Fun with Bases

Math 314—006
Application Mini-Project #4

Due: Tuesday, December 9

Special Policy: This is an "optional" project. If you choose not to do it, then the score I will give you for your mini-project #4 grade will be equal to the highest score you received on the first three mini-projects. If you choose to do it, I will grade it and give you a score as usual. (In the unfortunate situation that the score you receive is lower than one of your previous mini-project grades, I will record the highest score from the previous mini-projects as your grade on this mini-project.)

Goal: The goal of this mini-project is to show how one can use abstract vector spaces to solve problems from calculus.

Instructions: *This project is to be done in groups of two or three students, as assigned by the instructor.* Groups of size four or more are not permitted. Each group will turn in one solution to the tasks described below, and each member of the group will receive the same grade on the project. You should be careful to understand each part of the project – related questions may appear on exams.

Submission Guidelines: Please use the following guidelines when preparing your project for submission:

1. Include a cover page on which each member of the group *signs* their full name. Also, in the top right hand corner of each page submitted, write the names (first and last, written legibly) of each group member. You should not include student ID or social security numbers.

2. Only use one side of each sheet of paper and staple the pages together.

3. Presentation will be considered when grading this project. You should take care to write legibly and hand in full sheets of paper with no fringe, tears, etc. You should also clearly label each problem and submit these in order.

4. Give justification (in complete sentences!) for your answers.

5. Be academically honest. This means, for example, providing a list of sources other than the textbook (if any) that you used to do the assignment; stating clearly that you're copying or mimicking an example from the book in order to do the assignment (if appropriate).

6. The project is due at the beginning of class. Under certain circumstances, late submissions may be accepted, but they will be penalized.

Project Tasks:

1. Let n be any positive integer. Show that $\{1, \cos(t), \cos^2(t), \cos^3(t), \ldots, \cos^n(t)\}$ is a linearly independent subset of $\mathcal{F}$ (the vector space of all functions). Here is how you can do this: Learn about Vandermonde determinants. (These are discussed briefly on page 188 of your text, or you can learn about them from a variety of other sources.) Now consider a hypothetical dependence relation of the form
$$c_1 + c_2\cos(t) + c_3\cos^2(t) + \cdots + c_{n+1}\cos^n(t) = 0.$$
Pick $n+1$ values of t, say $t = t_1, t_2, \ldots, t_{n+1}$, such that $\cos(t_1), \ldots, \cos(t_{n+1})$ are distinct numbers. (We'll take it as obvious that such t_i's exist.) This gives $n+1$ equations involving the constants $c_1, \ldots, c_{n+1}$. Now use what you learned about Vandermonde determinants to conclude that we must have $c_1 = c_2 = \cdots = c_{n+1} = 0$.

2. Let V be the subspace of $\mathcal{F}$ spanned by $B = \{1, \cos(t), \cos^2(t), \cos^3(t), \cos^4(t), \cos^5(t)\}$. Since B is linearly independent (as you showed in (1)), we have that B is a basis of V. Using the trigonometric identities

$$\begin{aligned}
\cos(2t) &= -1 + 2\cos^2(t) \\
\cos(3t) &= -3\cos(t) + 4\cos^3(t) \\
\cos(4t) &= 1 - 8\cos^2(t) + 8\cos^4(t) \\
\cos(5t) &= 5\cos(t) - 20\cos^3(t) + 16\cos^5(t)
\end{aligned}$$

write the B-coordinate vector for each of the functions $1, \cos(t), \cos(2t), \cos(3t), \cos(4t), \cos(5t)$.

3. Use the calculations from the previous part to show that $C = \{1, \cos(t), \cos(2t), \cos(3t), \cos(4t), \cos(5t)\}$ is another basis of V.

4. Use the calculations from (2) to find the change of basis matrix $P_{B \leftarrow C}$ and then use a calculator to find $P_{C \leftarrow B}$.

5. Use $P_{C \leftarrow B}$ to calculate

$$\int \left(a_0 + a_1\cos(t) + a_2\cos^2(t) + a_3\cos^3(t) + a_4\cos^4(t) + a_5\cos^5(t)\right)\, dt$$

where $a_0, \ldots, a_5$ are arbitrary constants, by first transforming the integrand into a linear combination of the functions in C.