Problem Set 2

Due: Thursday, March 25

(1) Fix $\mathcal{H} := (1, 4, 6, 9, 10, 13, 13, \ldots)$ and let $S := k[x_1, x_2, x_3, x_4]$ where k is a field. Does there exist a homogeneous ideal $I \subset S$ such that $H(S/I) = \mathcal{H}$? Provide two reasons for your answer: one using an O-sequence approach and one using an order ideal of monomials approach.

(2) For this exercise we use the same notation that was set up in our discussion of lifting monomial ideals. Let $f = \mathbf{x}^{\boldsymbol{\alpha}} \in S = k[x_1, \ldots, x_n]$. Prove the following two facts:
   (a) $\overline{f}(\overline{\boldsymbol{\beta}}) = 0$ if and only if $\boldsymbol{\alpha} \not\leq \boldsymbol{\beta}$;
   (b) $\overline{f}(\overline{\boldsymbol{\gamma}}) = 0$ for all $\boldsymbol{\gamma}$ with $\deg(\boldsymbol{\gamma}) \leq \deg(\boldsymbol{\alpha})$ (except for $\boldsymbol{\alpha}$ itself).

(3) In this exercise we further explore Hilbert functions of distinct points in projective 2-space. Let $S = k[x_1, x_2]$, where k is an algebraically closed field of characteristic zero. Further, let $J \subset S$ be a homogeneous ideal such that $\sqrt{J} = (x_1, x_2)$. We set $\alpha(J)$ to be the least degree of a non-zero homogeneous polynomial in J.
   (a) Set $B = S/J$. Prove that

$$H(B, t) = \begin{cases} t+1 & \text{for } t < \alpha(J) \\ \leq \alpha(J) & \text{for } t \geq \alpha(J). \end{cases}$$

   (b) Let $V \subset S_t$ be a non-zero subspace of S_t. Denote by S_1V the subspace of S_{t+1} generated by $\{Lv \mid L \in S_1 \text{ and } v \in V\}$. Prove that

$$\dim_k(S_1V) \geq (\dim_k V) + 1.$$

   (c) Let $\mathbb{X} = \{P_1, \ldots, P_t\}$ be a set of distinct points in $\mathbb{P}^2$. We set $\alpha = \alpha(\mathbb{X})$ to be the least degree of a non-zero homogeneous polynomial in $I(\mathbb{X})$. Show that $\Delta H(\mathbb{X})$ has the form

$$\Delta H(\mathbb{X}) = \{1, 2, 3, \ldots, \alpha - 1, \alpha, \Delta H(\mathbb{X}, \alpha), \Delta H(\mathbb{X}, \alpha+1), \ldots\}$$

   where $\alpha \geq \Delta H(\mathbb{X}, \alpha) \geq \Delta H(\mathbb{X}, \alpha+1) \geq \Delta H(\mathbb{X}, \alpha+2) \geq \cdots$.

(4) Find all possible Hilbert functions for 9 distinct points in $\mathbb{P}^2$. Pick one of the Hilbert functions $\mathcal{H}$ and find a set $\mathbb{X} \subset \mathbb{P}^2$ of 9 distinct points in $\mathbb{P}^2$ such that $H(\mathbb{X}) = \mathcal{H}$. How do you know that the constructed set of points has the selected Hilbert function?

(5) Suppose that I is a homogeneous ideal in the ring $R = k[x_0, \ldots, x_n]$ where k is an algebraically closed field of characteristic 0. Suppose that $I_d \neq 0$ and that $H(R/I)$ has maximal growth in degree d. Prove that I_d and I_{d+1} have a greatest common divisor of positive degree in the following two cases:
   (a) $n = 1$ and $H(R/I, d) \geq 1$;
   (b) $n = 2$ and $H(R/I, d) \geq d + 1$.