Math 314/814: Matrix Theory **Dr. S. Cooper, Fall 2008**

Homework Solutions – Week of November 4

Section 5.3:

(7) In the last homework set (Week of October 28), we found that W has the orthogonal basis

$$\mathcal{B} = \left\{ \mathbf{v_1} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \mathbf{v_2} = \begin{bmatrix} -1/2 \\ 1/2 \\ 2 \end{bmatrix} \right\}.$$

We use $\mathcal{B}$ to find the orthogonal decomposition of $\mathbf{v}$. We calculate

$$\begin{aligned} proj_W(\mathbf{v}) &= \left(\frac{\mathbf{v_1}\cdot\mathbf{v}}{\mathbf{v_1}\cdot\mathbf{v_1}}\right)\mathbf{v_1} + \left(\frac{\mathbf{v_2}\cdot\mathbf{v}}{\mathbf{v_2}\cdot\mathbf{v_2}}\right)\mathbf{v_2} \\ &= \frac{0}{2}\mathbf{v_1} + \frac{2}{9/2}\mathbf{v_2} \\ &= \frac{4}{9}\mathbf{v_2} \\ &= \begin{bmatrix} -2/9 \\ 2/9 \\ 8/9 \end{bmatrix} \end{aligned}$$

and

$$\begin{aligned} perp_W(\mathbf{v}) &= \mathbf{v} - proj_W(\mathbf{v}) \\ &= \begin{bmatrix} 38/9 \\ -38/9 \\ 19/9 \end{bmatrix} \end{aligned}$$

So, the orthogonal decomposition of $\mathbf{v}$ with respect to W is

$$\mathbf{v} = proj_W(\mathbf{v}) + perp_W(\mathbf{v}) = \begin{bmatrix} -2/9 \\ 2/9 \\ 8/9 \end{bmatrix} + \begin{bmatrix} 38/9 \\ -38/9 \\ 19/9 \end{bmatrix}.$$

(8) In the last homework set (Week of October 28), we found that W has the orthogonal basis

$$\mathcal{B} = \left\{ \mathbf{v_1} = \begin{bmatrix} 2 \\ -1 \\ 1 \\ 2 \end{bmatrix}, \mathbf{v_2} = \begin{bmatrix} 0 \\ 1/2 \\ -3/2 \\ 1 \end{bmatrix} \right\}.$$

We use $\mathcal{B}$ to find the orthogonal decomposition of $\mathbf{v}$. We calculate

$$\begin{aligned} proj_W(\mathbf{v}) &= \left(\frac{\mathbf{v_1}\cdot\mathbf{v}}{\mathbf{v_1}\cdot\mathbf{v_1}}\right)\mathbf{v_1} + \left(\frac{\mathbf{v_2}\cdot\mathbf{v}}{\mathbf{v_2}\cdot\mathbf{v_2}}\right)\mathbf{v_2} \\ &= \frac{2}{10}\mathbf{v_1} + \frac{4}{14/4}\mathbf{v_2} \\ &= \frac{1}{5}\mathbf{v_1} + \frac{8}{7}\mathbf{v_2} \\ &= \begin{bmatrix} 2/5 \\ 13/35 \\ -53/35 \\ 54/35 \end{bmatrix} \end{aligned}$$

and

$$\begin{aligned} perp_W(\mathbf{v}) &= \mathbf{v} - proj_W(\mathbf{v}) \\ &= \begin{bmatrix} 3/5 \\ 127/35 \\ 53/35 \\ 16/35 \end{bmatrix} \end{aligned}$$

So, the orthogonal decomposition of $\mathbf{v}$ with respect to W is

$$\mathbf{v} = proj_W(\mathbf{v}) + perp_W(\mathbf{v}) = \begin{bmatrix} 2/5 \\ 13/35 \\ -53/35 \\ 54/35 \end{bmatrix} + \begin{bmatrix} 3/5 \\ 127/35 \\ 53/35 \\ 16/35 \end{bmatrix}.$$

(11) We start with a basis for $\mathbb{R}^3$ which includes the given vector. Let

$$\mathbf{x_1} = \begin{bmatrix} 3 \\ 1 \\ 5 \end{bmatrix}, \mathbf{x_2} = \begin{bmatrix} 0 \\ 35 \\ 0 \end{bmatrix}, \mathbf{x_3} = \begin{bmatrix} 1190 \\ 0 \\ 0 \end{bmatrix}.$$

it is easily verified that the matrix whose columns are $\mathbf{x_1}, \mathbf{x_2}$ and $\mathbf{x_3}$ row reduces to the 3×3 identity matrix. Thus, by the Fundamental Theorem for Invertible Matrices, $\mathcal{B} = \{\mathbf{x_1}, \mathbf{x_2}, \mathbf{x_3}\}$ is a basis for $\mathbb{R}^3$. We now apply GSOP to $\mathcal{B}$ to obtain an orthogonal basis for $\mathbb{R}^3$ which contains the given vector $\mathbf{x_1}$.

Let

$$\mathbf{v_1} = \mathbf{x_1} = \begin{bmatrix} 3 \\ 1 \\ 5 \end{bmatrix}.$$

Now let

$$\begin{aligned} \mathbf{v_2} &= \mathbf{x_2} - \left(\frac{\mathbf{v_1} \cdot \mathbf{x_2}}{\mathbf{v_1} \cdot \mathbf{v_1}} \right) \mathbf{v_1} \\ &= \mathbf{x_2} - \frac{35}{35} \mathbf{v_1} \\ &= \mathbf{x_2} - \mathbf{v_1} \\ &= \begin{bmatrix} -3 \\ 34 \\ -5 \end{bmatrix}. \end{aligned}$$

Finally, let

$$\begin{aligned} \mathbf{v_3} &= \mathbf{x_3} - \left(\frac{\mathbf{v_1} \cdot \mathbf{x_3}}{\mathbf{v_1} \cdot \mathbf{v_1}} \right) \mathbf{v_1} - \left(\frac{\mathbf{v_2} \cdot \mathbf{x_3}}{\mathbf{v_2} \cdot \mathbf{v_2}} \right) \mathbf{v_2} \\ &= \mathbf{x_3} - \frac{3570}{35} \mathbf{v_1} - \frac{-3570}{1190} \mathbf{v_2} \\ &= \mathbf{x_3} - 102\mathbf{v_1} + 3\mathbf{v_2} \\ &= \begin{bmatrix} 875 \\ 0 \\ -525 \end{bmatrix}. \end{aligned}$$

The set $\mathcal{C} = \{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$ is an orthogonal basis for $\mathbb{R}^3$ that contains the given vector.

Section 7.3:

(7) We need to find the least squares solution to the system

$$\begin{aligned} a + b &= 0 \\ a + 2b &= 1 \\ a + 3b &= 5 \end{aligned}$$

Let

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}$$

and

$$\mathbf{b} = \begin{bmatrix} 0 \\ 1 \\ 5 \end{bmatrix}.$$

We calculate

$$A^T A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 6 \\ 6 & 14 \end{bmatrix}.$$

Since $\det(A^T A) = 6 \neq 0$, the matrix $A^T A$ is invertible and the vector we seek equals $(A^T A)^{-1} A^T \mathbf{b}$. It is easy to verify that

$$(A^T A)^{-1} = \frac{1}{6} \begin{bmatrix} 14 & -6 \\ -6 & 3 \end{bmatrix}.$$

So,

$$\begin{aligned} \overline{\mathbf{x}} &= (A^T A)^{-1} A^T \mathbf{b} \\ &= \frac{1}{6} \begin{bmatrix} 14 & -6 \\ -6 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 5 \end{bmatrix} \\ &= \begin{bmatrix} -3 \\ 5/2 \end{bmatrix} \end{aligned}$$

So, the least squares approximating line is

$$y = -3 + \frac{5}{2} x.$$

The corresponding least squares error is:

$$||\mathbf{e}|| = ||\mathbf{b} - A\overline{\mathbf{x}}|| = \left|\left| \begin{bmatrix} 0 \\ 1 \\ 5 \end{bmatrix} - \begin{bmatrix} -1/2 \\ 2 \\ 9/2 \end{bmatrix} \right|\right| = \left|\left| \begin{bmatrix} 1/2 \\ -1 \\ 1/2 \end{bmatrix} \right|\right| = \sqrt{3/2} \approx 1.225.$$

(8) We need to find the least squares solution to the system

$$\begin{aligned} a + b &= 5 \\ a + 2b &= 3 \\ a + 3b &= 2 \end{aligned}$$

Let

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}$$

and

$$\mathbf{b} = \begin{bmatrix} 5 \\ 3 \\ 2 \end{bmatrix}.$$

We calculate

$$A^T A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 6 \\ 6 & 14 \end{bmatrix}.$$

Since $\det(A^T A) = 6 \neq 0$, the matrix $A^T A$ is invertible and the vector we seek equals $(A^T A)^{-1} A^T \mathbf{b}$. It is easy to verify that

$$(A^T A)^{-1} = \frac{1}{6} \begin{bmatrix} 14 & -6 \\ -6 & 3 \end{bmatrix}.$$

So,

$$\begin{aligned} \overline{\mathbf{x}} &= (A^T A)^{-1} A^T \mathbf{b} \\ &= \frac{1}{6} \begin{bmatrix} 14 & -6 \\ -6 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 5 \\ 3 \\ 2 \end{bmatrix} \\ &= \begin{bmatrix} 19/6 \\ -3/2 \end{bmatrix} \end{aligned}$$

So, the least squares approximating line is

$$y = \frac{19}{6} - \frac{3}{2}x.$$

The corresponding least squares error is:

$$||\mathbf{e}|| = ||\mathbf{b} - A\overline{\mathbf{x}}|| = \left\| \begin{bmatrix} 5 \\ 3 \\ 2 \end{bmatrix} - \begin{bmatrix} -5/3 \\ 1/6 \\ -4/3 \end{bmatrix} \right\| = \left\| \begin{bmatrix} 10/3 \\ 17/6 \\ 10/3 \end{bmatrix} \right\| = \sqrt{1089/36} = \frac{33}{6} = 5.5.$$

(11) We need to find the least squares solution to the system

$$\begin{aligned} a - 5b &= -1 \\ a + 0b &= 1 \\ a + 5b &= 2 \\ a + 10b &= 4 \end{aligned}$$

Let

$$A = \begin{bmatrix} 1 & -5 \\ 1 & 0 \\ 1 & 5 \\ 1 & 10 \end{bmatrix}$$

and

$$\mathbf{b} = \begin{bmatrix} -1 \\ 1 \\ 2 \\ 4 \end{bmatrix}.$$

We calculate

$$A^T A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -5 & 0 & 5 & 10 \end{bmatrix} \begin{bmatrix} 1 & -5 \\ 1 & 0 \\ 1 & 5 \\ 1 & 10 \end{bmatrix} = \begin{bmatrix} 4 & 10 \\ 10 & 150 \end{bmatrix}.$$

Since $\det(A^T A) = 500 \neq 0$, the matrix $A^T A$ is invertible and the vector we seek equals $(A^T A)^{-1} A^T \mathbf{b}$. It is easy to verify that

$$(A^T A)^{-1} = \frac{1}{500} \begin{bmatrix} 150 & -10 \\ -10 & 4 \end{bmatrix}.$$

So,

$$\begin{aligned}\overline{\mathbf{x}} &= (A^TA)^{-1}A^T\mathbf{b} \\ &= \frac{1}{500}\begin{bmatrix} 150 & -10 \\ -10 & 4 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 & 1 \\ -5 & 0 & 5 & 10 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \\ 2 \\ 4 \end{bmatrix} \\ &= \begin{bmatrix} 7/10 \\ 8/25 \end{bmatrix}\end{aligned}$$

So, the least squares approximating line is

$$y = \frac{7}{10} + \frac{8}{25}x.$$

The corresponding least squares error is:

$$||\mathbf{e}|| = ||\mathbf{b} - A\overline{\mathbf{x}}|| = \left\| \begin{bmatrix} -1 \\ 1 \\ 2 \\ 4 \end{bmatrix} - \begin{bmatrix} -9/10 \\ 7/10 \\ 23/10 \\ 39/10 \end{bmatrix} \right\| = \left\| \begin{bmatrix} -1/10 \\ 3/10 \\ -3/10 \\ 1/10 \end{bmatrix} \right\| = \sqrt{20/100} \approx 0.447.$$

(12) We need to find the least squares solution to the system

$$\begin{aligned} a - 5b &= 3 \\ a + 0b &= 3 \\ a + 5b &= 2 \\ a + 10b &= 0 \end{aligned}$$

Let

$$A = \begin{bmatrix} 1 & -5 \\ 1 & 0 \\ 1 & 5 \\ 1 & 10 \end{bmatrix}$$

and

$$\mathbf{b} = \begin{bmatrix} 3 \\ 3 \\ 2 \\ 0 \end{bmatrix}.$$

We calculate

$$A^TA = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -5 & 0 & 5 & 10 \end{bmatrix} \begin{bmatrix} 1 & -5 \\ 1 & 0 \\ 1 & 5 \\ 1 & 10 \end{bmatrix} = \begin{bmatrix} 4 & 10 \\ 10 & 150 \end{bmatrix}.$$

Since $\det(A^TA) = 500 \neq 0$, the matrix A^TA is invertible and the vector we seek equals $(A^TA)^{-1}A^T\mathbf{b}$. It is easy to verify that

$$(A^TA)^{-1} = \frac{1}{500}\begin{bmatrix} 150 & -10 \\ -10 & 4 \end{bmatrix}.$$

So,

$$\begin{aligned} \overline{\mathbf{x}} &= (A^TA)^{-1}A^T\mathbf{b} \\ &= \frac{1}{500}\begin{bmatrix} 150 & -10 \\ -10 & 4 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 & 1 \\ -5 & 0 & 5 & 10 \end{bmatrix}\begin{bmatrix} 3 \\ 3 \\ 2 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 5/2 \\ -1/5 \end{bmatrix} \end{aligned}$$

So, the least squares approximating line is

$$y = \frac{5}{2} - \frac{1}{5}x.$$

The corresponding least squares error is:

$$||\mathbf{e}|| = ||\mathbf{b} - A\overline{\mathbf{x}}|| = \left\| \begin{bmatrix} 3 \\ 3 \\ 2 \\ 0 \end{bmatrix} - \begin{bmatrix} 7/2 \\ 5/2 \\ 3/2 \\ 1/2 \end{bmatrix} \right\| = \left\| \begin{bmatrix} -1/2 \\ 1/2 \\ 1/2 \\ -1/2 \end{bmatrix} \right\| = \sqrt{1} = 1.$$

(19) We calculate

$$A^TA = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 11 & 6 \\ 6 & 6 \end{bmatrix}$$

and

$$A^T\mathbf{b} = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \end{bmatrix}.$$

Thus, the normal equations $A^T A\overline{\mathbf{x}} = A^T\mathbf{b}$ are

$$\begin{bmatrix} 11 & 6 \\ 6 & 6 \end{bmatrix} \overline{\mathbf{x}} = \begin{bmatrix} 5 \\ 4 \end{bmatrix}.$$

Since $\det(A^T A) = 1/30 \neq 0$, the matrix $A^T A$ is invertible. So, the least squares solution is

$$\begin{aligned} \overline{\mathbf{x}} &= (A^T A)^{-1} A^T\mathbf{b} \\ &= \frac{1}{30} \begin{bmatrix} 6 & -6 \\ -6 & 11 \end{bmatrix} \begin{bmatrix} 5 \\ 4 \end{bmatrix} \\ &= \begin{bmatrix} 1/5 \\ 7/15 \end{bmatrix} \end{aligned}$$

(20) We calculate

$$A^T A = \begin{bmatrix} 3 & 1 & 2 \\ -2 & -2 & 1 \end{bmatrix} \begin{bmatrix} 3 & -2 \\ 1 & -2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 14 & -6 \\ -6 & 9 \end{bmatrix}$$

and

$$A^T\mathbf{b} = \begin{bmatrix} 3 & 1 & 2 \\ -2 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ -3 \end{bmatrix}.$$

Thus, the normal equations $A^T A\overline{\mathbf{x}} = A^T\mathbf{b}$ are

$$\begin{bmatrix} 14 & -6 \\ -6 & 9 \end{bmatrix} \overline{\mathbf{x}} = \begin{bmatrix} 6 \\ -3 \end{bmatrix}.$$

Since $\det(A^T A) = 1/90 \neq 0$, the matrix $A^T A$ is invertible. So, the least squares solution is

$$\begin{aligned}\overline{\mathbf{x}} &= (A^TA)^{-1}A^T\mathbf{b}\\ &= \frac{1}{90}\begin{bmatrix} 9 & 6 \\ 6 & 14 \end{bmatrix}\begin{bmatrix} 6 \\ -3 \end{bmatrix}\\ &= \begin{bmatrix} 2/5 \\ -1/15 \end{bmatrix}\end{aligned}$$

(23) We calculate

$$A^TA = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & -1 & -1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & -1 & 1 & 1 \\ 1 & -1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 2 & 1 \\ 0 & 3 & -2 & -1 \\ 2 & -2 & 3 & 2 \\ 1 & -1 & 2 & 2 \end{bmatrix}$$

and

$$A^T\mathbf{b} = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & -1 & -1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ -3 \\ 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 2 \\ -5 \\ 3 \\ -1 \end{bmatrix}.$$

Thus, we need to solve the normal equations $A^TA\overline{\mathbf{x}} = A^T\mathbf{b}$:

$$\begin{bmatrix} 3 & 0 & 2 & 1 \\ 0 & 3 & -2 & -1 \\ 2 & -2 & 3 & 2 \\ 1 & -1 & 2 & 2 \end{bmatrix}\overline{\mathbf{x}} = \begin{bmatrix} 2 \\ -5 \\ 3 \\ -1 \end{bmatrix}.$$

To solve for the least squares solutions $\overline{\mathbf{x}}$ we form the augmented matrix and row-reduce:

$$\begin{bmatrix} 3 & 0 & 2 & 1 & | & 2 \\ 0 & 3 & -2 & -1 & | & -5 \\ 2 & -2 & 3 & 2 & | & 3 \\ 1 & -1 & 2 & 2 & | & -1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -1 & 2 & 2 & | & -1 \\ 0 & 3 & -4 & -5 & | & 5 \\ 0 & 0 & 1 & 2 & | & -5 \\ 0 & 0 & 0 & 0 & | & 0 \end{bmatrix}.$$

Since there are free variables, we see that there are infinitely many least squares solutions. Solving we let $x_4 = t$ and so $x_3 = -2t - 5, x_2 = -t - 5$ and $x_1 = t + 4$.

Thus, the set of least squares solutions is:

$$\left\{ \begin{bmatrix} 4+t \\ -5-t \\ -5-2t \\ t \end{bmatrix} : t \in \mathbb{R} \right\}.$$

(25) Let

$$A = \begin{bmatrix} 1 & 1 & -1 \\ 0 & -1 & 2 \\ 3 & 2 & -1 \\ -1 & 0 & 1 \end{bmatrix}$$

and

$$\mathbf{b} = \begin{bmatrix} 2 \\ 6 \\ 11 \\ 0 \end{bmatrix}.$$

Then the given system is equivalent to $A\mathbf{x} = \mathbf{b}$. We calculate

$$A^T A = \begin{bmatrix} 1 & 0 & 3 & -1 \\ 1 & -1 & 2 & 0 \\ -1 & 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & -1 \\ 0 & -1 & 2 \\ 3 & 2 & -1 \\ -1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 11 & 7 & -5 \\ 7 & 6 & -5 \\ -5 & -5 & 7 \end{bmatrix}$$

and

$$A^T \mathbf{b} = \begin{bmatrix} 1 & 0 & 3 & -1 \\ 1 & -1 & 2 & 0 \\ -1 & 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 6 \\ 11 \\ 0 \end{bmatrix} = \begin{bmatrix} 35 \\ 18 \\ -1 \end{bmatrix}.$$

Using Maple, we see that the matrix $A^T A$ is invertible. So, the least squares solution is

$$
\begin{aligned}
\overline{\mathbf{x}} &= (A^T A)^{-1} A^T \mathbf{b} \\
&= \begin{bmatrix} 17/44 & -6/11 & -5/44 \\ -6/11 & 13/11 & 5/11 \\ -5/44 & 5/11 & 17/44 \end{bmatrix} \begin{bmatrix} 35 \\ 18 \\ -1 \end{bmatrix} \\
&= \begin{bmatrix} 42/11 \\ 19/11 \\ 42/11 \end{bmatrix}
\end{aligned}
$$

(26) Let

$$
A = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 1 & 1 \\ -1 & 1 & -1 \\ 0 & 2 & 1 \end{bmatrix}
$$

and

$$
\mathbf{b} = \begin{bmatrix} 21 \\ 7 \\ 14 \\ 0 \end{bmatrix}.
$$

Then the given system is equivalent to $A\mathbf{x} = \mathbf{b}$. We calculate

$$
A^T A = \begin{bmatrix} 2 & 1 & -1 & 0 \\ 3 & 1 & 1 & 2 \\ 1 & 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 & 1 \\ 1 & 1 & 1 \\ -1 & 1 & -1 \\ 0 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 6 & 4 \\ 6 & 15 & 5 \\ 4 & 5 & 4 \end{bmatrix}
$$

and

$$
A^T \mathbf{b} = \begin{bmatrix} 2 & 1 & -1 & 0 \\ 3 & 1 & 1 & 2 \\ 1 & 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 21 \\ 7 \\ 14 \\ 0 \end{bmatrix} = \begin{bmatrix} 35 \\ 84 \\ 14 \end{bmatrix}.
$$

Using Maple, we see that the matrix $A^T A$ is invertible. So, the least squares solution is

$$
\begin{aligned}
\overline{\mathbf{x}} &= (A^TA)^{-1}A^T\mathbf{b} \\
&= \begin{bmatrix} 35/66 & -2/33 & -5/11 \\ -2/33 & 4/33 & -1/11 \\ -5/11 & -1/11 & 9/11 \end{bmatrix} \begin{bmatrix} 35 \\ 84 \\ 14 \end{bmatrix} \\
&= \begin{bmatrix} 469/66 \\ 224/33 \\ -133/11 \end{bmatrix}
\end{aligned}
$$

(29) We have the data points $(20, 14.5), (40, 31), (48, 36), (60, 45.5), (80, 59), (100, 73.5)$. We need to find the least squares solution to the system

$$
\begin{aligned}
a + 20b &= 14.5 \\
a + 40b &= 31 \\
a + 48b &= 36 \\
a + 60 &= 45.5 \\
a + 80b &= 59 \\
a + 100b &= 73.5
\end{aligned}
$$

Let

$$
A = \begin{bmatrix} 1 & 20 \\ 1 & 40 \\ 1 & 48 \\ 1 & 60 \\ 1 & 80 \\ 1 & 100 \end{bmatrix}
$$

and

$$
\mathbf{b} = \begin{bmatrix} 14.5 \\ 31 \\ 36 \\ 45.5 \\ 59 \\ 73.5 \end{bmatrix}.
$$

We calculate

$$A^TA = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 20 & 40 & 48 & 60 & 80 & 100 \end{bmatrix} \begin{bmatrix} 1 & 20 \\ 1 & 40 \\ 1 & 48 \\ 1 & 60 \\ 1 & 80 \\ 1 & 100 \end{bmatrix} = \begin{bmatrix} 6 & 348 \\ 342 & 24304 \end{bmatrix}.$$

Since $\det(A^TA) \neq 0$, the matrix A^TA is invertible and the vector we seek equals $(A^TA)^{-1}A^T\mathbf{b}$. We have

$$\begin{aligned} \overline{\mathbf{x}} &= (A^TA)^{-1}A^T\mathbf{b} \\ &= \begin{bmatrix} 1519/1545 & -29/2060 \\ -29/2060 & 1/4120 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 20 & 40 & 48 & 60 & 80 & 100 \end{bmatrix} \begin{bmatrix} 14.5 \\ 31 \\ 36 \\ 45.5 \\ 59 \\ 73.5 \end{bmatrix} \\ &= \begin{bmatrix} 0.9184466019417 \\ 0.729854368932 \end{bmatrix} \end{aligned}$$

So, the least squares approximating line is

$$b = 0.92 + 0.73h.$$

(32) (a) We need to solve the system

$$\begin{aligned} s_0 + 0.5v_0 + 0.125g &= 11 \\ s_0 + v_0 + 0.5g &= 17 \\ s_0 + 1.5v_0 + 1.125g &= 21 \\ s_0 + 2v_0 + 2g &= 23 \\ s_0 + 3v_0 + 4.5g &= 18 \end{aligned}$$

Let

$$A = \begin{bmatrix} 1 & 0.5 & 0.125 \\ 1 & 1 & 0.5 \\ 1 & 1.5 & 1.125 \\ 1 & 2 & 2 \\ 1 & 3 & 4.5 \end{bmatrix}$$

and

$$\mathbf{b} = \begin{bmatrix} 11 \\ 17 \\ 21 \\ 23 \\ 18 \end{bmatrix}.$$

We calculate

$$A^T A = \begin{bmatrix} 5 & 8 & 8.25 \\ 8 & 16.5 & 19.75 \\ 8.25 & 19.75 & 25.78125 \end{bmatrix}.$$

Since $\det(A^T A) \neq 0$, the matrix $A^T A$ is invertible and the vector we seek equals $(A^T A)^{-1} A^T \mathbf{b}$. We have (using Maple)

$$\begin{aligned} \overline{\mathbf{x}} &= (A^T A)^{-1} A^T \mathbf{b} \\ &= \begin{bmatrix} 1.92 \\ 20.31 \\ -9.94 \end{bmatrix} \end{aligned}$$

So, the least squares approximating quadratic is

$$s(t) = 1.92 + 20.31t - \frac{9.94}{2} t^2.$$

(b) $s_0 \approx 1.92$ m, $v_0 \approx 20.31$ m/s and $g \approx -9.94$ m/s^2

(c) The object will hit the ground when $s = 0$. We use Maple to factor $s(t)$ as

$$s(t) = -4.97(t + 0.09244349461)(t - 4.178962609).$$

Since t cannot be negative, we conclude that the object hits the ground at approximately $t = 4.12$ s.

Section 6.1:

(1) The set $V = \left\{ \begin{bmatrix} x \\ x \end{bmatrix} : x \in \mathbb{R} \right\}$ is a vector space. Axioms 2, 3, 7, 8, 9, and 10 all hold on the larger vector space $\mathbb{R}^2$, and so V inherits these properties. We verify axioms 1, 4, 5 and 6:

Note that

$$\begin{bmatrix} a \\ a \end{bmatrix} + \begin{bmatrix} b \\ b \end{bmatrix} = \begin{bmatrix} a+b \\ a+b \end{bmatrix} \in V$$

showing axiom (1) holds.

We have

$$\begin{bmatrix} a \\ a \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ a \end{bmatrix}$$

and

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} \in V$$

which verifies axiom (4).

We have

$$\begin{bmatrix} a \\ a \end{bmatrix} + \begin{bmatrix} -a \\ -a \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

and

$$\begin{bmatrix} -a \\ -a \end{bmatrix} \in V$$

which verifies axiom (5).

Note that for any scalar c

$$c \begin{bmatrix} a \\ a \end{bmatrix} = \begin{bmatrix} ca \\ ca \end{bmatrix} \in V$$

showing axiom (6) holds.

(2) $V = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 : x \geq 0, y \geq 0 \right\}$ with the usual vector addition and scalar multiplication is not a vector space. Axioms 5 and 6 do not hold. For example,

$$\mathbf{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

is in V, but the only vector for which $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$ is

$$\begin{bmatrix} -1 \\ -1 \end{bmatrix}$$

which is not in V.

Also, $-3\mathbf{u}$ is not in V.

(3) $V = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 : xy \geq 0 \right\}$ with the usual vector addition and scalar multiplication is not a vector space. Axiom 1 does not hold. For example,

$$\mathbf{u} = \begin{bmatrix} -1 \\ -1 \end{bmatrix} \in V$$

and

$$\mathbf{v} = \begin{bmatrix} 0 \\ 8 \end{bmatrix} \in V$$

but

$$\mathbf{u} + \mathbf{v} = \begin{bmatrix} -1 \\ 7 \end{bmatrix} \notin V.$$

(5) $\mathbb{R}^2$ with the usual addition but the given scalar multiplication is not a vector space. Axiom 8 does not hold. For example,

$$(2+3)\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \end{bmatrix} \neq 2\begin{bmatrix} 1 \\ 1 \end{bmatrix} + 3\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}.$$

(9) The set $V = \left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} : a, b, c \in \mathbb{R} \right\}$ is a vector space. Axioms 2, 3, 7, 8, 9, and 10 all hold on the larger vector space $\mathbb{R}^2$, and so V inherits these properties. We verify axioms 1, 4, 5 and 6:

Note that

$$\begin{bmatrix} a_1 & b_1 \\ 0 & c_1 \end{bmatrix} + \begin{bmatrix} a_2 & b_2 \\ 0 & c_2 \end{bmatrix} = \begin{bmatrix} a_1 + a_2 & b_1 + b_2 \\ 0 & c_1 + c_2 \end{bmatrix} \in V$$

showing axiom (1) holds.

We have

$$\begin{bmatrix} a & b \\ 0 & c \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$$

and

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in V$$

which verifies axiom (4).

We have

$$\begin{bmatrix} a & b \\ 0 & c \end{bmatrix} + \begin{bmatrix} -a & -b \\ 0 & -c \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

and

$$\begin{bmatrix} -a & -b \\ 0 & -c \end{bmatrix} \in V$$

which verifies axiom (5).

Note that for any scalar s

$$s \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} = \begin{bmatrix} sa & sb \\ 0 & sc \end{bmatrix} \in V$$

showing axiom (6) holds.

(25) W is a subspace of V. To see this, let $\mathbf{x}$ and $\mathbf{y}$ be two vectors in W. Then $\mathbf{x}$ and $\mathbf{y}$ are of the form

$$\mathbf{x} = \begin{bmatrix} a \\ -a \\ 2a \end{bmatrix}$$

and

$$\mathbf{y} = \begin{bmatrix} b \\ -b \\ 2b \end{bmatrix}.$$

So,

$$\mathbf{x} + \mathbf{y} = \begin{bmatrix} a+b \\ -(a+b) \\ 2(a+b) \end{bmatrix} \in W.$$

Also, let $c \in \mathbb{R}$ and $\mathbf{x} \in W$ be as above. Then

$$c\mathbf{x} = \begin{bmatrix} ca \\ -ca \\ 2ca \end{bmatrix} \in W.$$

So, by Theorem 6.2, W is a subspace of $\mathbb{R}^3$.

(26) W is not a subspace of $\mathbb{R}^3$. For example, note that

$$\mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} \in W$$

and

$$\mathbf{y} = \begin{bmatrix} 2 \\ 2 \\ 5 \end{bmatrix} \in W$$

but

$$\mathbf{x} + \mathbf{y} = \begin{bmatrix} 3 \\ 3 \\ 8 \end{bmatrix} \notin W$$

since $8 \neq 3 + 3 + 1$.

(35) W is a subspace of $\mathcal{P}_2$. To see this, let f and g be in W. Then

$$\begin{aligned} f &= a + bx + cx^2 \\ g &= o + px + qx^2 \end{aligned}$$

where a, b, c, o, p, q are scalars such that $a + b + c = 0$ and $o + p + q = 0$. So

$$f + g = (a + o) + (b + p)x + (c + q)x^2$$

and

$$(a + o) + (b + p) + (c + q) = (a + b + c) + (o + p + q) = 0 + 0 = 0.$$

This shows that $f + g \in W$.

Also, let m be a scalar and $f \in W$ as above. Then

$$mf = ma + mbx + mcx^2$$

where

$$ma + mb + mc = m(a + b + c) = m(0) = 0.$$

We conclude that $mf \in W$.

(36) W is not a subspace of $\mathcal{P}_2$. To see this, note that

$$f = 0 + x + x^2$$

and

$$g = 1 + 0x + x^2$$

are both in W. But,

$$f + g = 1 + x + 2x^2 \notin W$$

since $1(1)(2) = 2 \neq 0$.

(61) Let $a + bx + cx^2 \in \mathcal{P}_2$. We need to determine if there exist scalars c_1, c_2, c_3 such that

$$a + bx + cx^2 = c_1(1 + x) + c_2(x + x^2) + c_3(1 + x^2).$$

This is true iff

$$a + bx + cx^2 = (c_1 + c_3) + (c_1 + c_2)x + (c_2 + c_3)x^2.$$

Comparing coefficients, we see that we must have

$$\begin{aligned} c_1 + c_3 &= a \\ c_1 + c_2 &= b \\ c_2 + c_3 &= c \end{aligned}$$

We form the associated augmented matrix and row-reduce:

$$\begin{bmatrix} 1 & 0 & 1 & | & a \\ 1 & 1 & 0 & | & b \\ 0 & 1 & 1 & | & c \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 1 & | & a \\ 0 & 1 & -1 & | & b - a \\ 0 & 0 & 2 & | & c - b + a \end{bmatrix}.$$

We see that the system is always consistent. Thus, yes the given polynomials span $\mathcal{P}_2$.

(62) Let $a + bx + cx^2 \in \mathcal{P}_2$. We need to determine if there exist scalars c_1, c_2, c_3 such that

$$a + bx + cx^2 = c_1(1 + x + 2x^2) + c_2(2 + x + 2x^2) + c_3(-1 + x + 2x^2).$$

This is true iff

$$a + bx + cx^2 = (c_1 + 2c_2 - c_3) + (c_1 + c_2 + c_3)x + (2c_1 + 2c_2 + 2c_3)x^2.$$

Comparing coefficients, we see that we must have

$$\begin{aligned} c_1 + 2c_2 - c_3 &= a \\ c_1 + c_2 + c_3 &= b \\ 2c_1 + 2c_2 + 2c_3 &= c \end{aligned}$$

We form the associated augmented matrix and start to row-reduce:

$$\begin{bmatrix} 1 & 2 & -1 & | & a \\ 1 & 1 & 1 & | & b \\ 2 & 2 & 2 & | & c \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & -1 & | & a \\ 1 & 1 & 1 & | & b \\ 0 & 0 & 0 & | & c - 2b \end{bmatrix}.$$

We see that the system is inconsistent. Thus, no the given polynomials do not span $\mathcal{P}_2$.