Fun with Bases

Math 314—002

Application Mini-Project #4

Solutions

1. Let n be any positive integer. Show that $\{1, \cos(t), \cos^2(t), \cos^3(t), \ldots, \cos^n(t)\}$ is a linearly independent subset of $\mathcal{F}$ (the vector space of all functions). Here is how you can do this: Learn about Vandermonde determinants. (These are discussed briefly on page 288 of your text, or you can learn about them from a variety of other sources.) Now consider a hypothetical dependence relation of the form
$$c_1 + c_2\cos(t) + c_3\cos^2(t) + \cdots + c_{n+1}\cos^n(t) = 0.$$
Pick $n+1$ values of t, say $t = t_1, t_2, \ldots, t_{n+1}$, such that $\cos(t_1), \ldots, \cos(t_{n+1})$ are distinct numbers. (We'll take it as obvious that such t_i's exist.) This gives $n+1$ equations involving the constants $c_1, \ldots, c_{n+1}$. Now use what you learned about Vandermonde determinants to conclude that we must have $c_1 = c_2 = \cdots = c_{n+1} = 0$.

Solution: Suppose we have scalars $c_1, c_2, \ldots, c_{n+1}$ such that
$$c_1(1) + c_2\cos(t) + c_3\cos^2(t) + \cdots + c_{n+1}\cos^n(t) = 0,$$
where 0 denotes the zero function f defined by $f(t) = 0$ for all t.

As suggested, we pick $n+1$ values of t, say $t = t_1, t_2, \ldots, t_{n+1}$ such that $\cos(t_1), \cos(t_2), \ldots, \cos(t_{n+1})$ are distinct numbers. Plugging these values of t into the above equation gives us the following $n+1$ equations in the variables $c_1, c_2, \ldots, c_{n+1}$:

$$\begin{array}{rcl}
0 &=& (1)c_1 + \cos(t_1)(c_2) + \cos^2(t_1)(c_3) + \cdots + \cos^n(t_1)(c_{n+1}) \\
0 &=& (1)c_1 + \cos(t_2)(c_2) + \cos^2(t_2)(c_3) + \cdots + \cos^n(t_2)(c_{n+1}) \\
\cdots &=& \cdots\cdots\cdots \\
0 &=& (1)c_1 + \cos(t_n)(c_2) + \cos^2(t_n)(c_3) + \cdots + \cos^n(t_n)(c_{n+1}) \\
0 &=& (1)c_1 + \cos(t_{n+1})(c_2) + \cos^2(t_{n+1})(c_3) + \cdots + \cos^n(t_{n+1})(c_{n+1})
\end{array}$$

We can express this system of linear equations in the matrix equation $A\mathbf{c} = \mathbf{0}$ where

$$A = \begin{bmatrix} 1 & \cos(t_1) & \cos^2(t_1) & \cdots & \cos^n(t_1) \\ 1 & \cos(t_2) & \cos^2(t_2) & \cdots & \cos^n(t_2) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \cos(t_n) & \cos^2(t_n) & \cdots & \cos^n(t_n) \\ 1 & \cos(t_{n+1}) & \cos^2(t_{n+1}) & \cdots & \cos^n(t_{n+1}) \end{bmatrix}$$

and

$$\mathbf{c} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \\ c_{n+1} \end{bmatrix}.$$

Using Vandermonde determinants, we have that

$$\begin{array}{rcl}
\det(A) &=& \displaystyle\prod_{1 \le i < j \le n+1} (\cos(t_j) - \cos(t_i)) \\
&=& (\cos(t_2) - \cos(t_1)) \cdots (\cos(t_{n+1}) - \cos(t_1))(\cos(t_3) - \cos(t_2)) \cdots (\cos(t_{n+1}) - \cos(t_2)) \cdots \\
&& \cdots (\cos(t_{n+1}) - \cos(t_n))
\end{array}$$

Since the numbers $\cos(t_1), \cos(t_2), \ldots, \cos(t_{n+1})$ are distinct, the number $\cos(t_j) - \cos(t_i) \neq 0$ for all $1 \leq i < j \leq n+1$. Thus, $\det(A) \neq 0$. We conclude that the $(n+1) \times (n+1)$ matrix A is invertible. This means that the system $A\mathbf{c} = \mathbf{0}$ has the unique solution

$$\mathbf{c} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ \vdots \\ c_n \\ c_{n+1} \end{bmatrix} = A^{-1}\mathbf{0} = \mathbf{0}.$$

Since $\mathbf{c} = \mathbf{0}$, we must have that

$$c_1 = c_2 = c_3 = \cdots = c_n = c_{n+1} = 0.$$

Therefore,

$$B = \{1, \cos(t), \cos^2(t), \ldots, \cos^n(t)\}$$

is a linearly independent subset of $\mathcal{F}$.

2. Let V be the subspace of $\mathcal{F}$ spanned by $B = \{1, \cos(t), \cos^2(t), \cos^3(t), \cos^4(t), \cos^5(t)\}$. Since B is linearly independent (as you showed in (1)), we have that B is a basis of V. Using the trigonometric identities

$$\begin{aligned} \cos(2t) &= -1 + 2\cos^2(t) \\ \cos(3t) &= -3\cos(t) + 4\cos^3(t) \\ \cos(4t) &= 1 - 8\cos^2(t) + 8\cos^4(t) \\ \cos(5t) &= 5\cos(t) - 20\cos^3(t) + 16\cos^5(t) \end{aligned}$$

write the B-coordinate vector for each of the functions $1, \cos(t), \cos(2t), \cos(3t), \cos(4t), \cos(5t)$.

Solution: Using the trig identities, we express the functions $1, \cos(t), \cos(2t), \cos(3t), \cos(4t), \cos(5t)$ as linear combinations of the functions in B:

$$\begin{array}{rcl} 1 &=& 1(1) + 0\cos(t) + 0\cos^2(t) + 0\cos^3(t) + 0\cos^4(t) + 0\cos^5(t) \\ \cos(t) &=& 0(1) + 1\cos(t) + 0\cos^2(t) + 0\cos^3(t) + 0\cos^4(t) + 0\cos^5(t) \\ \cos(2t) &=& -1(1) + 0\cos(t) + 2\cos^2(t) + 0\cos^3(t) + 0\cos^4(t) + 0\cos^5(t) \\ \cos(3t) &=& 0(1) - 3\cos(t) + 0\cos^2(t) + 4\cos^3(t) + 0\cos^4(t) + 0\cos^5(t) \\ \cos(4t) &=& 1(1) + 0\cos(t) - 8\cos^2(t) + 0\cos^3(t) + 8\cos^4(t) + 0\cos^5(t) \\ \cos(5t) &=& 0(1) + 5\cos(t) + 0\cos^2(t) - 20\cos^3(t) + 0\cos^4(t) + 16\cos^5(t) \end{array}$$

Thus, by definition, we have the following B-coordinate vectors:

$$[1]_B = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, [\cos(t)]_B = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, [\cos(2t)]_B = \begin{bmatrix} -1 \\ 0 \\ 2 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

$$[\cos(3t)]_B = \begin{bmatrix} 0 \\ -3 \\ 0 \\ 4 \\ 0 \\ 0 \end{bmatrix}, [\cos(4t)]_B = \begin{bmatrix} 1 \\ 0 \\ -8 \\ 0 \\ 8 \\ 0 \end{bmatrix}, [\cos(5t)]_B = \begin{bmatrix} 0 \\ 5 \\ 0 \\ -20 \\ 0 \\ 16 \end{bmatrix}.$$

3. Use the calculations from the previous part to show that $C = \{1, \cos(t), \cos(2t), \cos(3t), \cos(4t), \cos(5t)\}$ is another basis of V.

Solution: Let A be the 6×6 matrix whose columns are the coordinate vectors we found in Task 2. That is, let

$$A = \begin{bmatrix} 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -3 & 0 & 5 \\ 0 & 0 & 2 & 0 & -8 & 0 \\ 0 & 0 & 0 & 4 & 0 & -20 \\ 0 & 0 & 0 & 0 & 8 & 0 \\ 0 & 0 & 0 & 0 & 0 & 16 \end{bmatrix}.$$

We see that A is already in row-echelon form. Moreover, since there is a pivot (i.e., a leading entry) in each column of A, we know that the column vectors of A form a linearly independent set in $\mathbb{R}^6$. Therefore, by Theorem 6.7, the set $C = \{1, \cos(t), \cos(2t), \cos(3t), \cos(4t), \cos(5t)\}$ is a linearly independent set of functions.

Knowing that B is a basis for V, we see that $\dim(V) = 6$. Since the functions in C are linear combinations of the functions in B, we know that the functions in C are also in V. Therefore, since C has $6 = \dim(V)$ linearly independent functions in V, C must also be a basis for V by Theorem 6.10 (c).

4. Use the calculations from (2) to find the change of basis matrix $P_{B \leftarrow C}$ and then use a calculator to find $P_{C \leftarrow B}$.

Solution: By definition,

$$\begin{aligned} P_{B \leftarrow C} &= \begin{bmatrix} [1]_B & [\cos(t)]_B & [\cos(2t)]_B & [\cos(3t)]_B & [\cos(4t)]_B & [\cos(5t)]_B \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -3 & 0 & 5 \\ 0 & 0 & 2 & 0 & -8 & 0 \\ 0 & 0 & 0 & 4 & 0 & -20 \\ 0 & 0 & 0 & 0 & 8 & 0 \\ 0 & 0 & 0 & 0 & 0 & 16 \end{bmatrix}. \end{aligned}$$

By Theorem 6.12 (c), $P_{C \leftarrow B} = (P_{B \leftarrow C})^{-1}$. Using a CAS we find

$$P_{C \leftarrow B} = (P_{B \leftarrow C})^{-1} = \begin{bmatrix} 1 & 0 & \frac{1}{2} & 0 & \frac{3}{8} & 0 \\ 0 & 1 & 0 & \frac{3}{4} & 0 & \frac{5}{8} \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{4} & 0 & \frac{5}{16} \\ 0 & 0 & 0 & 0 & \frac{1}{8} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{16} \end{bmatrix}.$$

5. Use $P_{C \leftarrow B}$ to calculate

$$\int \left(a_0 + a_1 \cos(t) + a_2 \cos^2(t) + a_3 \cos^3(t) + a_4 \cos^4(t) + a_5 \cos^5(t)\right) dt$$

where $a_0, \ldots, a_5$ are arbitrary constants, by first transforming the integrand into a linear combination of the functions in C.

Solution: Let $f(t) = a_0 + a_1 \cos(t) + a_2 \cos^2(t) + a_3 \cos^3(t) + a_4 \cos^4(t) + a_5 \cos^5(t)$. Then

$$[f(t)]_B = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix}.$$

By Theorem 6.12 (a),

$$\begin{aligned}
[f(t)]_C &= P_{C\leftarrow B}[f(t)]_B \\
&= \begin{bmatrix} 1 & 0 & \frac{1}{2} & 0 & \frac{3}{8} & 0 \\ 0 & 1 & 0 & \frac{3}{4} & 0 & \frac{5}{8} \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{4} & 0 & \frac{5}{16} \\ 0 & 0 & 0 & 0 & \frac{1}{8} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{16} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix} \\
&= \begin{bmatrix} a_0 + \frac{1}{2}a_2 + \frac{3}{8}a_4 \\ a_1 + \frac{3}{4}a_3 + \frac{5}{8}a_5 \\ \frac{1}{2}a_2 + \frac{1}{2}a_4 \\ \frac{1}{4}a_3 + \frac{5}{16}a_5 \\ \frac{1}{8}a_4 \\ \frac{1}{16}a_5 \end{bmatrix}
\end{aligned}$$

This means that

$$\begin{aligned}
f(t) &= \left(a_0 + \frac{1}{2}a_2 + \frac{3}{8}a_4\right)(1) + \left(a_1 + \frac{3}{4}a_3 + \frac{5}{8}a_5\right)\cos(t) + \left(\frac{1}{2}a_2 + \frac{1}{2}a_4\right)\cos(2t) \\
&\quad + \left(\frac{1}{4}a_3 + \frac{5}{16}a_5\right)\cos(3t) + \left(\frac{1}{8}a_4\right)\cos(4t) + \left(\frac{1}{16}a_5\right)\cos(5t).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\int f(t)\,dt &= \left(a_0 + \frac{1}{2}a_2 + \frac{3}{8}a_4\right)(t) + \left(a_1 + \frac{3}{4}a_3 + \frac{5}{8}a_5\right)\sin(t) + \left(\frac{1}{2}a_2 + \frac{1}{2}a_4\right)\frac{1}{2}\sin(2t) \\
&\quad + \left(\frac{1}{4}a_3 + \frac{5}{16}a_5\right)\frac{1}{3}\sin(3t) + \left(\frac{1}{8}a_4\right)\frac{1}{4}\sin(4t) + \left(\frac{1}{16}a_5\right)\frac{1}{5}\sin(5t) + C,
\end{aligned}$$

where C is the constant of integration.