Math 314/814: Matrix Theory **Dr. S. Cooper, Fall 2008**

Homework Solutions – Week of November 18

Section 6.3:

(17) The sets $\mathcal{B} = \{1, x-1, (x-1)^2\}$ and $\mathcal{C} = \{1, x, x^2\}$ are both bases for $\mathcal{P}_2$. We need to find $[p(x)]_{\mathcal{B}}$. Observe that

$$\begin{aligned} 1 &= 1(1) + 0(x-1) + 0(x-1)^2 \\ x &= 1(1) + 1(x-1) + 0(x-1)^2 \\ x^2 &= 1(1) + 2(x-1) + 1(x-1)^2 \end{aligned}$$

Thus,

$$P_{\mathcal{B}\leftarrow\mathcal{C}} = \left[\begin{array}{ccc} [1]_{\mathcal{B}} & [x]_{\mathcal{B}} & [x^2]_{\mathcal{B}} \end{array}\right] = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}.$$

By Theorem 6.12,

$$[p(x)]_{\mathcal{B}} = P_{\mathcal{B}\leftarrow\mathcal{C}}[p(x)]_{\mathcal{C}} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ -5 \end{bmatrix} = \begin{bmatrix} -2 \\ -8 \\ -5 \end{bmatrix}.$$

We conclude that the Taylor polynomial of $p(x)$ about $a = 1$ is

$$p(x) = -2 - 8(x-1) - 5(x-1)^2.$$

(18) The sets $\mathcal{B} = \{1, x+2, (x+2)^2\}$ and $\mathcal{C} = \{1, x, x^2\}$ are both bases for $\mathcal{P}_2$. We need to find $[p(x)]_{\mathcal{B}}$. Observe that

$$\begin{aligned} 1 &= 1(1) + 0(x+2) + 0(x+2)^2 \\ x &= -2(1) + 1(x+2) + 0(x+2)^2 \\ x^2 &= 4(1) - 4(x+2) + (x+2)^2 \end{aligned}$$

Thus,

$$P_{\mathcal{B}\leftarrow\mathcal{C}} = \left[\begin{array}{ccc} [1]_{\mathcal{B}} & [x]_{\mathcal{B}} & [x^2]_{\mathcal{B}} \end{array}\right] = \begin{bmatrix} 1 & -2 & 4 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{bmatrix}.$$

By Theorem 6.12,

$$[p(x)]_{\mathcal{B}} = P_{\mathcal{B}\leftarrow\mathcal{C}}[p(x)]_{\mathcal{C}} = \begin{bmatrix} 1 & -2 & 4 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ -5 \end{bmatrix} = \begin{bmatrix} -23 \\ 22 \\ -5 \end{bmatrix}.$$

We conclude that the Taylor polynomial of $p(x)$ about $a = 1$ is

$$p(x) = -23 + 22(x+2) - 5(x+2)^2.$$

Section 6.4:

(2) T is not a linear transformation. For example,

$$\begin{aligned} 2T\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} &= 2\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 2 & 2 \\ 0 & 2 \end{bmatrix} \\ &\neq T\left(2\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\right) \\ &= T\begin{bmatrix} 2 & 2 \\ 2 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}. \end{aligned}$$

(3) T is a linear transformation. Let $A, C \in M_{nn}$ and α be a scalar. Then

$$T(A+C) = (A+C)B = AB + CB = T(A) + T(C)$$

and

$$T(\alpha A) = (\alpha A)B = \alpha(AB) = \alpha T(A).$$

(5) The transformation $T : M_{nn} \to \mathbb{R}$ defined by

$$T(A) = T\left(\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}\right) = tr(A) = a_{11} + a_{22} + \cdots + a_{nn}$$

is linear.

Suppose that $A = [a_{ij}]$ and $B = [b_{ij}]$ are both $n \times n$ matrices and α is a scalar. Then

$$\begin{aligned} T(A+B) &= T([a_{ij} + b_{ij}]) \\ &= (a_{11} + b_{11}) + (a_{22} + b_{22}) + \cdots + (a_{nn} + b_{nn}) \\ &= (a_{11} + a_{22} + \cdots + a_{nn}) + (b_{11} + b_{22} + \cdots + b_{nn}) \\ &= tr(A) + tr(B) \\ &= T(A) + T(B) \end{aligned}$$

and

$$\begin{aligned} T(\alpha A) &= tr(\alpha A) \\ &= \alpha a_{11} + \alpha a_{22} + \cdots + \alpha a_{nn} \\ &= \alpha(a_{11} + a_{22} + \cdots + a_{nn}) \\ &= \alpha tr(A) \\ &= \alpha T(A). \end{aligned}$$

(7) T is not linear. For example, let

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and

$$B = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Then

$$A + B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

We see that

$$T(A) + T(B) = rank(A) + rank(B) = 2 + 2 = 4 \neq T(A+B) = rank(A+B) = 0.$$

(8) T is not linear. For example,

$$\begin{aligned} 2T(1+x+x^2) &= 2(2+2x+2x^2) \\ &= 4+4x+4x^2 \\ &\neq T(2(1+x+x^2)) \\ &= T(2+2x+2x^2) \\ &= 3+3x+3x^2. \end{aligned}$$

(16) Using the fact that T is linear, we have

$$\begin{aligned} T(6+x-4x^2) &= 6T(1)+1T(x)-4T(x^2) \\ &= 6(3-2x)+(4x-x^2)-4(2+2x^2) \\ &= 10-8x-9x^2 \end{aligned}$$

and

$$\begin{aligned} T(a+bx+cx^2) &= aT(1)+bT(x)+cT(x^2) \\ &= a(3-2x)+b(4x-x^2)+c(2+2x^2) \\ &= (3a+2c)+(-2a+4b)x+(-b+2c)x^2 \end{aligned}$$

(17) First note that

$$4-x+3x^2 = 0(1+x)-1(x+x^2)+4(1+x^2).$$

So, using the fact that T is linear, we have

$$\begin{aligned} T(4-x+3x^2) &= T(0(1+x)-1(x+x^2)+4(1+x^2)) \\ &= 0T(1+x)-1T(x+x^2)+4T(1+x^2) \\ &= 0(1+x^2)-(x-x^2)+4(1+x+x^2) \\ &= 4+3x+5x^2 \end{aligned}$$

To find $T(a+bx+c^2)$ in general, we need to write $a+bx+cx^2$ as a linear combination of $1+x$, $x+x^2$ and $1+x^2$:

$$\begin{aligned} a+bx+cx^2 &= c_1(1+x)+c_2(x+x^2)+c_3(1+x^2) \\ &= (c_1+c_3)+(c_1+c_2)x+(c_2+c_3)x^2 \end{aligned}$$

Comparing coefficients this gives the linear system

$$\begin{aligned} c_1 + c_3 &= a \\ c_1 + c_2 &= b \\ c_2 + c_3 &= c \end{aligned}$$

We form the augmented matrix and row-reduce:

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & a \\ 1 & 1 & 0 & b \\ 0 & 1 & 1 & c \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & a \\ 0 & 1 & -1 & b-a \\ 0 & 0 & 2 & c-b+a \end{array}\right].$$

Solving the system, we obtain

$$\begin{aligned} c_1 &= \frac{a+b-c}{2} \\ c_2 &= \frac{-a+b+c}{2} \\ c_3 &= \frac{a-b+c}{2} \end{aligned}$$

Thus, using the linearity of T, we have

$$\begin{aligned} & T(a+bx+cx^2) \\ = \;& T\left(\left(\frac{a+b-c}{2}\right)(1+x) + \left(\frac{-a+b+c}{2}\right)(x+x^2) + \left(\frac{a-b+c}{2}\right)(1+x^2)\right) \\ = \;& \left(\frac{a+b-c}{2}\right)T(1+x) + \left(\frac{-a+b+c}{2}\right)T(x+x^2) + \left(\frac{a-b+c}{2}\right)T(1+x^2) \\ = \;& \left(\frac{a+b-c}{2}\right)(1+x^2) + \left(\frac{-a+b+c}{2}\right)(x-x^2) + \left(\frac{a-b+c}{2}\right)(1+x+x^2) \\ = \;& a + cx + \left(\frac{3a-b-c}{2}\right)x^2 \end{aligned}$$

(19) Let $\mathcal{B} = \{E_{11}, E_{12}, E_{21}, E_{22}\}$ be the standard basis for M_{22}. Then

$$\begin{bmatrix} w & x \\ y & z \end{bmatrix} = wE_{11} + xE_{12} + yE_{21} + zE_{22}.$$

Since $T : M_{22} \to \mathbb{R}$, we know that the images of the standard basis vectors are simply real numbers. Let $T(E_{11}) = a, T(E_{12}) = b, T(E_{21}) = c$ and $T(E_{22}) = d$ where a, b, c and d are real numbers. Then, by the linearity of T, we have

$$\begin{aligned} T\begin{bmatrix} w & x \\ y & z \end{bmatrix} &= T(wE_{11} + xE_{12} + yE_{21} + zE_{22}) \\ &= wT(E_{11}) + xT(E_{12}) + yT(E_{21}) + zT(E_{22}) \\ &= aw + bx + cy + dx \end{aligned}$$

(20) Observe that

$$\begin{bmatrix} 0 \\ 6 \\ -8 \end{bmatrix} = 6\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} - 4\begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix}.$$

Since

$$\begin{aligned}
T\begin{bmatrix} 0 \\ 6 \\ -8 \end{bmatrix} &= -2+2x^2 \\
&\neq 6T\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} - 4T\begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix} \\
&= 6(1+x)-4(2-x+x^2) \\
&= -2+10x-4x^2
\end{aligned}$$

the transformation T with the given properties cannot be linear.

(25) We have

$$\begin{aligned}
(S\circ T)\begin{bmatrix} 2 \\ 1 \end{bmatrix} &= S\left(T\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right) \\
&= S\begin{bmatrix} 5 \\ -1 \end{bmatrix} \\
&= \begin{bmatrix} 4 & -1 \\ 0 & 6 \end{bmatrix}
\end{aligned}$$

and

$$\begin{aligned}
(S\circ T)\begin{bmatrix} x \\ y \end{bmatrix} &= S\left(T\begin{bmatrix} x \\ y \end{bmatrix}\right) \\
&= S\begin{bmatrix} 2x+y \\ -y \end{bmatrix} \\
&= \begin{bmatrix} 2x & -y \\ 0 & 2x+2y \end{bmatrix}
\end{aligned}$$

Finally, $(T\circ S)\begin{bmatrix} x \\ y \end{bmatrix}$ is not defined since $S\begin{bmatrix} x \\ y \end{bmatrix}$ is a 2×2 matrix, but the domain of T is $\mathbb{R}^2$.

(29) We have

$$\begin{aligned}
(S\circ T)\begin{bmatrix} x \\ y \end{bmatrix} &= S\left(T\begin{bmatrix} x \\ y \end{bmatrix}\right) \\
&= S\begin{bmatrix} x-y \\ -3x+4y \end{bmatrix} \\
&= \begin{bmatrix} 4(x-y)+(-3x+4y) \\ 3(x-y)+(-3x+4y) \end{bmatrix} \\
&= \begin{bmatrix} x \\ y \end{bmatrix}
\end{aligned}$$

and

$$
\begin{aligned}
(T \circ S)\begin{bmatrix} x \\ y \end{bmatrix} &= T\left(S\begin{bmatrix} x \\ y \end{bmatrix}\right) \\
&= T\begin{bmatrix} 4x+y \\ 3x+y \end{bmatrix} \\
&= \begin{bmatrix} (4x+y)-(3x+y) \\ -3(4x+y)+4(3x+y) \end{bmatrix} \\
&= \begin{bmatrix} x \\ y \end{bmatrix}
\end{aligned}
$$

By definition, since $S \circ T = I_{\mathbb{R}^2}$ and $T \circ S = I_{\mathbb{R}^2}$, S and T are inverses.

Section 6.5:

(1) (a) (i) Since

$$T\begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \neq \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

the given matrix is not in $\ker(T)$.

(ii) Since

$$T\begin{bmatrix} 0 & 4 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

the given matrix is in $\ker(T)$.

(iii) Since

$$T\begin{bmatrix} 3 & 0 \\ 0 & -3 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & -3 \end{bmatrix} \neq \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

the given matrix is not in $\ker(T)$.

(b) (i) Any matrix in range(T) must have zeros for the (1,2) and (2,1) entries. Thus, the given matrix is not in range(T).

(ii) For the same reason as in (i), the given matrix is not in range(T).

(iii) Since the (1,2) and (2,1) of the given matrix are 0, this matrix is in range(T). In fact,

$$T\begin{bmatrix} 3 & 1 \\ 1 & -3 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & -3 \end{bmatrix}.$$

(c) We have

$$
\begin{aligned}
ker(T) &= \left\{\begin{bmatrix} a & b \\ c & d \end{bmatrix} : T\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\right\} \\
&= \left\{\begin{bmatrix} a & b \\ c & d \end{bmatrix} : \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\right\} \\
&= \left\{\begin{bmatrix} a & b \\ c & d \end{bmatrix} : a = d = 0\right\} \\
&= \left\{\begin{bmatrix} 0 & b \\ c & 0 \end{bmatrix}\right\}
\end{aligned}
$$

and

$$\begin{aligned} range(T) &= \left\{ T \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right\} \\ &= \left\{ \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} \right\} \end{aligned}$$

(3) (a) (i) Since

$$T(1+x) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

the given polynomial is not in $\ker(T)$.

(ii) Since

$$T(x-x^2) = \begin{bmatrix} -1 \\ 0 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

the given polynomial is not in $\ker(T)$.

(iii) Since

$$T(1+x-x^2) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

the given polynomial is in $\ker(T)$.

(b) (i) Since

$$T(1+x-x^2) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

the given vector is in range(T).

(ii) Since

$$T(2+x-x^2) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

the given vector is in range(T).

(iii) Since

$$T(1+x) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

the given vector is in range(T).

(c) We have

$$\begin{aligned} ker(T) &= \left\{ a+bx+cx^2 : T(a+bx+cx^2) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\} \\ &= \left\{ a+bx+cx^2 : \begin{bmatrix} a-b \\ b+c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\} \\ &= \left\{ a+bx+cx^2 : a=b, b=-c \right\} \\ &= \left\{ a+ax-ax^2 \right\} \end{aligned}$$

and

$$\begin{aligned} range(T) &= \left\{ T(a+bx+cx^2) : a+bx+cx^2 \in \mathcal{P}_2 \right\} \\ &= \left\{ \begin{bmatrix} a-b \\ b+c \end{bmatrix} \right\} \\ &= \mathbb{R}^2 \end{aligned}$$

(5) From Exercise 1 we have that

$$\begin{aligned} ker(T) &= \left\{ \begin{bmatrix} 0 & b \\ c & 0 \end{bmatrix} \right\} \\ &= \left\{ c \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right\} \\ &= span \left(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right) \end{aligned}$$

Suppose we have scalars c_1 and c_2 such that

$$c_1 \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

It is easy to see that $c_1 = c_2 = 0$. Thus the spanning set is also linearly independent. We conclude that

$$\mathcal{B} = \left\{ \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right\}$$

is a basis for $\ker(T)$ and so $\text{nullity}(T) = 2$.

Similarly,

$$\begin{aligned} range(T) &= \left\{ \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} \right\} \\ &= \left\{ a \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + d \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\} \\ &= span \left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right) \end{aligned}$$

Suppose we have scalars c_1 and c_2 such that

$$c_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

It is easy to see that $c_1 = c_2 = 0$. Thus the spanning set is also linearly independent. We conclude that

$$\mathcal{B}' = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

is a basis for $\text{range}(T)$ and so $\text{rank}(T) = 2$.

By the Rank Theorem,

$$\dim(M_{22}) = 4 = 2 + 2 = nullity(T) + rank(T).$$

(7) From Exercise 3, we have that

$$ker(T) = \{a + ax - ax^2\} = \{a(1 + x - x^2)\} = span(1 + x - x^2).$$

Thus, since there is only only spanning vector in this case, we see that

$$\mathcal{B} = \{1 + x - x^2\}$$

is a basis for $\ker(T)$ and so $\text{nullity}(T) = 1$.

Since $\text{range}(T) = \mathbb{R}^2$, we can take the standard basis

$$\mathcal{B}' = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$$

as a basis for $\text{range}(T)$. We conclude that $\text{rank}(T) = 2$.

By the Rank Theorem,

$$\dim(\mathcal{P}_2) = 3 = 1 + 2 = nullity(T) + rank(T).$$

(11) We have

$$\begin{aligned}
ker(T) &= \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right\} \\
&= \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : \begin{bmatrix} a-b & -a+b \\ c-d & -c+d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right\} \\
&= \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a = b, c = d \right\} \\
&= \left\{ \begin{bmatrix} a & a \\ c & c \end{bmatrix} \right\} \\
&= \left\{ a \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \right\} \\
&= span \left(\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \right)
\end{aligned}$$

Furthermore, if c_1 and c_2 are scalars such that

$$c_1 \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

then clearly $c_1 = c_2 = 0$ which shows that the spanning set is also linearly independent. Thus

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \right\}$$

is a basis for $\ker(T)$. Thus, $\text{nullity}(T) = 2$.

By the Rank Theorem

$$\dim(M_{22}) = 4 = nullity(T) + rank(T) = 2 + rank(T) \implies rank(T) = 4 - 2 = 2.$$

(17) (a) We apply Theorem 6.20 to conclude that the transformation T is not 1-1. To see that $\ker(T) \neq \{\mathbf{0}\}$ observe that $1 + 2x + x^2 \neq 0 + 0x + 0x^2$ and

$$T(1 + 2x + x^2) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

(b) Since $\dim(\mathcal{P}_2) = 3 = \dim(\mathbb{R}^3)$ and T is not 1-1, we apply Theorem 6.21 to conclude that T is also not onto.

(21) The vector space

$$V = D_3 = \left\{ \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix} \right\}$$

has basis

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}$$

(you should check that these matrices span D_3 and are linearly independent). Thus,

$$\dim(D_3) = 3 = \dim(\mathbb{R}^3).$$

By Theorem 6.25, the vector spaces D_3 and $\mathbb{R}^3$ are isomorphic.

Define the linear transformation $T : D_3 \to \mathbb{R}^3$ by

$$T \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}.$$

Note that

$$T \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \implies a = b = c = 0$$

and so

$$ker(T) = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right\}$$

which shows that T is 1-1.

Also, if $\mathbf{x} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ is any vector in $\mathbb{R}^3$, then

$$T \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix} = \mathbf{x}$$

which shows that range$(T) = \mathbb{R}^3$. Thus, T is onto.

We conclude that T is an isomorphism.

(23) Note that

$$V = \{A \in M_{33} : A^T = A\}$$

and

$$W = \{B \in M_{22} : B^T = -B\}.$$

We observe that

$$
\begin{aligned}
V &= \left\{ \begin{bmatrix} a & b & c \\ b & e & f \\ c & f & s \end{bmatrix} \right\} \\
&= span \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right)
\end{aligned}
$$

It is also easy to see that these spanning matrices are linearly independent, and hence form a basis for V.

Similarly,

$$
\begin{aligned}
W &= \left\{ \begin{bmatrix} 0 & b & c \\ -b & 0 & f \\ -c & -f & 0 \end{bmatrix} \right\} \\
&= span \left(\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \right)
\end{aligned}
$$

It is also easy to see that these spanning matrices are linearly independent, and hence form a basis for W.

We conclude that $\dim(V) = 6$ and $\dim(W) = 3$. So, by Theorem 6.25, V and W are not isomorphic.