Math 314/814: Matrix Theory **Dr. S. Cooper, Fall 2008**

Homework Solutions – Week of October 28

Section 5.1:

(16) The columns of the given matrix A are easily seen to be orthonormal. Thus, the matrix is orthogonal. By Theorem 5.5,

$$A^{-1} = A^T = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

(19) Let A be the given matrix. Then

$$\begin{aligned} A^T A &= \begin{bmatrix} \cos\theta\sin\theta & \cos^2\theta & \sin\theta \\ -\cos\theta & \sin\theta & 0 \\ -\sin^2\theta & -\cos\theta\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta\sin\theta & -\cos\theta & -\sin^2\theta \\ \cos^2\theta & \sin\theta & -\cos\theta\sin\theta \\ \sin\theta & 0 & \cos\theta \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

To see this, you will need to use the trig identity $\sin^2\theta + \cos^2\theta = 1$. So, for example,

$$\begin{aligned} \cos^2\theta\sin^2\theta + \cos^4\theta + \sin^2\theta &= \cos^2\theta(\sin^2\theta + \cos^2\theta) + \sin^2\theta \\ &= \cos^2\theta + \sin^2\theta = 1 \\ -\cos\theta\sin^3\theta - \cos^3\theta\sin\theta + \cos\theta\sin\theta &= \cos\theta\sin\theta(-\sin^2\theta - \cos^2\theta + 1) = 0 \end{aligned}$$

We see that $A^{-1} = A^T$. So, by Theorem 5.5 A is orthogonal and

$$A^{-1} = A^T = \begin{bmatrix} \cos\theta\sin\theta & \cos^2\theta & \sin\theta \\ -\cos\theta & \sin\theta & 0 \\ -\sin^2\theta & -\cos\theta\sin\theta & \cos\theta \end{bmatrix}.$$

(20) The columns of the given matrix A are easily seen to be orthonormal. Thus, the matrix is orthogonal. By Theorem 5.5,

$$A^{-1} = A^T = \begin{bmatrix} 1/2 & 1/2 & -1/2 & 1/2 \\ -1/2 & 1/2 & 1/2 & 1/2 \\ 1/2 & 1/2 & 1/2 & -1/2 \\ 1/2 & -1/2 & 1/2 & 1/2 \end{bmatrix}.$$

(26) Suppose Q is an orthogonal matrix. Let P be a matrix obtained from Q by interchanging rows of Q.

Since Q is orthogonal, the row vectors of Q form an orthonormal set (see Theorem 5.7). Since changing the order of a list of vectors doesn't change the length of the vectors nor the orthogonality of the set, the row vectors of P form an orthonormal set. Thus, the column vectors of P^T form an orthonormal set. We conclude, using the definition of orthogonal matrices, that P^T is an orthogonal matrix. Applying Theorem 5.5 to P^T yields that

$$(P^T)^{-1} = (P^T)^T = P.$$

Therefore,

$$P^{-1} = [(P^T)^{-1}]^{-1} = P^T.$$

Now applying Theorem 5.5 to P shows that P is an orthogonal matrix.

Section 5.2:

(1) Observe that

$$W = \left\{ \begin{bmatrix} x \\ 2x \end{bmatrix} \in \mathbb{R}^2 \right\} = span\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix} \right).$$

That is W equals the column space of the 2×1 matrix

$$A = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

By Theorem 5.10, $W^\perp = null(A^T)$. We have

$$A^T = \begin{bmatrix} 1 & 2 \end{bmatrix}.$$

After we solve the system $A^T\mathbf{x} = \mathbf{0}$, we see that

$$\begin{aligned} W^\perp = null(A^T) &= \left\{ \mathbf{x} = \begin{bmatrix} -2t \\ t \end{bmatrix} : t \in \mathbb{R} \right\} \\ &= \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : x + 2y = 0 \right\} \end{aligned}$$

Thus, a basis for $W^\perp$ is

$$\mathcal{B}^\perp = \left\{ \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right\}.$$

(2) Observe that

$$W = \left\{ \begin{bmatrix} x \\ (-3/4)x \end{bmatrix} \in \mathbb{R}^2 \right\} = span\left(\begin{bmatrix} 1 \\ -3/4 \end{bmatrix} \right).$$

That is W equals the column space of the 2×1 matrix

$$A = \begin{bmatrix} 1 \\ -3/4 \end{bmatrix}.$$

By Theorem 5.10, $W^\perp = null(A^T)$. We have

$$A^T = \begin{bmatrix} 1 & -3/4 \end{bmatrix}.$$

After we solve the system $A^T\mathbf{x} = \mathbf{0}$, we see that

$$\begin{aligned} W^\perp = null(A^T) &= \left\{ \mathbf{x} = \begin{bmatrix} (3/4)t \\ t \end{bmatrix} : t \in \mathbb{R} \right\} \\ &= \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : x - (3/4)y = 0 \right\} \end{aligned}$$

Thus, a basis for $W^\perp$ is

$$\mathcal{B}^\perp = \left\{ \begin{bmatrix} 3/4 \\ 1 \end{bmatrix} \right\}.$$

(3) Observe that

$$W = \left\{ \begin{bmatrix} x \\ y \\ x+y \end{bmatrix} \in \mathbb{R}^3 \right\} = span\left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right).$$

That is W equals the column space of the 3×2 matrix

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}.$$

By Theorem 5.10, $W^{\perp} = null(A^T)$. We have

$$A^T = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

After we solve the system $A^T\mathbf{x} = \mathbf{0}$, we see that

$$\begin{aligned} W^{\perp} = null(A^T) &= \left\{ \mathbf{x} = \begin{bmatrix} -t \\ -t \\ t \end{bmatrix} : t \in \mathbb{R} \right\} \\ &= \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : x = t, y = t, z = -t \right\} \end{aligned}$$

Thus, a basis for $W^{\perp}$ is

$$\mathcal{B}^{\perp} = \left\{ \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \right\}.$$

(6) Observe that

$$W = \left\{ \begin{bmatrix} 2t \\ 2t \\ -t \end{bmatrix} \in \mathbb{R}^3 \right\} = span\left(\begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix} \right).$$

That is W equals the column space of the 3×1 matrix

$$A = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}.$$

By Theorem 5.10, $W^{\perp} = null(A^T)$. We have

$$A^T = \begin{bmatrix} 2 & 2 & -1 \end{bmatrix}.$$

After we solve the system $A^T\mathbf{x} = \mathbf{0}$, we see that

$$\begin{aligned} W^{\perp} = null(A^T) &= \left\{ \mathbf{x} = \begin{bmatrix} -s + (t/2) \\ s \\ t \end{bmatrix} : s, t \in \mathbb{R} \right\} \\ &= \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : x = -s + (t/2), y = s, z = t \right\} \end{aligned}$$

Thus, a basis for $W^{\perp}$ is

$$\mathcal{B}^{\perp} = \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1/2 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

(7) We begin by finding RREF(A):

$$A \longrightarrow RREF(A) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

We see that a basis for $row(A)$ is

$$\left\{ \begin{bmatrix} 1 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & -2 \end{bmatrix} \right\}.$$

By solving the system $A\mathbf{x} = \mathbf{0}$, we find that $x_1 = -x_3 = -t, x_2 = 2x_3 = 2t, x_3 = t$. So, a basis for $null(A)$ is

$$\left\{ \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} \right\}.$$

Note that

$$\begin{aligned} 1(-1) + (0)(2) + (1)(1) &= 0 \\ (0)(-1) + (1)(2) + (-2)(1) &= 0 \end{aligned}$$

That is, the basis vectors for $row(A)$ are orthogonal to the basis vectors for $null(A)$. This is enough to show that every vector in $row(A)$ is orthogonal to every vector in $null(A)$.

(9) Using the RREF(A) from exercise 7, we see that a basis for $col(A)$ is

$$\left\{ \begin{bmatrix} 1 \\ 5 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 1 \\ -1 \end{bmatrix} \right\}.$$

To find a basis for $null(A^T)$, we need to row-reduce A^T:

$$A^T = \begin{bmatrix} 1 & 5 & 0 & -1 \\ -1 & 2 & 1 & -1 \\ 3 & 1 & -2 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 5 & 0 & -1 \\ 0 & 7 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

By solving $A^T\mathbf{x} = \mathbf{0}$, we find $x_1 = 5/7t - 3/7s, x_2 = -t/7 + 2/7s, x_3 = t, x_4 = s$. So, a basis for $null(A^T)$ is

$$\left\{ \begin{bmatrix} 5 \\ -1 \\ 7 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 2 \\ 0 \\ 7 \end{bmatrix} \right\}.$$

Note that

$$\begin{aligned}
(1)(5)+(5)(-1)+(0)(7)+(-1)(0) &= 0\\
(-1)(5)+(2)(-1)+(1)(7)+(-1)(0) &= 0\\
(1)(-3)+(5)(2)+(0)(0)+(-1)(7) &= 0\\
(-1)(-3)+(2)(2)+(1)(0)+(-1)(7) &= 0
\end{aligned}$$

That is, the basis vectors for $col(A)$ are orthogonal to the basis vectors for $null(A^T)$. This is enough to show that every vector in $col(A)$ is orthogonal to every vector in $null(A^T)$.

(11) We have that

$$W = span\left(\begin{bmatrix} 2\\ 1\\ -2 \end{bmatrix}, \begin{bmatrix} 4\\ 0\\ 1 \end{bmatrix}\right) = col(A)$$

where

$$A = \begin{bmatrix} 2 & 4\\ 1 & 0\\ -2 & 1 \end{bmatrix}.$$

So, by Theorem 5.10, $W^{\perp} = null(A^T)$. We have

$$A^T = \begin{bmatrix} 2 & 1 & -2\\ 4 & 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 2 & 1 & -2\\ 0 & 2 & -5 \end{bmatrix}.$$

Thus, when we solve the system $A^T\mathbf{x} = \mathbf{0}$, we have $x_1 = -1/4t, x_2 = 5/2t, x_3 = t$. So, a basis for $W^{\perp} = null(A^T)$ is

$$\left\{\begin{bmatrix} 1\\ -10\\ -4 \end{bmatrix}\right\}.$$

(12) We have that

$$W = span\left(\begin{bmatrix} 1\\ -1\\ 3\\ -2 \end{bmatrix}, \begin{bmatrix} 0\\ 1\\ -2\\ 1 \end{bmatrix}\right) = col(A)$$

where

$$A = \begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 3 & -2 \\ -2 & 1 \end{bmatrix}.$$

So, by Theorem 5.10, $W^\perp = null(A^T)$. We have

$$A^T = \begin{bmatrix} 1 & -1 & 3 & -2 \\ 0 & 1 & -2 & 1 \end{bmatrix}.$$

Thus, when we solve the system $A^T\mathbf{x} = \mathbf{0}$, we have $x_1 = x_2 - 3x_3 + 2x_4 = -s + t, x_2 = 2x_3 - x_4 = 2s - t, x_3 = s, x_4 = t$. So, a basis for $W^\perp = null(A^T)$ is

$$\left\{ \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

(16) By definition,

$$proj_W(\mathbf{v}) = \left(\frac{\mathbf{u_1} \cdot \mathbf{v}}{\mathbf{u_1} \cdot \mathbf{u_1}} \right) \mathbf{u_1} + \left(\frac{\mathbf{u_2} \cdot \mathbf{v}}{\mathbf{u_2} \cdot \mathbf{u_2}} \right) \mathbf{u_2}.$$

We calculate

$$\begin{aligned} \mathbf{u_1} \cdot \mathbf{v} &= 2 \\ \mathbf{u_2} \cdot \mathbf{v} &= 2 \\ \mathbf{u_1} \cdot \mathbf{u_1} &= 3 \\ \mathbf{u_2} \cdot \mathbf{u_2} &= 2 \end{aligned}$$

Thus,

$$proj_W(\mathbf{v}) = \frac{2}{3}\mathbf{u_1} + \frac{2}{2}\mathbf{u_2} = \begin{bmatrix} 5/3 \\ -1/3 \\ 2/3 \end{bmatrix}.$$

(17) By definition,

$$proj_W(\mathbf{v}) = \left(\frac{\mathbf{u_1}\cdot\mathbf{v}}{\mathbf{u_1}\cdot\mathbf{u_1}}\right)\mathbf{u_1} + \left(\frac{\mathbf{u_2}\cdot\mathbf{v}}{\mathbf{u_2}\cdot\mathbf{u_2}}\right)\mathbf{u_2}.$$

We calculate

$$\begin{aligned} \mathbf{u_1}\cdot\mathbf{v} &= 1 \\ \mathbf{u_2}\cdot\mathbf{v} &= 13 \\ \mathbf{u_1}\cdot\mathbf{u_1} &= 9 \\ \mathbf{u_2}\cdot\mathbf{u_2} &= 18 \end{aligned}$$

Thus,

$$proj_W(\mathbf{v}) = \frac{1}{9}\mathbf{u_1} + \frac{13}{18}\mathbf{u_2} = \begin{bmatrix} -1/2 \\ 1/2 \\ 3 \end{bmatrix}.$$

(21) Let $\mathbf{u_1} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ and $\mathbf{u_2} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$. Then $W = span(\mathbf{u_1}, \mathbf{u_2})$. We want to write $\mathbf{v}$ as

$$\mathbf{v} = \mathbf{w} + \mathbf{w}^{\perp},$$

where $\mathbf{w} \in W$ and $\mathbf{w}^{\perp} \in W^{\perp}$. Using the proof of Theorem 5.11, we see that

$$\mathbf{w} = proj_W(\mathbf{v}) = \left(\frac{\mathbf{u_1}\cdot\mathbf{v}}{\mathbf{u_1}\cdot\mathbf{u_1}}\right)\mathbf{u_1} + \left(\frac{\mathbf{u_2}\cdot\mathbf{v}}{\mathbf{u_2}\cdot\mathbf{u_2}}\right)\mathbf{u_2}.$$

We calculate

$$\begin{aligned} \mathbf{u_1}\cdot\mathbf{v} &= 3 \\ \mathbf{u_2}\cdot\mathbf{v} &= 9 \\ \mathbf{u_1}\cdot\mathbf{u_1} &= 6 \\ \mathbf{u_2}\cdot\mathbf{u_2} &= 3 \end{aligned}$$

Thus,

$$\mathbf{w} = proj_W(\mathbf{v}) = \frac{1}{2}\mathbf{u_1} + 3\mathbf{u_2} = \begin{bmatrix} 7/2 \\ -2 \\ 7/2 \end{bmatrix}.$$

We now let

$$\mathbf{w}^{\perp} = \mathbf{v} - \mathbf{w} = \begin{bmatrix} 1/2 \\ 0 \\ -1/2 \end{bmatrix}.$$

So, the orthogonal decomposition of $\mathbf{v}$ with respect to W is

$$\mathbf{v} = \mathbf{w} + \mathbf{w}^{\perp} = \begin{bmatrix} 7/2 \\ -2 \\ 7/2 \end{bmatrix} + \begin{bmatrix} 1/2 \\ 0 \\ -1/2 \end{bmatrix}.$$

(25) No, it is not necessarily true that $\mathbf{w}'$ is in $W^{\perp}$. For example, consider the subspace W of $\mathbb{R}^3$ from exercise 21. That is, let

$$W = span\left(\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right).$$

Take

$$\mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \in \mathbb{R}^3.$$

We observe that since W is a subspace, the vector

$$\mathbf{w} := \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

is in W. Moreover, we can write

$$\mathbf{v} = \mathbf{0} + \mathbf{v}.$$

Here the vector $\mathbf{v}$ is also playing the role of $\mathbf{w}'$. However,

$$\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = 6 \neq 0.$$

This shows that $\mathbf{w}'$ is not in $W^{\perp}$.

(29) To do this exercise, it is helpful to use a part of exercise (27); namely, if $\mathbf{u} \in W$, then $proj_W(\mathbf{u}) = \mathbf{u}$. Let's see why this is true.

By Theorem 5.11 and its proof, we can write

$$\mathbf{u} = \mathbf{w} + \mathbf{w}^{\perp} = proj_W(\mathbf{u}) + \mathbf{w}^{\perp},$$

where $\mathbf{w} = proj_W(\mathbf{u}) \in W$ and $\mathbf{w}^{\perp} \in W^{\perp}$ and these vectors are unique. But $\mathbf{0} \in W^{\perp}$ and $\mathbf{u} \in W$ and

$$\mathbf{u} = \mathbf{u} + \mathbf{0}.$$

Since the vectors $\mathbf{w}$ and $\mathbf{w}^{\perp}$ are unique, we must have that $\mathbf{w} = proj_W(\mathbf{u}) = \mathbf{u}$ and $\mathbf{w}^{\perp} = \mathbf{0}$. This shows that $proj_W(\mathbf{u}) = \mathbf{u}$.

Now let $\mathbf{x}$ be a vector in $\mathbb{R}^n$. Then, by definition, $proj_W(\mathbf{x})$ is a linear combination of vectors in W and so $proj_W(\mathbf{x})$ is a vector in W (since W is a subspace and thus closed under scalar multiplication and vector addition!). But the projection of any vector in W is just itself by the above observations. Therefore, we conlcude that

$$proj_W(proj_W(\mathbf{x})) = proj_W(\mathbf{x}).$$

Section 5.3:

(1) Let $\mathbf{v_1} = \mathbf{x_1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Then let

$$\begin{aligned} \mathbf{v_2} &= \mathbf{x_2} - \left(\frac{\mathbf{v_1} \cdot \mathbf{x_2}}{\mathbf{v_1} \cdot \mathbf{v_1}}\right)\mathbf{v_1} \\ &= \begin{bmatrix} 1 \\ 2 \end{bmatrix} - \frac{3}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} -1/2 \\ 1/2 \end{bmatrix} \end{aligned}$$

The set $\{\mathbf{v_1}, \mathbf{v_2}\}$ is an orthogonal basis for $\mathbb{R}^2$.

To obtain an orthonormal basis for $\mathbb{R}^2$ we normalize the vectors $\mathbf{v_1}$ and $\mathbf{v_2}$. We find

$$\begin{aligned} ||\mathbf{v_1}|| &= \sqrt{\mathbf{v_1} \cdot \mathbf{v_1}} = \sqrt{2} \\ ||\mathbf{v_2}|| &= \sqrt{\mathbf{v_2} \cdot \mathbf{v_2}} = 1/\sqrt{2} \end{aligned}$$

Let

$$\mathbf{w_1} = \frac{1}{||\mathbf{v_1}||}\mathbf{v_1} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

and

$$\mathbf{w_2} = \frac{1}{||\mathbf{v_2}||}\mathbf{v_2} \begin{bmatrix} -\sqrt{2}/2 \\ \sqrt{2}/2 \end{bmatrix}.$$

Then the set $\{\mathbf{w_1}, \mathbf{w_2}\}$ is an orthonormal basis for $\mathbb{R}^2$.

(4) Let $\mathbf{v_1} = \mathbf{x_1} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. Then let

$$\begin{aligned} \mathbf{v_2} &= \mathbf{x_2} - \left(\frac{\mathbf{v_1} \cdot \mathbf{x_2}}{\mathbf{v_1} \cdot \mathbf{v_1}}\right)\mathbf{v_1} \\ &= \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - \frac{2}{3}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 1/3 \\ 1/3 \\ -2/3 \end{bmatrix} \end{aligned}$$

and

$$\begin{aligned} \mathbf{v_3} &= \mathbf{x_3} - \left(\frac{\mathbf{v_1} \cdot \mathbf{x_3}}{\mathbf{v_1} \cdot \mathbf{v_1}}\right)\mathbf{v_1} - \left(\frac{\mathbf{v_2} \cdot \mathbf{x_3}}{\mathbf{v_2} \cdot \mathbf{v_2}}\right)\mathbf{v_2} \\ &= \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - \frac{1}{3}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \frac{1/3}{2/3}\begin{bmatrix} 1/3 \\ 1/3 \\ -2/3 \end{bmatrix} \\ &= \begin{bmatrix} 1/2 \\ -1/2 \\ 0 \end{bmatrix} \end{aligned}$$

The set $\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$ is an orthogonal basis for $\mathbb{R}^3$.

To obtain an orthonormal basis for $\mathbb{R}^3$ we normalize the vectors $\mathbf{v_1}, \mathbf{v_2}$ and $\mathbf{v_3}$. We find

$$\begin{aligned} ||\mathbf{v_1}|| &= \sqrt{\mathbf{v_1} \cdot \mathbf{v_1}} = \sqrt{3} \\ ||\mathbf{v_2}|| &= \sqrt{\mathbf{v_2} \cdot \mathbf{v_2}} = \sqrt{6}/3 \\ ||\mathbf{v_3}|| &= \sqrt{\mathbf{v_3} \cdot \mathbf{v_3}} = \sqrt{2}/2 \end{aligned}$$

Let

$$\mathbf{w_1} = \frac{1}{||\mathbf{v_1}||}\mathbf{v_1} = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$$

and

$$\mathbf{w_2} = \frac{1}{||\mathbf{v_2}||}\mathbf{v_2} \begin{bmatrix} 1/\sqrt{6} \\ 1/\sqrt{6} \\ -2/\sqrt{6} \end{bmatrix}$$

and

$$\mathbf{w_3} = \frac{1}{||\mathbf{v_3}||}\mathbf{v_3} \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{bmatrix}$$

Then the set $\{\mathbf{w_1}, \mathbf{w_2}, \mathbf{w_3}\}$ is an orthonormal basis for $\mathbb{R}^3$.

(5) Let $\mathbf{v_1} = \mathbf{x_1} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$. Then let

$$\begin{aligned} \mathbf{v_2} &= \mathbf{x_2} - \left(\frac{\mathbf{v_1} \cdot \mathbf{x_2}}{\mathbf{v_1} \cdot \mathbf{v_1}}\right)\mathbf{v_1} \\ &= \begin{bmatrix} 3 \\ 4 \\ 2 \end{bmatrix} - \frac{7}{2}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} -1/2 \\ 1/2 \\ 2 \end{bmatrix} \end{aligned}$$

Then the set $\{\mathbf{v_1}, \mathbf{v_2}\}$ is an orthogonal basis for W.

(6) Let $\mathbf{v_1} = \mathbf{x_1} = \begin{bmatrix} 2 \\ -1 \\ 1 \\ 2 \end{bmatrix}$. Then let

$$\begin{aligned} \mathbf{v_2} &= \mathbf{x_2} - \left(\frac{\mathbf{v_1} \cdot \mathbf{x_2}}{\mathbf{v_1} \cdot \mathbf{v_1}}\right) \mathbf{v_1} \\ &= \begin{bmatrix} 3 \\ -1 \\ 0 \\ 4 \end{bmatrix} - \frac{15}{10} \begin{bmatrix} 2 \\ -1 \\ 1 \\ 2 \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ 1/2 \\ -3/2 \\ 1 \end{bmatrix} \end{aligned}$$

Then the set $\{\mathbf{v_1}, \mathbf{v_2}\}$ is an orthogonal basis for W.

(9) We first need to find a basis for $col(A)$ by row-reducing A. We have

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

Thus, every original column of A is a basis vector for $col(A)$. More precisely, we let

$$\mathbf{x_1} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{x_2} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{x_3} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

Then $\mathcal{B} = \{\mathbf{x_1}, \mathbf{x_2}, \mathbf{x_3}\}$ is a basis for $col(A)$.

We now apply the Gram-Schmidt process on the vectors in $\mathcal{B}$ to obtain an orthogonal basis for $col(A)$.

Let $\mathbf{v_1} = \mathbf{x_1} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$. Then let

$$\begin{aligned}
\mathbf{v_2} &= \mathbf{x_2} - \left(\frac{\mathbf{v_1} \cdot \mathbf{x_2}}{\mathbf{v_1} \cdot \mathbf{v_1}} \right) \mathbf{v_1} \\
&= \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 \\ -1/2 \\ 1/2 \end{bmatrix}
\end{aligned}$$

and

$$\begin{aligned}
\mathbf{v_3} &= \mathbf{x_3} - \left(\frac{\mathbf{v_1} \cdot \mathbf{x_3}}{\mathbf{v_1} \cdot \mathbf{v_1}} \right) \mathbf{v_1} - \left(\frac{\mathbf{v_2} \cdot \mathbf{x_3}}{\mathbf{v_2} \cdot \mathbf{v_2}} \right) \mathbf{v_2} \\
&= \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - \frac{1/2}{3/2} \begin{bmatrix} 1 \\ -1/2 \\ 1/2 \end{bmatrix} \\
&= \begin{bmatrix} 2/3 \\ 2/3 \\ -2/3 \end{bmatrix}
\end{aligned}$$

The set $\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$ is an orthogonal basis for $col(A)$.