**Math 314/814 - 006 Exam 1**

**Salmon Exam Solutions**

October 9, 2008

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**Name:** ______________________________

Read each question carefully.

Be sure to show all of your work and not just your final conclusion.

You may not use your notes or text for this exam, but you may use a calculator.

*Good Luck!*

| Problem | Points | Score |
|---|---|---|
| 1 | 10 | |
| 2 | 10 | |
| 3 | 12 | |
| 4 | 8 | |
| 5 | 6 | |
| 6 | 12 | |
| 7 | 11 | |
| 8 | 15 | |
| 9 | 4 | |
| 10 | 12 | |
| $\sum$ | 100 | |

(1) (a) Let $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_k}$ be vectors in $\mathbb{R}^n$. Complete the definition: [3 pts]

$\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_k}$ are *linearly dependent* if

*Solution:* there exist scalars $c_1, \ldots c_k$ <u>not all zero</u> such that

$$c_1\mathbf{v_1} + c_2\mathbf{v_2} + \cdots + c_k\mathbf{v_k} = \mathbf{0}.$$

(b) Let $\mathbf{v_1} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \mathbf{v_2} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$, and $\mathbf{v_3} = \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix}$.

Are $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}$ linearly dependent? If so, state a *dependence relation.* [7 pts]

*Hint:* You may assume that $\begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 4 \\ 0 & -1 & 2 \end{bmatrix}$ row reduces to $\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$.

*Solution:* Since the system $[\mathbf{v_1}\ \ \mathbf{v_2}\ \ \mathbf{v_3}]\mathbf{x} = \mathbf{0}$ has a nontrivial solution (i.e. a free variable exists), the vectors $\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$ are linearly dependent. We can read off a dependence relation from the reduced matrix:

$$\mathbf{v_3} = 3\mathbf{v_1} - 2\mathbf{v_2}.$$

(2) Let $\mathbf{v_1} = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}, \mathbf{v_2} = \begin{bmatrix} 5 \\ 1 \\ -7 \end{bmatrix}, \mathbf{v_3} = \begin{bmatrix} -3 \\ -2 \\ 0 \end{bmatrix}$, and $\mathbf{w} = \begin{bmatrix} -4 \\ -1 \\ 5 \end{bmatrix}$.

Is $\mathbf{w}$ in Span$(\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3})$? If yes, then express $\mathbf{w}$ as a linear combination of $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}$. [10 pts]

*Solution:* We want to know if there exist scalars $c_1, c_2, c_3$ such that

$$\mathbf{w} = c_1\mathbf{v_1} + c_2\mathbf{v_2} + c_3\mathbf{v_3}.$$

This is the same as asking if the system $A\mathbf{x} = \mathbf{w}$ is solvable, where $A$ is the $3 \times 3$ matrix whose columns are $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}$ and $\mathbf{x} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$. We row reduce the augmented matrix of this system:

$$\begin{bmatrix} 1 & 5 & -3 & | & -4 \\ 0 & 1 & -2 & | & -1 \\ -2 & -7 & 0 & | & 5 \end{bmatrix} \xrightarrow{R_3 \to R_3 + 2R_1} \begin{bmatrix} 1 & 5 & -3 & | & -4 \\ 0 & 1 & -2 & | & -1 \\ 0 & 3 & -6 & | & -3 \end{bmatrix}$$

$$\xrightarrow{R_3 \to R_3 - 3R_2} \begin{bmatrix} 1 & 5 & -3 & | & -4 \\ 0 & 1 & -2 & | & -1 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

We see that the system is consistent. Thus, $\mathbf{w}$ is in Span$\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$.

To write $\mathbf{w}$ as a linear combination of $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}$, we simply solve the system. Note that $c_3$ is a free variable. So, let $c_3 = t \in \mathbb{R}$. Then

$$\begin{aligned} c_2 &= 2c_3 - 1 = 2t - 1 \\ c_1 &= -5c_2 + 3c_3 - 4 = -7t + 1 \end{aligned}$$

Letting $t = 1$, we see that

$$\mathbf{w} = -6\mathbf{v_1} + \mathbf{v_2} + \mathbf{v_3}.$$

(3) Find all the values of $k$ such that the linear system

$$\begin{aligned} y + 2kz &= 0 \\ x + 2y + 6z &= 2 \\ kx + 2z &= 1 \end{aligned}$$

has (i) no solution; (ii) a unique solution; (iii) infinitely many solutions. *Show all of your work, labeling each row operation you use.* [12 pts]

*Solution:* We row reduce the augmented matrix associated with the system:

$$\begin{bmatrix} 0 & 1 & 2k & | & 0 \\ 1 & 2 & 6 & | & 2 \\ k & 0 & 2 & | & 1 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & 2 & 6 & | & 2 \\ 0 & 1 & 2k & | & 0 \\ k & 0 & 2 & | & 1 \end{bmatrix}$$

$$\xrightarrow{R_3 \to R_3 - kR_1} \begin{bmatrix} 1 & 2 & 6 & | & 2 \\ 0 & 1 & 2k & | & 0 \\ 0 & -2k & (2-6k) & | & (1-2k) \end{bmatrix}$$

$$\xrightarrow{R_3 \to R_3 + 2kR_2} \begin{bmatrix} 1 & 2 & 6 & | & 2 \\ 0 & 1 & 2k & | & 0 \\ 0 & 0 & (4k^2-6k+2) & | & (1-2k) \end{bmatrix}$$

Note that $4k^2 - 6k + 2 = 2(2k-1)(k-1)$.

*Answers:*

No Solution: $\underline{k = 1}$

Unique Solution: $\underline{\text{all } k \text{ in } \mathbb{R} \text{ except } k = 1/2, k = 1}$

Infinitely Many Solutions: $\underline{k = 1/2}$

(4) Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation defined by

$$T\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3x - 2y \\ 6x + 5y \end{bmatrix}.$$

Find the standard matrix of $T$. [8 pts]

*Solution:* We have

$$T\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}, \quad T\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 5 \end{bmatrix}, \quad T\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

So, the standard matrix of $T$ is

$$\begin{bmatrix} 3 & -2 & 0 \\ 6 & 5 & 0 \end{bmatrix}.$$

(5) Consider the transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$ defined by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x - y \\ 2xy \\ x + y \end{bmatrix}.$$

Is $T$ *linear*? If yes, then verify this using the definition. If not, give an explicit example for which one of the linearity properties fails. [6 pts]

*Solution:* $T$ is not linear. Note that

$$T\left(2\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = T\begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 8 \\ 4 \end{bmatrix} \neq 2T\begin{bmatrix} 1 \\ 1 \end{bmatrix} = 2\begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \\ 4 \end{bmatrix}.$$

(6) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the invertible linear transformation with standard matrix

$$A = \begin{bmatrix} 1 & 0 & 3 \\ -1 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}.$$

(a) Find $A^{-1}$. [8 pts]

*Solution:* We row reduce the multi-augmented matrix $[A|I_3]$ to $[I_3|A^{-1}]$:

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 1 & 0 & 0 \\ -1 & 1 & 4 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 1 & 0 & 0 \\ 0 & 1 & 7 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & -3 \\ 0 & 1 & 7 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & -3 \\ 0 & 1 & 0 & 1 & 1 & -7 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

We conclude that

$$A^{-1} = \begin{bmatrix} 1 & 0 & -3 \\ 1 & 1 & -7 \\ 0 & 0 & 1 \end{bmatrix}$$

(b) Find $T^{-1}\left(\begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}\right)$. [4 pts]

*Solution:* The standard matrix of $T^{-1}$ is $A^{-1}$. Thus.

$$T^{-1}\left(\begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}\right) = A^{-1}\begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -3 \\ 1 & 1 & -7 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 5 \\ 10 \\ -1 \end{bmatrix}.$$

(7) (a) Define what it means for a subset $H$ to be a *subspace* of $\mathbb{R}^n$. [3 pts]

*Solution:* $H$ is a *subspace* if it is satisfies 3 properties:

(i) The zero vector of $\mathbb{R}^n$ is in $H$;

(ii) For every $\mathbf{x}, \mathbf{y}$ in $H$, the vector $\mathbf{x} + \mathbf{y}$ is also in $H$ (i.e. $H$ is closed under vector addition);

(iii) For every $\mathbf{x}$ in $H$ and every $c \in \mathbb{R}$, the vector $c\mathbf{x}$ is also in $H$ (i.e. $H$ is closed under scalar multiplication).

(b) Is $H = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 : y = 4z \right\}$ a *subspace* of $\mathbb{R}^3$?

Justify your answer (i.e. either verify the definition or give a specific example where one of the conditions fails). [8 pts]

*Solution:* $H$ is a subspace of $\mathbb{R}^3$. We verify the 3 properties of the definition above.

• $\mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ is in $H$ since $0 = 4(0)$.

• Let $\mathbf{u} = \begin{bmatrix} x \\ 4z \\ z \end{bmatrix}, \mathbf{v} = \begin{bmatrix} s \\ 4t \\ t \end{bmatrix}$ be two vectors in $H$. Then

$$\mathbf{u} + \mathbf{v} = \begin{bmatrix} x + s \\ 4z + 4t \\ z + t \end{bmatrix}.$$

Since $4z + 4t = 4(z + t)$, the vector $\mathbf{u} + \mathbf{v}$ is also in $H$.

• Let $d$ be any real scalar and $\mathbf{u} = \begin{bmatrix} x \\ 4z \\ z \end{bmatrix}$ be any vector in $H$. Then

$$d\mathbf{u} = \begin{bmatrix} dx \\ d4z \\ dz \end{bmatrix}.$$

Since $d4z = 4(dz)$, the vector $d\mathbf{u}$ is also in $H$.

(8) Let $A = \begin{bmatrix} 1 & 2 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & -1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 \end{bmatrix}$. You may assume that $RREF(A) = \begin{bmatrix} 1 & 0 & -1 & 0 & -1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

(a) Find a basis for the null space of $A$. [7 pts]

*Solution:* We solve the system $A\mathbf{x} = \mathbf{0}$. We see that $x_3$ and $x_5$ are free variables. Let $x_3 = s$ and $x_5 = t$, where $s, t$ are arbitrary real numbers. Then $x_1 = x_3 + x_5 = s + t, x_2 = -x_3 - x_5 = -s - t$ and $x_4 = -x_5 = -t$. Thus

$$Null(A) = \left\{ s \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ -1 \\ 0 \\ -1 \\ 1 \end{bmatrix} : s, t \in \mathbb{R} \right\}.$$

So, a basis for the null space of $A$ is

$$\left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \\ -1 \\ 1 \end{bmatrix} \right\}.$$

(b) Find a basis for the column space of $A$. [5 pts]

*Solution:* We take the columns of $A$ which correspond to pivot columns of $B$. Thus, a basis for col$(A)$ is

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

(c) $\text{rank}(A) = \underline{3}$ [1 pt]

(d) $\dim(\text{Null}(A)) = \underline{2}$ [1 pt]

(e) $\dim(\text{row}(A)) = \underline{3}$ [1 pt]

(9) Let $A$ be an $m \times n$ matrix. Prove that $AA^T$ is symmetric. [4 pts]

*Solution:* We need to show that the transpose of $AA^T$ is itself. Using properties from class, we have

$$(AA^T)^T = (A^T)^T A^T = AA^T.$$

(10) Are the following statements *true* or *false*? Carefully justify your answers. [3 pts each]

(a) Let $B$ be an invertible matrix. Then $(B^T B)^{-1} = B^{-1}(B^{-1})^T$.

*Solution:* This statement is true. Using properties from class, we have

$$(B^T B)^{-1} = B^{-1}(B^T)^{-1} = B^{-1}(B^{-1})^T.$$

(b) Let $A$ be a $99 \times 99$ matrix. It is possible that the null space and column space of $A$ are the same subspaces of $\mathbb{R}^{99}$.

*Solution:* If $null(A) = col(A)$, then $nullity(A) = dim(col(A)) = rank(A)$. If this were the case, then

$$99 = nullity(A) + Rank(A) = 2Rank(A)$$

by the Rank Theorem. Since 99 is odd and the dimension of a subspace must be an integer, this is impossible.

(c) Let $A$ be a $3 \times 4$ matrix. If $col(A) = \mathbb{R}^3$, then the column vectors of $A$ must be linearly independent.

*Solution:* This statement is false. If $col(A) = \mathbb{R}^3$, then the rank of $A$ must equal 3. This means that there is a free variable when we consider the system $A\mathbf{x} = \mathbf{0}$. Thus, the columns of $A$ must be linearly dependent. For an example, consider the matrix

$$A = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

(d) Suppose that $A$ and $B$ are $n \times n$ invertible matrices. Then the rank of $AB$ equals $n$.

*Solution:* This statement is true. If $A$ and $B$ are $n \times n$ invertible matrices, then $AB$ is an $n \times n$ invertible matrix. So, by the Fundamental Theorem for Invertible Matrices, $rank(AB) = n$.