Math 314/814: Matrix Theory **Dr. S. Cooper, Fall 2008**

Homework Solutions – Week of November 25

Section 6.5:

(29) We first show that T is 1-1. By Theorem 6.20, it suffices to show that $ker(T) = \{\mathbf{0}\}$.

Let $p(x) = a_0 + a_1x + a_2x^2 + \cdots + a_na^n$ such that $p(x)$ is in the kernel of T. Then

$$\begin{aligned} 0 + 0x + 0x^2 + \cdots + 0x^n &= T(p(x)) \\ &= x^n p\left(\frac{1}{x}\right) \\ &= x^n \left[a_0 + a_1\left(\frac{1}{x}\right) + a_2\left(\frac{1}{x}\right)^2 + \cdots + a_n\left(\frac{1}{x}\right)^n\right] \\ &= a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_n \end{aligned}$$

Comparing coefficients, we see that $a_0 = a_1 = \cdots = a_n = 0$. That is, $p(x)$ is the zero polynomial. We conclude that $ker(T) = \{\mathbf{0}\}$, and so T is 1-1. By Theorem 6.21, T must also be onto. Therefore, T is an isomorphism.

(31) First recall that $\mathcal{C}[a,b]$ denotes the set of all continous functions from $[a,b]$ to $\mathbb{R}$.

Define $T : \mathcal{C}[0,1] \to \mathcal{C}[0,2]$ by $T(f) = g$ where g is the function such that $g(x) = f\left(\frac{x}{2}\right)$.

Let $f \in \mathcal{C}[0,1]$ such that $T(f) = g$ is the zero function. Then for all x we have

$$g(x) = f\left(\frac{x}{2}\right) = 0 \implies f(x) = 0.$$

That is, f is the zero function. By Theorem 6.20, we have that T is 1-1.

Now let $g \in \mathcal{C}[0,2]$. Let $f \in \mathcal{C}[0,1]$ be the function defined by $f(x) = g(2x)$. Then $T(f) = g$ which shows that T is also onto.

Since T is an isomorphism, we conclude that $\mathcal{C}[0,1]$ and $\mathcal{C}[0,2]$ are isomorphic.

(33) (a) Let $\mathbf{x}$ be in $ker(S \circ T)$. Then $(S \circ T)(\mathbf{x}) = S(T(\mathbf{x})) = \mathbf{0}$. Since S is 1-1, we must have that $T(\mathbf{x}) = \mathbf{0}$. But, since T is 1-1, we then must have $\mathbf{x} = \mathbf{0}$. Therefore, $ker(S \circ T) = \{\mathbf{0}\}$ showing that $S \circ T$ is 1-1.

(b) Let $\mathbf{x} \in W$. Since S is onto, there exists $\mathbf{y} \in V$ such that $S(\mathbf{y}) = \mathbf{x}$. Also, since T is onto, there exists $\mathbf{m} \in U$ such that $T(\mathbf{m}) = \mathbf{y}$. So,

$$(S \circ T)(\mathbf{m}) = S(T(\mathbf{m})) = S(\mathbf{y}) = \mathbf{x}.$$

That is, $\mathbf{x}$ is in the range of $S \circ T$. Since $\mathbf{x}$ was arbitrarily chosen, $S \circ T$ is onto.

(35) (a) Suppose that T is onto. Then $range(T) = W$ and so $rank(T) = \dim(range(T)) = \dim(W)$. Thus, by the Rank Theorem

$$\dim(V) = Rank(T) + Nullity(T) = \dim(W) + Nullity(T).$$

So, since $\dim(V) < \dim(W)$,

$$\dim(V) + Nullity(T) < \dim(W) + Nullity(T) = \dim(V)$$

implying that $Nullity(T) < 0$, a contradiction. We conclude that T cannot be onto.

(b) Suppose that T is 1-1. Then, $ker(T) = \{\mathbf{0}\}$ and so $Nullity(T) = \dim(ker(T)) = 0$. Thus, by the Rank Theorem,

$$\dim(V) = Rank(T) + Nullity(T) = Rank(T) = \dim(Range(T)).$$

But, since $Range(T)$ is a subspace of W, $\dim(Range(T)) \leq \dim(W)$. Putting this together with the above, we conclude that $\dim(V) \leq \dim(W)$, a contradition to our assumption. Therefore, T cannot be 1-1.

(37) First we show that $ker(T) = ker(T^2)$. To see this, let $\mathbf{x} \in ker(T)$. Then

$$T(\mathbf{x}) = \mathbf{0} \implies T(T(\mathbf{x})) = T(\mathbf{0}) = \mathbf{0}$$

which shows that $\mathbf{x} \in ker(T^2)$ (i.e., $ker(T)$ is a subset of $ker(T^2)$). Moreover, by the Rank Theorem, we have

$$\begin{aligned} \dim(V) &= Nullity(T) + Rank(T) \\ \dim(V) &= Nullity(T^2) + Rank(T^2) \end{aligned}$$

Since $rank(T) = rank(T^2)$ this implies that $nullity(T) = nullity(T^2)$. Thus, $ker(T) \subseteq ker(T^2)$ and $\dim(ker(T)) = \dim(ker(T^2))$, which shows that $ker(T) = ker(T^2)$.

Now let $\mathbf{v} \in range(T) \cap ker(T)$. Then $T(\mathbf{v}) = \mathbf{0}$ and there exists $\mathbf{y} \in V$ such that $T(\mathbf{y}) = \mathbf{v}$. Thus,

$$\mathbf{0} = T(\mathbf{v}) = T(T(\mathbf{y})) = T^2(\mathbf{y}).$$

This shows that $\mathbf{y} \in ker(T^2) = ker(T)$. Thus, $\mathbf{v} = T(\mathbf{y}) = \mathbf{0}$. Observe that since $T(\mathbf{0}) = \mathbf{0}$, $\mathbf{0} \in range(T) \cap ker(T)$.

We conclude that $range(T) \cap ker(T) = \{\mathbf{0}\}$.

Section 6.6:

(1) We calculate that

$$\begin{aligned} T(1) &= 0(1) - x \\ T(x) &= 1 + 0x \end{aligned}$$

So,

$$A = [T]_{\mathcal{C} \leftarrow \mathcal{B}} = \left[\begin{array}{cc} [T(1)]_{\mathcal{C}} & [T(x)]_{\mathcal{C}} \end{array} \right] = \left[\begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right].$$

Now $T(\mathbf{v}) = 2 - 4x$. We verify this using Theorem 6.26:

$$[T(\mathbf{v})]_{\mathcal{C}} = A[\mathbf{v}]_{\mathcal{B}} = \left[\begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right] \left[\begin{array}{c} 4 \\ 2 \end{array} \right] = \left[\begin{array}{c} 2 \\ -4 \end{array} \right],$$

i.e., $T(\mathbf{v}) = 2(1) - 4(x) = 2 - 4x$.

(2) We calculate that

$$\begin{aligned} T(1+x) &= 1 - x \\ T(1-x) &= -1 - x \end{aligned}$$

So,

$$A = [T]_{\mathcal{C}\leftarrow\mathcal{B}} = \left[\ [T(1+x)]_{\mathcal{C}} \quad [T(1-x)]_{\mathcal{C}}\ \right] = \begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix}.$$

Now $\mathbf{v} = 4 + 2x = 3(1+x) + 1(1-x)$ and $T(\mathbf{v}) = 2 - 4x$. We verify this using Theorem 6.26:

$$[T(\mathbf{v})]_{\mathcal{C}} = A[\mathbf{v}]_{\mathcal{B}} = \begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -4 \end{bmatrix},$$

i.e., $T(\mathbf{v}) = 2(1) - 4(x) = 2 - 4x$.

(5) We calculate

$$\begin{aligned} T(1) &= \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \mathbf{e_1} + \mathbf{e_2} \\ T(x) &= \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 0\mathbf{e_1} + \mathbf{e_2} \\ T(x^2) &= \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 0\mathbf{e_1} + \mathbf{e_2} \end{aligned}$$

So,

$$A = [T]_{\mathcal{C}\leftarrow\mathcal{B}} = \left[\ [T(1)]_{\mathcal{C}} \quad [T(x)]_{\mathcal{C}} \quad [T(x^2)]_{\mathcal{C}}\ \right] = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}.$$

Now $T(\mathbf{v}) = \begin{bmatrix} a \\ a+b+c \end{bmatrix}$. We verify this using Theorem 6.26:

$$[T(\mathbf{v})]_{\mathcal{C}} = A[\mathbf{v}]_{\mathcal{B}} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} a \\ a+b+c \end{bmatrix},$$

i.e., $T(\mathbf{v}) = a\mathbf{e_1} + (a+b+c)\mathbf{e_2} = \begin{bmatrix} a \\ a+b+c \end{bmatrix}$.

(6) We calculate

$$\begin{aligned} T(x^2) &= \begin{bmatrix} 0 \\ 1 \end{bmatrix} = -\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ T(x) &= \begin{bmatrix} 0 \\ 1 \end{bmatrix} = -\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ T(1) &= \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 0\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \end{aligned}$$

So,

$$A = [T]_{\mathcal{C}\leftarrow\mathcal{B}} = \left[\ [T(x^2)]_{\mathcal{C}} \quad [T(x)]_{\mathcal{C}} \quad [T(1)]_{\mathcal{C}}\ \right] = \begin{bmatrix} -1 & -1 & 0 \\ 1 & 1 & 1 \end{bmatrix}.$$

Now $T(\mathbf{v}) = \begin{bmatrix} a \\ a+b+c \end{bmatrix}$. We verify this using Theorem 6.26:

$$[T(\mathbf{v})]_{\mathcal{C}} = A[\mathbf{v}]_{\mathcal{B}} = \begin{bmatrix} -1 & -1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} c \\ b \\ a \end{bmatrix} = \begin{bmatrix} -c-b \\ a+b+c \end{bmatrix},$$

i.e., $T(\mathbf{v}) = (-c-b)\begin{bmatrix} 1 \\ 0 \end{bmatrix} + (a+b+c)\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} a \\ a+b+c \end{bmatrix}$.

(13) (a) Let $f(x) = a\sin x + b\cos x \in W$. Then $D(f(x)) = f'(x) = a\cos x - b\sin x \in W$. Thus, D maps W into itself.

(b) We calculate that

$$\begin{aligned} D(\sin x) &= 0(\sin x) + \cos x \\ D(\cos x) &= -\sin x + 0\cos x \end{aligned}$$

So,

$$A = [D]_{\mathcal{B}} = \begin{bmatrix} [D(\sin x)]_{\mathcal{B}} & [D(\cos x)]_{\mathcal{B}} \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

(c) We have $D(f(x)) = f'(x) = 3\cos x + 5\sin x$. We verify this using Theorem 6.26:

$$[D(f(x))]_{\mathcal{B}} = A[f(x)]_{\mathcal{B}} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 3 \\ -5 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \end{bmatrix},$$

i.e., $D(f(x)) = 5\sin x + 3\cos x$.