Problem Set 1

Due: Tuesday, February 16

(1) Let $S = k[x_1, \ldots, x_n]$ where k is a field. Fix a monomial order $>_\sigma$ on $\mathbb{Z}^n_{\geq 0}$.
  (a) Show that multideg$(fg) =$ multideg$(f)+$ multideg(g) for non-zero polynomials $f, g \in S$.
  (b) A special case of a *weight order* is constructed as follows. Fix $\mathbf{u} \in \mathbb{Z}^n_{\geq 0}$. Then, for $\boldsymbol{\alpha}, \boldsymbol{\beta}$ in $\mathbb{Z}^n_{\geq 0}$, define $\boldsymbol{\alpha} >_{\mathbf{u},\sigma} \boldsymbol{\beta}$ if and only if

$$\mathbf{u} \cdot \boldsymbol{\alpha} > \mathbf{u} \cdot \boldsymbol{\beta}, \quad \text{or} \quad \mathbf{u} \cdot \boldsymbol{\alpha} = \mathbf{u} \cdot \boldsymbol{\beta} \quad \text{and} \quad \boldsymbol{\alpha} >_\sigma \boldsymbol{\beta},$$

  where $\cdot$ denotes the usual dot product of vectors. Verify that $>_{\mathbf{u},\sigma}$ is a monomial order.
  (c) A particular example of a weight order is the *elimination order* which was introduced by Bayer and Stillman. Fix an integer $1 \leq i \leq n$ and let $\mathbf{u} = (1, \ldots, 1, 0, \ldots, 0)$, where there are i 1's and $n - i$ 0's. Then the *ith elimination order* $>_i$ is the weight order $>_{\mathbf{u},grevlex}$. Prove that $>_i$ has the following property: if $\mathbf{x}^{\boldsymbol{\alpha}}$ is a monomial in which one of $x_1, \ldots, x_i$ appears, then $\mathbf{x}^{\boldsymbol{\alpha}} >_i \mathbf{x}^{\boldsymbol{\beta}}$ for *any* monomial $\mathbf{x}^{\boldsymbol{\beta}}$ involving only $x_{i+1}, \ldots, x_n$. Does this property hold for the graded reverse lexicographic order?

(2) Let I be a non-zero ideal in $k[x_1, \ldots, x_n]$. Let $G = \{g_1, \ldots, g_t\}$ and $F = \{f_1, \ldots, f_r\}$ be two minimal Gröbner bases for I with respect to some fixed monomial order. Show that $\{LT(g_1), \ldots, LT(g_t)\} = \{LT(f_1), \ldots, LT(f_r)\}$.

(3) Suppose that $I = (g_1, \ldots, g_t)$ is a non-zero ideal of $k[x_1, \ldots, x_n]$ and fix a monomial order on $\mathbb{Z}^n_{\geq 0}$. Suppose that for all f in I we obtain a zero remainder upon dividing f by $G = \{g_1, \ldots, g_t\}$ using the Division Algorithm. Prove that G is a Gröbner basis for I. (We showed the converse of this statement in class.)

(4) Consider the ideal $I = (xy + z - xz, x^2 - z) \subset k[x, y, z]$. For what follows, use the graded reverse lexicographic order with $x > y > z$. You are not permitted to use a computer algebra system for this exercise. Be sure to show all of your work.
  (a) Apply Buchberger's Algorithm to find a Gröbner basis for I. Is the result a reduced Gröbner basis for I?
  (b) Use your answer from part (a) to determine if $f = xy^3z - z^3 + xy$ is in I.

(5) Consider the affine variety $V = \mathbf{V}(x^2+y^2+z^2-4, x^2+2y^2-5, xz-1)$ in $\mathbb{C}^3$. Use a computer algebra system and Gröbner bases to find all the points of V.