1}{4} & -\frac{2}{3} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{3} & 0 & 0 & 0 & -\frac{1}{2} \end{bmatrix} \qquad \textbf{(17)}$$

ReducedRowEchelonForm(*B*);

$$\begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -\frac{3}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -\frac{3}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{3}{2} \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -\frac{5}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -\frac{3}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -\frac{3}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{18}$$

We see this time that the null space is two-dimensional, spanned by the probability vectors

$$z := \frac{1}{\left(1 + \frac{3}{2} + \frac{3}{2} + 1\right)} \mathit{Vector}[\mathit{column}]\left(\left[1, \frac{3}{2}, \frac{3}{2}, 1, 0, 0, 0, 0, 0, 0, 0, 0\right]\right);$$

$$\begin{bmatrix} \frac{1}{5} \\ \frac{3}{10} \\ \frac{3}{10} \\ \frac{1}{5} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad (19)$$

and

$$w := \frac{1}{\left(\frac{3}{2} + \frac{5}{2} + 2 + \frac{3}{2} + 2 + 2 + \frac{3}{2} + 1\right)} \mathit{Vector}[\mathit{column}]\left(\left[0, 0, 0, 0, \frac{3}{2}, \frac{5}{2}, 2, \frac{3}{2}, 2, 2, \frac{3}{2}, 1\right]\right);$$

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \frac{3}{28} \\ \frac{5}{28} \\ \frac{1}{7} \\ \frac{3}{28} \\ \frac{1}{7} \\ \frac{1}{7} \\ \frac{3}{28} \\ \frac{1}{14} \end{bmatrix} \quad (20)$$

Thus we see that if we start in rooms 1, 2, 3 or 4 we can only end in rooms 1, 2, 3 or 4. And if we start in rooms 5, 6, 7, 8, 9, 10, 11 or 12, we can only end in rooms 5, 6, 7, 8, 9, 10, 11 or 12.