Problem Set 3

Due: Thursday, April 15

This problem set involves choices! Submit solutions to 2 exercises from Part I and 1 exercise from Part II.

Part I - Exercises Related to Hilbert Functions & Regular Sequences

(1) For parts (b) - (d) of this exercise use reverse-lexicographic order with $x_1 >_{revlex} > x_2 >_{revlex} \cdots$.
  (a) Find a $(3,4,5)$-lex-plus-powers ideal $L \subset S = k[x_1,x_2,x_3]$ such that $H(S/L,3) = 9$ and $H(S/L,6) = 5$.
  (b) Fix m to be a monomial of degree d in $S = k[x_1,x_2,x_3,x_4]/(x_1^5,x_2^4,x_3^4,x_4^3)$. Recall that $L(m)$ denotes the set of all degree d monomials in S which are greater than or equal to m. Decompose $|L(x_1^3x_2^3x_4^2)|$ in terms of integers of the form $\binom{e_1,\ldots,e_j}{l}$. Give an algebraic description of each term in the decomposition.
  (c) Assume $I \subset S = k[x_1,x_2,x_3,x_4]$ is a homogeneous ideal containing $\{x_1^5,x_2^4,x_3^4,x_4^3\}$. If $H(S/I,8) = 17$, then what is the largest value possible for $H(S/I,9)$?
  (d) Assume that the EGH Conjecture is true. Can there be a homogeneous $(3,4,4,5)$-ideal $I \subset S = k[x_1,x_2,x_3,x_4]$ with $H(S/I) = (1,4,10,18,24,29,\ldots)$?

(2) *EGH Points Conjecture in* $\mathbb{P}^2$: Fix integers $2 \le d_1 \le d_2$. Let $\Delta\mathcal{H} = \{h_t\}_{t\ge 0}$ be the first difference Hilbert function of some finite set of distinct points in $\mathbb{P}^2$ such that $h_t \le H(k[x_1,x_2]/(x_1^{d_1},x_2^{d_2}),t)$ for all $t \ge 0$. Prove that there exist finite sets of distinct points $\mathbb{X} \subseteq \mathbb{Y} \subset \mathbb{P}^2$ where $\mathbb{Y}$ is a complete intersection of type $\{d_1,d_2\}$ and $\Delta H(\mathbb{X}) = \Delta\mathcal{H}$ if and only if $h_{t+1} \le h_t^{(t)}$ for all $t \ge 1$.

(3) *Classical Cayley-Bacharach Theorem*: Let $\mathbb{X} = \{P_1,\ldots,P_9\}$ be the complete intersection of two cubics in $\mathbb{P}^2$. Use the Cayley-Bacharach Theorem to show that any cubic passing through 8 of the 9 points of $\mathbb{X}$ must also pass through the remaining 9th point.

Part II - Exercises From Group Presentations

(1) *From Croll-Gibbons-Johnson*: Our exercise outlines a proof of the following lemma due to Buchsbaum and Eisenbud:

Lemma. *Let R be a ring, $x \in R$, and $S = R/(x)$. Let B be an S-module, and let*

$$\mathcal{F}: \qquad F_2 \xrightarrow{\phi_2} F_1 \xrightarrow{\phi_1} F_0$$

be an exact sequence of S-modules with $\operatorname{coker}(\phi_1) \cong B$*. Suppose that*

$$\mathcal{G}: \qquad G_2 \xrightarrow{\psi_2} G_1 \xrightarrow{\psi_1} G_0$$

is a complex of R-modules such that

(i) x is a non-zero divisor on each G_i,
(ii) $G_i \otimes_R S \cong F_i$, and
(iii) $\psi_i \otimes_R S = \phi_i$.

Then $A = \operatorname{coker}(\psi_1)$ is a lifting of B to R.

(a) With the conditions of the lemma and $i \in \{0,1,2\}$, prove that the sequence

$$0 \longrightarrow G_i \xrightarrow{\cdot x} G_i \xrightarrow{q} G_i/xG_i \longrightarrow 0$$

is exact, where $\cdot x$ is the map given by multiplication by x and q is the canonical quotient map.

(b) In the diagram below, show that each square of the diagram commutes.

$$\begin{array}{ccccccccccccc}
 & & & & 0 & & 0 & & 0 & & & & \\
 & & & & \downarrow & & \downarrow & & \downarrow & & & & \\
\cdots & \longrightarrow & 0 & \longrightarrow & G_2 & \xrightarrow{\psi_2} & G_1 & \xrightarrow{\psi_1} & G_0 & \longrightarrow & 0 & \longrightarrow & \cdots \\
 & & & & \downarrow{\cdot x} & & \downarrow{\cdot x} & & \downarrow{\cdot x} & & & & \\
\cdots & \longrightarrow & 0 & \longrightarrow & G_2 & \xrightarrow{\psi_2} & G_1 & \xrightarrow{\psi_1} & G_0 & \longrightarrow & 0 & \longrightarrow & \cdots \\
 & & & & \downarrow & & \downarrow & & \downarrow & & & & \\
\cdots & \longrightarrow & 0 & \longrightarrow & F_2 & \xrightarrow{\phi_2} & F_1 & \xrightarrow{\phi_1} & F_0 & \longrightarrow & 0 & \longrightarrow & \cdots \\
 & & & & \downarrow & & \downarrow & & \downarrow & & & & \\
 & & & & 0 & & 0 & & 0 & & & &
\end{array}$$

Conclude that

$$0 \longrightarrow \mathcal{G} \xrightarrow{\cdot x} \mathcal{G} \longrightarrow \mathcal{F} \longrightarrow 0$$

is an exact sequence of complexes (briefly explain why each column is exact).

(c) Given any exact sequence of complexes $0 \longrightarrow D_. \xrightarrow{\cdot x} D_. \longrightarrow C_. \longrightarrow 0$, there is a corresponding long exact sequence in homology given by

$$\begin{array}{ccccc}
\cdots & \longrightarrow & H_2(D_.) & \longrightarrow & H_2(C_.) \\
H_1(D_.) & \xrightarrow{\cdot x} & H_1(D_.) & \longrightarrow & H_1(C_.) \\
H_0(D_.) & \xrightarrow{\cdot x} & H_0(D_.) & \longrightarrow & H_0(C_.) \longrightarrow 0.
\end{array}$$

Use the long exact sequence in homology with the exact sequence of complexes to determine that $A/xA \cong B$ and x is a non-zero divisor on A. Conclude that A is a lifting of B to R.

(2) *From Brase-Denkert-Janssen*: Accept that any monomial ordering $>$ on $k[x_1, \ldots, x_n]$ can be obtained by taking pairwise orthogonal vectors $\mathbf{v_1}, \ldots, \mathbf{v_r} \in k^n$ where $\mathbf{v_1}$ has only non-negative entries and where $\mathbf{x}^{\boldsymbol{\alpha}} > \mathbf{x}^{\boldsymbol{\beta}}$ if and only if there exists $t \leq r$ such that $\mathbf{v_i} \cdot \boldsymbol{\alpha} = \mathbf{v_i} \cdot \boldsymbol{\beta}$ for all $i \leq t-1$ and $\mathbf{v_t} \cdot \boldsymbol{\alpha} > \mathbf{v_t} \cdot \boldsymbol{\beta}$.

(a) Let $r = n$ and $\mathbf{v_i} = \mathbf{e_i}$ for all i where $\mathbf{e_i}$ is the ith standard basis vector for k^n. Show that $>$ is the lexicographic order.

(b) Let $r = n$ and define vectors as follows:

$$\mathbf{v_1} = (1, \ldots, 1)$$
$$\mathbf{v_i} = (1, 1, \ldots, 1, i - (n+1), 0, 0, \ldots, 0)$$

where the entry $i - (n+1)$ is in the $(n + 2 - i)$th position for $i \in \{2, \ldots, n\}$. Show that $>$ is the graded reverse-lexicographic order.