

Fun with Bases

Math 314—006

Application Mini-Project #4

Solutions

1. Let n be any positive integer. Show that $\{1, \cos(t), \cos^2(t), \cos^3(t), \dots, \cos^n(t)\}$ is a linearly independent subset of \mathcal{F} (the vector space of all functions). Here is how you can do this: Learn about Vandermonde determinants. (These are discussed briefly on page 288 of your text, or you can learn about them from a variety of other sources.) Now consider a hypothetical dependence relation of the form

$$c_1 + c_2 \cos(t) + c_3 \cos^2(t) + \dots + c_{n+1} \cos^n(t) = 0.$$

Pick $n + 1$ values of t , say $t = t_1, t_2, \dots, t_{n+1}$, such that $\cos(t_1), \dots, \cos(t_{n+1})$ are distinct numbers. (We'll take it as obvious that such t_i 's exist.) This gives $n + 1$ equations involving the constants c_1, \dots, c_{n+1} . Now use what you learned about Vandermonde determinants to conclude that we must have $c_1 = c_2 = \dots = c_{n+1} = 0$.

Solution: Suppose we have scalars c_1, c_2, \dots, c_{n+1} such that

$$c_1(1) + c_2 \cos(t) + c_3 \cos^2(t) + \dots + c_{n+1} \cos^n(t) = 0,$$

where 0 denotes the zero function f defined by $f(t) = 0$ for all t .

As suggested, we pick $n + 1$ values of t , say $t = t_1, t_2, \dots, t_{n+1}$ such that $\cos(t_1), \cos(t_2), \dots, \cos(t_{n+1})$ are distinct numbers. Plugging these values of t into the above equation gives us the following $n + 1$ equations in the variables c_1, c_2, \dots, c_{n+1} :

$$\begin{aligned} 0 &= (1)c_1 + \cos(t_1)(c_2) + \cos^2(t_1)(c_3) + \dots + \cos^n(t_1)(c_{n+1}) \\ 0 &= (1)c_1 + \cos(t_2)(c_2) + \cos^2(t_2)(c_3) + \dots + \cos^n(t_2)(c_{n+1}) \\ \dots &= \dots\dots\dots \\ 0 &= (1)c_1 + \cos(t_n)(c_2) + \cos^2(t_n)(c_3) + \dots + \cos^n(t_n)(c_{n+1}) \\ 0 &= (1)c_1 + \cos(t_{n+1})(c_2) + \cos^2(t_{n+1})(c_3) + \dots + \cos^n(t_{n+1})(c_{n+1}) \end{aligned}$$

We can express this system of linear equations in the matrix equation $A\mathbf{c} = \mathbf{0}$ where

$$A = \begin{bmatrix} 1 & \cos(t_1) & \cos^2(t_1) & \dots & \cos^n(t_1) \\ 1 & \cos(t_2) & \cos^2(t_2) & \dots & \cos^n(t_2) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \cos(t_n) & \cos^2(t_n) & \dots & \cos^n(t_n) \\ 1 & \cos(t_{n+1}) & \cos^2(t_{n+1}) & \dots & \cos^n(t_{n+1}) \end{bmatrix}$$

and

$$\mathbf{c} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \\ c_{n+1} \end{bmatrix}.$$

Using Vandermonde determinants, we have that

$$\begin{aligned} \det(A) &= \prod_{1 \leq i < j \leq n+1} (\cos(t_j) - \cos(t_i)) \\ &= (\cos(t_2) - \cos(t_1)) \dots (\cos(t_{n+1}) - \cos(t_1)) (\cos(t_3) - \cos(t_2)) \dots (\cos(t_{n+1}) - \cos(t_2)) \dots \\ &\quad \dots (\cos(t_{n+1}) - \cos(t_n)) \end{aligned}$$

Since the numbers $\cos(t_1), \cos(t_2), \dots, \cos(t_{n+1})$ are distinct, the number $\cos(t_j) - \cos(t_i) \neq 0$ for all $1 \leq i < j \leq n+1$. Thus, $\det(A) \neq 0$. We conclude that the $(n+1) \times (n+1)$ matrix A is invertible. This means that the system $A\mathbf{c} = \mathbf{0}$ has the unique solution

$$\mathbf{c} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ \vdots \\ c_n \\ c_{n+1} \end{bmatrix} = A^{-1}\mathbf{0} = \mathbf{0}.$$

Since $\mathbf{c} = \mathbf{0}$, we must have that

$$c_1 = c_2 = c_3 = \dots = c_n = c_{n+1} = 0.$$

Therefore,

$$B = \{1, \cos(t), \cos^2(t), \dots, \cos^n(t)\}$$

is a linearly independent subset of \mathcal{F} .

2. Let V be the subspace of \mathcal{F} spanned by $B = \{1, \cos(t), \cos^2(t), \cos^3(t), \cos^4(t), \cos^5(t)\}$. Since B is linearly independent (as you showed in (1)), we have that B is a basis of V . Using the trigonometric identities

$$\begin{aligned} \cos(2t) &= -1 + 2\cos^2(t) \\ \cos(3t) &= -3\cos(t) + 4\cos^3(t) \\ \cos(4t) &= 1 - 8\cos^2(t) + 8\cos^4(t) \\ \cos(5t) &= 5\cos(t) - 20\cos^3(t) + 16\cos^5(t) \end{aligned}$$

write the B -coordinate vector for each of the functions $1, \cos(t), \cos(2t), \cos(3t), \cos(4t), \cos(5t)$.

Solution: Using the trig identities, we express the functions $1, \cos(t), \cos(2t), \cos(3t), \cos(4t), \cos(5t)$ as linear combinations of the functions in B :

$$\begin{aligned} 1 &= 1(1) + 0\cos(t) + 0\cos^2(t) + 0\cos^3(t) + 0\cos^4(t) + 0\cos^5(t) \\ \cos(t) &= 0(1) + 1\cos(t) + 0\cos^2(t) + 0\cos^3(t) + 0\cos^4(t) + 0\cos^5(t) \\ \cos(2t) &= -1(1) + 0\cos(t) + 2\cos^2(t) + 0\cos^3(t) + 0\cos^4(t) + 0\cos^5(t) \\ \cos(3t) &= 0(1) - 3\cos(t) + 0\cos^2(t) + 4\cos^3(t) + 0\cos^4(t) + 0\cos^5(t) \\ \cos(4t) &= 1(1) + 0\cos(t) - 8\cos^2(t) + 0\cos^3(t) + 8\cos^4(t) + 0\cos^5(t) \\ \cos(5t) &= 0(1) + 5\cos(t) + 0\cos^2(t) - 20\cos^3(t) + 0\cos^4(t) + 16\cos^5(t) \end{aligned}$$

Thus, by definition, we have the following B -coordinate vectors:

$$\begin{aligned} [1]_B &= \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, [\cos(t)]_B = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, [\cos(2t)]_B = \begin{bmatrix} -1 \\ 0 \\ 2 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \\ [\cos(3t)]_B &= \begin{bmatrix} 0 \\ -3 \\ 0 \\ 4 \\ 0 \\ 0 \end{bmatrix}, [\cos(4t)]_B = \begin{bmatrix} 1 \\ 0 \\ -8 \\ 0 \\ 8 \\ 0 \end{bmatrix}, [\cos(5t)]_B = \begin{bmatrix} 0 \\ 5 \\ 0 \\ -20 \\ 0 \\ 16 \end{bmatrix}. \end{aligned}$$

3. Use the calculations from the previous part to show that $C = \{1, \cos(t), \cos(2t), \cos(3t), \cos(4t), \cos(5t)\}$ is another basis of V .

Solution: Let A be the 6×6 matrix whose columns are the coordinate vectors we found in Task 2. That is, let

$$A = \begin{bmatrix} 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -3 & 0 & 5 \\ 0 & 0 & 2 & 0 & -8 & 0 \\ 0 & 0 & 0 & 4 & 0 & -20 \\ 0 & 0 & 0 & 0 & 8 & 0 \\ 0 & 0 & 0 & 0 & 0 & 16 \end{bmatrix}.$$

We see that A is already in row-echelon form. Moreover, since there is a pivot (i.e., a leading entry) in each column of A , we know that the column vectors of A form a linearly independent set in \mathbb{R}^6 . Therefore, by Theorem 6.7, the set $C = \{1, \cos(t), \cos(2t), \cos(3t), \cos(4t), \cos(5t)\}$ is a linearly independent set of functions.

Knowing that B is a basis for V , we see that $\dim(V) = 6$. Since the functions in C are linear combinations of the functions in B , we know that the functions in C are also in V . Therefore, since C has $6 = \dim(V)$ linearly independent functions in V , C must also be a basis for V by Theorem 6.10 (c).

4. Use the calculations from (2) to find the change of basis matrix $P_{B \leftarrow C}$ and then use a calculator to find $P_{C \leftarrow B}$.

Solution: By definition,

$$\begin{aligned} P_{B \leftarrow C} &= \begin{bmatrix} [1]_B & [\cos(t)]_B & [\cos(2t)]_B & [\cos(3t)]_B & [\cos(4t)]_B & [\cos(5t)]_B \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -3 & 0 & 5 \\ 0 & 0 & 2 & 0 & -8 & 0 \\ 0 & 0 & 0 & 4 & 0 & -20 \\ 0 & 0 & 0 & 0 & 8 & 0 \\ 0 & 0 & 0 & 0 & 0 & 16 \end{bmatrix}. \end{aligned}$$

By Theorem 6.12 (c), $P_{C \leftarrow B} = (P_{B \leftarrow C})^{-1}$. Using a CAS we find

$$P_{C \leftarrow B} = (P_{B \leftarrow C})^{-1} = \begin{bmatrix} 1 & 0 & \frac{1}{2} & 0 & \frac{3}{8} & 0 \\ 0 & 1 & 0 & \frac{3}{4} & 0 & \frac{5}{8} \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{4} & 0 & \frac{5}{16} \\ 0 & 0 & 0 & 0 & \frac{1}{8} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{16} \end{bmatrix}.$$

5. Use $P_{C \leftarrow B}$ to calculate

$$\int (a_0 + a_1 \cos(t) + a_2 \cos^2(t) + a_3 \cos^3(t) + a_4 \cos^4(t) + a_5 \cos^5(t)) dt$$

where a_0, \dots, a_5 are arbitrary constants, by first transforming the integrand into a linear combination of the functions in C .

Solution: Let $f(t) = a_0 + a_1 \cos(t) + a_2 \cos^2(t) + a_3 \cos^3(t) + a_4 \cos^4(t) + a_5 \cos^5(t)$. Then

$$[f(t)]_B = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix}.$$

By Theorem 6.12 (a),

$$\begin{aligned}
 [f(t)]_C &= P_{C \leftarrow B} [f(t)]_B \\
 &= \begin{bmatrix} 1 & 0 & \frac{1}{2} & 0 & \frac{3}{8} & 0 \\ 0 & 1 & 0 & \frac{3}{4} & 0 & \frac{5}{8} \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{4} & 0 & \frac{5}{16} \\ 0 & 0 & 0 & 0 & \frac{1}{8} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{16} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix} \\
 &= \begin{bmatrix} a_0 + \frac{1}{2}a_2 + \frac{3}{8}a_4 \\ a_1 + \frac{3}{4}a_3 + \frac{5}{8}a_5 \\ \frac{1}{2}a_2 + \frac{1}{2}a_4 \\ \frac{1}{4}a_3 + \frac{5}{16}a_5 \\ \frac{1}{8}a_4 \\ \frac{1}{16}a_5 \end{bmatrix}
 \end{aligned}$$

This means that

$$\begin{aligned}
 f(t) &= \left(a_0 + \frac{1}{2}a_2 + \frac{3}{8}a_4 \right) (1) + \left(a_1 + \frac{3}{4}a_3 + \frac{5}{8}a_5 \right) \cos(t) + \left(\frac{1}{2}a_2 + \frac{1}{2}a_4 \right) \cos(2t) \\
 &\quad + \left(\frac{1}{4}a_3 + \frac{5}{16}a_5 \right) \cos(3t) + \left(\frac{1}{8}a_4 \right) \cos(4t) + \left(\frac{1}{16}a_5 \right) \cos(5t).
 \end{aligned}$$

Therefore,

$$\begin{aligned}
 \int f(t) dt &= \left(a_0 + \frac{1}{2}a_2 + \frac{3}{8}a_4 \right) (t) + \left(a_1 + \frac{3}{4}a_3 + \frac{5}{8}a_5 \right) \sin(t) + \left(\frac{1}{2}a_2 + \frac{1}{2}a_4 \right) \frac{1}{2} \sin(2t) \\
 &\quad + \left(\frac{1}{4}a_3 + \frac{5}{16}a_5 \right) \frac{1}{3} \sin(3t) + \left(\frac{1}{8}a_4 \right) \frac{1}{4} \sin(4t) + \left(\frac{1}{16}a_5 \right) \frac{1}{5} \sin(5t) + C,
 \end{aligned}$$

where C is the constant of integration.