Problem Set 4

Due: Thursday, April 22

This problem set involves choices! Submit solutions to 2 exercises from Part I and 1 exercise from Part II.

Part I - Exercises Related to Borel-Fixed and Generic Initial Ideals

The following exercises are taken from the Chapter 2 Exercises of "Combinatorial Commutative Algebra" by E. Miller and B. Sturmfels.

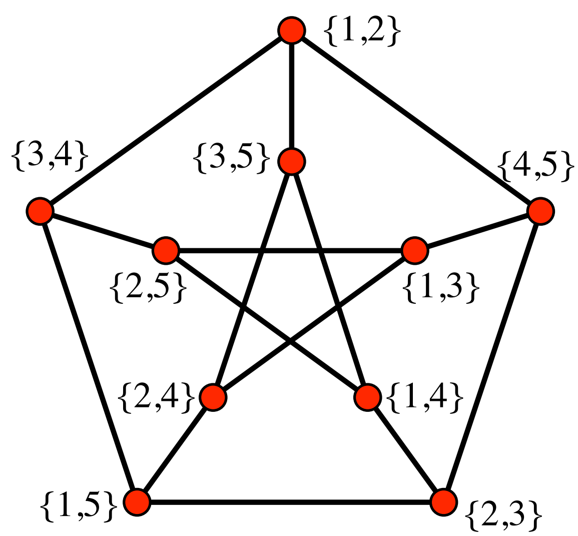
(1) [Exercise 2.2] Can you find a general formula for the number $\mathcal{B}(r,d)$ of Borel-fixed ideals generated by r monomials of degree d in three unknowns $\{x_1, x_2, x_3\}$?
(2) [Exercise 2.4] Is the class of Borel-fixed ideals closed under the ideal-theoretic operations of taking intersections, sums, and products? Either prove your claims or give counter-examples.
(3) [Modified Exercise 2.11] Let $I = (x_1x_2, x_2x_3, x_1x_3) \subseteq S = k[x_1, x_2, x_3]$. Compute the generic initial ideal $\mathbf{gin}_<(I)$ for the lexicographic and reverse lexicographic monomial orders. Also, compute the lex-segment ideal $L \subseteq S$ with $H(S/I) = H(S/L)$. (Note: Although you can use a computer algebra program to support your solution, you should avoid finding the generic initial ideals by using the pre-defined function.)

Part II - Exercises From Group Presentations

(1) *From Boeckner-Stolee*: Recall the definition of a *perfect graph* is a graph for which every induced subgraph, we have the chromatic number equal to the clique number.

It is well known that the *Petersen graph*, described as follows and shown below, is not perfect.

The Petersen graph is the graph on 10 vertices, given by subsets of size 2 from a set of 5 elements. The edges are formed if the two vertices (as subsets) are disjoint.



(a) Show that the chromatic number of the Petersen graph is 3, but the clique number is 2.
(b) Find an odd hole.
(c) Let $J := I(G)^\vee$, where G is the Petersen graph. Give an associated prime of height > 3 in $\mathrm{Ass}(J^2)$.

(2) *From DeVries-Yu*: Let $K_{n,d}$ be the complete bipartite graph on n and d vertices (i.e. let L be a set of n vertices and R a set of d vertices with $L \cap R = \emptyset$. Then the vertex set of $K_{n,d}$ is $L \cup R$, and the edge set of $K_{n,d}$ is the set of all pairs with one element from L and one element from R). Let $I(K_{n,d})$ denote the edge ideal of $K_{n,d}$. Write a recursive formula for $\beta_{i,j}(I(K_{n,d}))$ in terms of the Betti numbers of $I(K_{m,d})$ for $m < n$. Use your formula to compute $\beta_{1,j}(I(K_{n,d}))$ for all j.