Problem Set 4 Solutions

Due: Thursday, April 22

This problem set involves choices! Submit solutions to 2 exercises from Part I and 1 exercise from Part II.

Part I - Exercises Related to Borel-Fixed and Generic Initial Ideals

The following exercises are taken from the Chapter 2 Exercises of "Combinatorial Commutative Algebra" by E. Miller and B. Sturmfels.

(1) [Exercise 2.2] Can you find a general formula for the number $\mathcal{B}(r,d)$ of Borel-fixed ideals generated by r monomials of degree d in three unknowns $\{x_1, x_2, x_3\}$?

Solution: There were a variety of approaches to this exercise. I highlight two below.

- *Approach 1*: The first approach is a recurrence relation:

$$\mathcal{B}(r,d) = \begin{cases} 0 & \text{if } r < 0 \\ 1 & \text{if } 0 \le r \le 2 \\ \mathcal{B}(r, d-1) + \mathcal{B}(r-d-1, d-1) & \text{if } r \ge 3. \end{cases}$$

The following explanation is based on the submission from Derrick Stolee.

The cases of $r \le 0$ are obvious. If $r = 1$ then the only Borel-fixed ideal in $k[x_1, x_2, x_3]$ generated by 1 monomial of degree d is $I = (x_1^d)$. Similarly, if $r = 2$, the only Borel-fixed ideal in $k[x_1, x_2, x_3]$ generated by 2 monomials of degree d is $I = (x_1^d, x_1^{d-1}x_2)$.

For the recurrence relation, we consider all Borel-fixed ideals in $k[x_1, x_2, x_3]$ which are generated by r monomials of degree d. These ideals can be partitioned into two types:

  - Type 1: those ideals containing x_2^d;
  - Type 2: those ideals not containing x_2^d.

By Proposition 2.3 in Miller and Sturmfels' book, any Borel-fixed ideal I of Type 1 must also contain the monomials of the form $x_1^j x_2^{d-j}$. There are $d+1$ such monomials which leaves $r-d-1$ monomial generators for I, say $m_1, \ldots, m_{r-d-1}$. Note that m_j must involve a positive power of x_3 for $1 \le j \le r-d-1$. Define the ideal J to be the ideal generated by the degree $d-1$ monomials $\frac{m_1}{x_3}, \ldots, \frac{m_{r-d-1}}{x_3}$. Using Proposition 2.3, it is straightforward to check that J is a Borel-fixed ideal generated by $r-d-1$ monomials of degree $d-1$. Since there is a 1-1 correspondence between ideals of the form I and J, this allows us to conclude that the number of Type 1 ideals is equal to $\mathcal{B}(r-d-1, d-1)$.

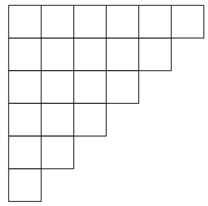
We now focus on Type 2 ideals. Again by Proposition 2.3, a Borel-fixed ideal $I = (m_1, \ldots, m_r)$ of Type 2 cannot contain any monomial of the form $x_2^j x_3^{d-j}$ (otherwise x_2^d would be forced into the ideal I). So, any monomial generator of I must involve a positive power of x_1. Define the ideal J to be the ideal generated by the degree $d-1$ monomials $\frac{m_1}{x_1}, \ldots, \frac{m_r}{x_1}$. It is straightforward to see that J is a Borel-fixed ideal generated by r monomials of degree $d-1$. Since there is a 1-1 correspondence between ideals of the form I and J, this allows us to conclude that the number of Type 2 ideals is equal to $\mathcal{B}(r, d-1)$.

By summing the cardinalities of the ideals of Type 1 and 2, we see that

$$\mathcal{B}(r,d) = \mathcal{B}(r, d-1) + \mathcal{B}(r-d-1, d-1).$$

- *Approach 2*: The second approach involves partitions. The following is Justin DeVries' solution.

The basic idea behind the following solution is to arrange the monomials of degree d into a Young tableau:



Along the top row, the $d+1$ monomials with exponent equal to 0 on x_3 are arranged in the order:
$$x_1^d x_2^0 x_3^0, x_1^{d-1} x_2^1 x_3^0, x_1^{d-2} x_2^2 x_3^0, \ldots, x_1^0 x_2^d x_3^0.$$
Along the second row, the d monomials with exponent equal to 1 on x_3 are arranged in the order:
$$x_1^{d-1} x_2^0 x_3^1, x_1^{d-2} x_2^1 x_3^1, x_1^{d-3} x_2^2 x_3^1, \ldots, x_1^0 x_2^{d-1} x_3^1.$$
The remaining rows are labeled similarly, with row i (starting the index at $i=0$) getting the $d+1-i$ monomials with i as the exponent on x_3 and in decreasing order of the exponents on x_1.

With this labeling, the action of the Borel subgroup defines a "flow" on the tableau:

– any monomial can be transformed to a monomial on the same row that is one step to the *left* on the tableau,
– any monomial can be transformed to a monomial on the row *above* that is one step to the *right*.

Of course, the Borel group provides other transformations, e.g. transforming a monomial to the one directly above it in the tableau, but the virtue of these two transformations is that an ideal generated by degree d monomials is Borel-fixed if and only if the monomials in I arranged in the tableau are closed under these transformations.

The sub-tableau that are closed under the above flow are those that have decreasing rows extending all the way to the left and that have distinct row lengths. When the ideal has exactly r generators, there must be exactly r boxes altogether in the tableau. Combinatorialists will recognize this description as a partition of r into distinct parts, and this viewpoint is much easier to prove statements with, so we'll adopt it for the remainder of the solution.

Recall that a *partition* of r is a sum $r = \sum n_i$ with $n_i \in \mathbb{N}_{>0}$. The numbers n_i are called the *parts* of the partition.

We'll show that $\mathcal{B}(r,d)$ also counts the number of partitions of the integer r with distinct parts that are at most $d+1$. Taking $d \geq r$, a general formula for $\mathcal{B}(r,d)$ would give a general formula for counting the number of partitions of r with distinct parts, and since there is no known formula for this latter number we will be satisfied with this description[1].

Let I be a Borel-fixed ideal with generators $x_1^{a_1} x_2^{b_1} x_3^{c_1}, \ldots, x_1^{a_r} x_2^{b_r} x_3^{c_r}$ with $a_i + b_i + c_i = d$ for all i. Define a partition of r by setting
$$n_i = \max\{b_j + 1 \mid 1 \leq j \leq r \text{ and } c_j = i\},$$
and $n_i = 0$ if $c_j \neq i$ for all j. First, we claim that $n_j \neq 0$ implies $n_i > n_j$ for $i < j$. If $n_j \neq 0$ then there is a generator $x_1^{a_\ell} x_2^{n_j - 1} x_3^j \in I$. Because I is Borel-fixed, we also have a generator $x_1^{a_\ell} x_2^{n_j - 1 + (j-i)} x_3^i \in I$. So $n_i \geq n_j + (j-i) > n_j$. From this we know there is an integer N such that $n_i \neq 0$ for $i \leq N$ and $n_i = 0$ for $i > N$. We claim that

(a) $n_i \leq d+1$ for $0 \leq i \leq N$,
(b) $n_i > n_j$ for $0 \leq i < j \leq N$,
(c) $r = \sum_{i=0}^N n_i$.

[1] a description of the generating function for $\mathcal{B}(r,d)$ would be nice however

The first claim follows simply from the fact that $b_j \leq d$ for all j as the generators of I have degree d. The second claim follows from the above argument since $n_N \neq 0$. For the third claim, when $n_i \neq 0$ we have a generator $x_1^{a_\ell} x_2^{n_i-1} x_3^i \in I$. Because I is Borel-fixed, the n_i monomials $x_1^{a_\ell+1} x_2^{n_i-2} x_3^i, \ldots, x_1^{a_\ell+n_i} x_2^0 x_3^i$ are also generators of I. Define

$$S(i) = \{x_1^{a_\ell} x_2^{n_i-1} x_3^i, x_1^{a_\ell+1} x_2^{n_i-2} x_3^i, \ldots, x_1^{a_\ell+n_i-1} x_2^0 x_3^i\}.$$

Note that $S(i)$ is contained in the generating set of I for all i.

For $i \neq j$, the sets $S(i)$ and $S(j)$ are disjoint because the exponents on x_3 are distinct. Moreover, every generator of I is in some $S(i)$. Indeed, take a generator $x_1^{a_j} x_2^{b_j} x_3^{c_j}$. Then $b_j \leq n_{c_j} - 1$ by the definition of n_{c_j}. So this generator is in $S(c_j)$.

Thus the disjoint sum $\bigcup_{0 \leq i \leq N} S(i)$ gives the generating set of I. Phrased in terms of cardinalities, $r = \sum_{i=0}^N n_i$. This establishes the third claim.

So given a Borel-fixed ideal I generated by r monomials of degree d, we can produce a partition of r with distinct parts that are at most $d+1$. Now suppose we have such a partition: $r = \sum_{i=0}^N n_i$ with $n_i > n_j$ for $i < j$. Define the sets $S(i)$ as before:

$$S(i) = \{x_1^{d-(n_i-1)-i} x_2^{n_i-1} x_3^i, x_1^{d-(n_i-1)-i+1} x_2^{n_i-2} x_3^i, \ldots, x_1^{d-(n_i-1)-i+n_i-1} x_2^0 x_3^i\}.$$

We claim that the ideal I generated by $\bigcup_{0 \leq i \leq N} S(i)$ is a Borel-fixed ideal of the desired form. First note that every generator has degree d and that there are exactly r of them since $r = \sum_{i=0}^N n_i$ and $S(i) \cap S(j) = \emptyset$ for $i \neq j$. So it suffices to show that I is Borel-fixed. As I is monomial, it suffices to show that if m is a generator of I and x_j divides m then $m\frac{x_i}{x_j} \in I$ for $i < j$.

Take a generator $m = x_1^a x_2^b x_3^c$ of I. We have $m \in S(c)$, so the definition of $S(c)$ shows that $m\frac{x_1}{x_2} \in I$ if x_2 divides m. Suppose that x_3 divides m so that $c \geq 1$. We have $b \leq n_c - 1$ and $n_c \leq n_{c-1} - 1$ by definition of I and the partition, so $b+1 \leq n_{c-1} - 1$. Thus

$$m\frac{x_2}{x_3} = x_1^a x_2^{b+1} x_3^{c-1}$$

is in $S(c-1)$, and therefore is in I. Finally, note that $m\frac{x_1}{x_3} = m\frac{x_1}{x_2}\frac{x_2}{x_3}$, and the right-hand side is in I by what we have shown. So I is a Borel-fixed ideal.

Examining the two constructions used we see that they are inverses: given a Borel-fixed ideal we stratify the generators into sets $S(i)$ which give a partition by their cardinalities, and given a partition we form the same sets $S(i)$ to determine the generators of a Borel-fixed ideal. So we have a bijective correspondence between Borel-fixed ideals generated by r monomials of degree d and partitions of r that have distinct parts that are at most $d+1$.

(2) [Exercise 2.4] Is the class of Borel-fixed ideals closed under the ideal-theoretic operations of taking intersections, sums, and products? Either prove your claims or give counter-examples.

Solution: Yes, the class of Borel-fixed ideals is closed under all three operations of taking intersections, sums, and products.

<u>Intersections</u>: Let $\{J_\alpha\}_{\alpha \in \Gamma}$ be a collection of Borel-fixed ideals in $S = k[x_1, \ldots, x_n]$. It is clear that $J := \cap_{\alpha \in \Gamma} J_\alpha$ will be a monomial ideal. Let m be any monomial in J such that x_j divides m. Then for all $\alpha \in \Gamma$, since $m \in J_\alpha$ and J_α is Borel-fixed, we must have

$$m\frac{x_i}{x_j} \in J_\alpha$$

for all $i < j$. We conclude that

$$m\frac{x_i}{x_j} \in J$$

for all $i < j$. Thus, the intersection $\cap_{\alpha \in \Gamma} J_\alpha$ is Borel-fixed.

Sums: For ease of notation, we work below with two ideals. The proof for the sum of more than two ideals uses the same ideas. Suppose $J_1 = (f_1, \dots, f_r)$ and $J_2 = (g_1, \dots, g_l)$ are Borel-fixed ideals in $S = k[x_1, \dots, x_n]$. We have that $J_1 + J_2 = (f_1, \dots, f_r, g_1, \dots, g_l)$. Since J_1 and J_2 are both Borel-fixed, if x_j divides f_b then

$$f_b \frac{x_i}{x_j} \in J_1 \subset J_1 + J_2$$

and if x_j divides g_c then

$$g_c \frac{x_i}{x_j} \in J_2 \subset J_1 + J_2$$

for all $i < j$. Since it is enough to check the Borel-condition on (minimal) generators this shows that the ideal $J_1 + J_2$ must be Borel-fixed.

Products: Again, for ease of notation, we work below with two ideals. The general proof for the product of finitely many ideals uses the same ideas. Suppose $J_1 = (f_1, \dots, f_r)$ and $J_2 = (g_1, \dots, g_l)$ are Borel-fixed ideals in $S = k[x_1, \dots, x_n]$. We have that $J_1 J_2 = (f_a g_b : 1 \le a \le r, 1 \le b \le l)$. Assume x_j divides $f_a g_b$. Then x_j divides f_a or g_b. If x_j divides f_a then

$$f_a \frac{x_i}{x_j} \in J_1$$

and so

$$f_a g_b \frac{x_i}{x_j} \in J_1 J_2$$

for all $i < j$. Similarly, if x_j divides g_b then

$$g_b \frac{x_i}{x_j} \in J_2$$

and so

$$f_a g_b \frac{x_i}{x_j} \in J_1 J_2$$

for all $i < j$. Since it is enough to check the Borel-condition on (minimal) generators this shows that $J_1 J_2$ must be Borel-fixed.

(3) [Modified Exercise 2.11] Let $I = (x_1x_2, x_2x_3, x_1x_3) \subseteq S = k[x_1, x_2, x_3]$. Compute the generic initial ideal $\mathbf{gin}_<(I)$ for the lexicographic and reverse lexicographic monomial orders. Also, compute the lex-segment ideal $L \subseteq S$ with $H(S/I) = H(S/L)$. (Note: Although you can use a computer algebra program to support your solution, you should avoid finding the generic initial ideals by using pre-defined functions.)

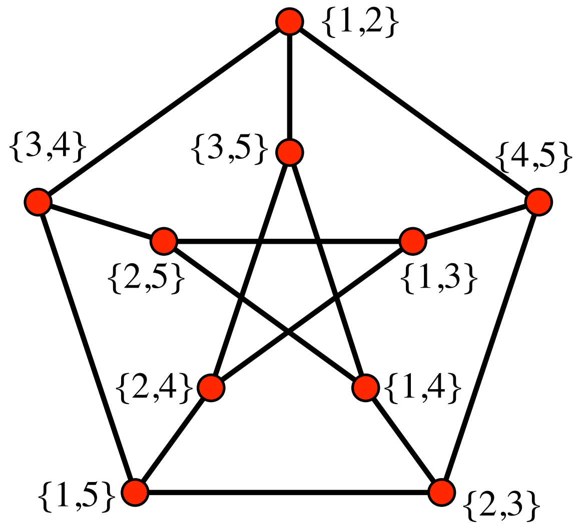
Solution: Only 1 student submitted a solution for this exercise. So that others can continue to work on the exercise, I only provide here the final outcomes. The computer algebra system CoCoA gives $\mathbf{gin}_{grevlex}(I) = (x_1^2, x_1x_2, x_2^2)$ and $\mathbf{gin}_{lex}(I) = (x_1^2, x_1x_2, x_1x_3, x_2^3)$. Also, the lex-segment ideal L is $L = (x_1x_2, x_1x_3, x_1^2, x_2^3)$.

Part II - Exercises From Group Presentations

(1) *From Boekner-Stolee*: Recall the definition of a *perfect graph* is a graph for which every induced subgraph, we have the chromatic number equal to the clique number.

It is well known that the *Petersen graph*, described as follows and shown below, is not perfect.

The Petersen graph is the graph on 10 vertices, given by subsets of size 2 from a set of 5 elements. The edges are formed if the two vertices (as subsets) are disjoint.



(a) Show that the chromatic number of the Petersen graph is 3, but the clique number is 2.

Solution: The outer (or inner) 5-cycle subgraph needs three colors (which was demonstrated in class). Since this subgraph requires at least three colors, so does the entire graph. It is straightforward to see that the 3-coloring on the subgraph forces a 3-coloring of the Petersen graph.

There are no cliques of size 3 since there are no 3 subsets of size two from a set of 5 elements which are mutually disjoint (this would require 6 elements).

(b) Find an odd hole.

Solution: There are many odd holes, in fact any 5 cycle you find is an odd hole since the graph is triangle free.

(c) Let $J := I(G)^\vee$, where G is the Petersen graph. Give an associated prime of height > 3 in $\mathrm{Ass}(J^2)$.

Solution: Use the variables corresponding to the vertices from your odd hole in part (b) to generate an ideal. Let x_1, x_2, x_3, x_4, x_5 be the variables for the vertices in the odd hole (so that $(x_1^2, x_2^2, x_3^2, x_4^2, x_5^2)$ is part of a primary decomposition of J). By a theorem from the presentation

$$\{x_{i_1}, \ldots, x_{i_s} \mid x_{i_1}, \ldots, x_{i_s} \text{ induces an odd cycle}\} \subseteq \mathrm{Ass}(J^2).$$

Thus, $(x_1, x_2, x_3, x_4, x_5) \in \mathrm{Ass}(J^2)$. Since we have the series of inclusions

$$(x_1) \subsetneq (x_1, x_2) \subsetneq (x_1, x_2, x_3) \subsetneq (x_1, x_2, x_3, x_4) \subsetneq (x_1, x_2, x_3, x_4, x_5)$$

we know that $(x_1, x_2, x_3, x_4, x_5)$ has height > 3 (in fact, the height is 5).

(2) *From DeVries-Yu*: Let $K_{n,d}$ be the complete bipartite graph on n and d vertices (i.e. let L be a set of n vertices and R a set of d vertices with $L \cap R = \emptyset$. Then the vertex set of $K_{n,d}$ is $L \cup R$, and the edge set of $K_{n,d}$ is the set of all pairs with one element from L and one element from R). Let $I(K_{n,d})$ denote the edge ideal of $K_{n,d}$. Write a recursive formula for $\beta_{i,j}(I(K_{n,d}))$ in terms of the Betti numbers of $I(K_{m,d})$ for $m < n$. Use your formula to compute $\beta_{1,j}(I(K_{n,d}))$ for all j.

Solution: The complement of $K_{n,d}$ is a disjoint union of K_n and K_d, so the complement of $K_{n,d}$ is chordal, and thus $I(K_{n,d})$ has a linear resolution. So it suffices to find $\beta_{i,i+2}(I(K_{n,d}))$.

In class we computed $K_{1,d}$, so assume $n > 1$. Pick a vertex v in the partition of size n. Deleting v leaves the graph $K_{n-1,d}$, which has at least one edge. So v is a splitting vertex. Therefore the recursive formula is

$$\beta_{i,i+2}(I(K_{n,d})) = \beta_{i,i+2}(I(K_{1,d})) + \beta_{i,i+2}(I(K_{n-1,d})) + \beta_{i-1,i+1}(I(K_{n-1,d})).$$

To compute $\beta_{1,3}(I(K_{n,d})$ we first compute $\beta_{0,2}(I(K_{n,d}))$. For $i = 0$ we have

$$\beta_{0,2}(I(K_{n,d})) = \beta_{0,2}(I(K_{1,d})) + \beta_{0,2}(I(K_{n-1,d})) + 0 = d + \beta_{0,2}(I(K_{n-1,d})).$$

As $\beta_{0,2}(I(K_{1,d})) = d$, summing to solve the recurrence gives $\beta_{0,2}(I(K_{n,d})) = \sum_{\ell=1}^{n} d = nd$.

Now we turn to $\beta_{1,3}(I(K_{n,d}))$. We have

$$\beta_{1,3}(I(K_{n,d})) = \binom{d}{2} + \beta_{1,3}(I(K_{n-1,d})) + \beta_{0,2}(K_{n-1,d}) = \binom{d}{2} + \beta_{1,3}(I(K_{n-1,d})) + (n-1)d.$$

Using $\beta_{1,3}(I(K_{1,d})) = \binom{d}{2}$ and solving the recurrence by summation gives

$$\beta_{1,3}(I(K_{n,d})) = \sum_{\ell=0}^{n-1} \left(\binom{d}{2} + \ell d \right) = n\binom{d}{2} + d\sum_{\ell=0}^{n-1} \ell = n\binom{d}{2} + d\binom{n}{2}.$$

Since $I(K_{n,d})$ has a linear resolution, this determines $\beta_{1,j}(I(K_{n,d}))$ for all j.