Math 314/814: Matrix Theory **Dr. S. Cooper, Fall 2008**

Homework Solutions – Week of December 2

Section 6.6:

(17) (a) By definition,

$$(S \circ T)(p(x)) = S(T(p(x))) = S \begin{bmatrix} p(0) \\ p(1) \end{bmatrix} = \begin{bmatrix} p(0) - 2p(1) \\ 2p(0) - p(1) \end{bmatrix}.$$

So,

$$\begin{aligned} (S \circ T)(1) &= \begin{bmatrix} -1 \\ 1 \end{bmatrix} = -\mathbf{e_1} + \mathbf{e_2} \\ (S \circ T)(x) &= \begin{bmatrix} -2 \\ -1 \end{bmatrix} = -2\mathbf{e_1} - \mathbf{e_2} \end{aligned}$$

Thus, by definition,

$$[S \circ T]_{\mathcal{D} \leftarrow \mathcal{B}} = \begin{bmatrix} [(S \circ T)(1)]_{\mathcal{D}} & [(S \circ T)(x)]_{\mathcal{D}} \end{bmatrix} = \begin{bmatrix} -1 & -2 \\ 1 & -1 \end{bmatrix}.$$

(b) Note that

$$\begin{aligned} T(1) &= \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \mathbf{e_1} + \mathbf{e_2} \\ T(x) &= \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 0\mathbf{e_1} + \mathbf{e_2} \end{aligned}$$

Thus,

$$[T]_{\mathcal{C} \leftarrow \mathcal{B}} = \begin{bmatrix} [T(1)]_{\mathcal{C}} & [T(x)]_{\mathcal{C}} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}.$$

In addition,

$$\begin{aligned} S(\mathbf{e_1}) &= \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \mathbf{e_1} + 2\mathbf{e_2} \\ S(\mathbf{e_2}) &= \begin{bmatrix} -2 \\ -1 \end{bmatrix} = -2\mathbf{e_1} - \mathbf{e_2} \end{aligned}$$

Thus,

$$[S]_{\mathcal{D} \leftarrow \mathcal{C}} = \begin{bmatrix} [S(\mathbf{e_1})]_{\mathcal{D}} & [S(\mathbf{e_2})]_{\mathcal{D}} \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 2 & -1 \end{bmatrix}.$$

Therefore, by Theorem 6.27,

$$[S \circ T]_{\mathcal{D} \leftarrow \mathcal{B}} = [S]_{\mathcal{D} \leftarrow \mathcal{C}}[T]_{\mathcal{C} \leftarrow \mathcal{B}} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} -1 & -2 \\ 1 & -1 \end{bmatrix}.$$

(18) (a) We have, by definition,

$$(S \circ T)(p(x)) = S(T(p(x))) = S(p(x+1)) = p(x+2).$$

Thus,

$$\begin{aligned} (S \circ T)(1) &= 1 = 1(1) + 0(x) + 0(x^2) \\ (S \circ T)(x) &= x + 2 = 2(1) + 1(x) + 0(x^2) \end{aligned}$$

Therefore,

$$[S \circ T]_{\mathcal{D} \leftarrow \mathcal{B}} = \left[\ [(S \circ T)(1)]_{\mathcal{D}} \quad [(S \circ T)(x)]_{\mathcal{D}} \ \right] = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

(b) Note that

$$\begin{aligned} T(1) &= 1 = 1(1) + 0(x) + 0(x^2) \\ T(x) &= x + 1 = 1(1) + 1(x) + 0(x^2) \end{aligned}$$

Thus,

$$[T]_{\mathcal{C} \leftarrow \mathcal{B}} = \left[\ [T(1)]_{\mathcal{C}} \quad [T(x)]_{\mathcal{C}} \ \right] = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

In addition,

$$\begin{aligned} S(1) &= 1 = 1(0) + 0(x) + 0(x^2) \\ S(x) &= x + 1 = 1(1) + 1(x) + 0(x^2) \\ S(x^2) &= (x+1)^2 = 1(1) + 2(x) + 1(x^2) \end{aligned}$$

Thus,

$$[S]_{\mathcal{D} \leftarrow \mathcal{C}} = \left[\ [S(1)]_{\mathcal{D}} \quad [S(x)]_{\mathcal{D}} \quad [S(x^2)]_{\mathcal{D}} \ \right] = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}.$$

Therefore, by Theorem 6.27,

$$[S \circ T]_{\mathcal{D} \leftarrow \mathcal{B}} = [S]_{\mathcal{D} \leftarrow \mathcal{C}} [T]_{\mathcal{C} \leftarrow \mathcal{B}} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

(19) Let $\mathcal{B} = \{1, x\}$ be the standard basis for $\mathcal{P}_1$. From Exercise (1), we have

$$[T]_{\mathcal{B}} = A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Since $\det(A) = 1 \neq 0$, we know that A is invertible. Thus, by Theorem 6.28, T is invertible and

$$[T^{-1}]_{\mathcal{B}} = A^{-1} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

So,

$$[T^{-1}(a + bx)]_{\mathcal{B}} = A^{-1}[a + bx]_{\mathcal{B}} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -b \\ a \end{bmatrix}.$$

We conclude that $T^{-1}(a + bx) = -b + ax$.

(20) In Exercise 5 we found that

$$A = [T]_{\mathcal{C} \leftarrow \mathcal{B}} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

where $\mathcal{B} = \{1, x, x^2\}$ and $\mathcal{C} = \{\mathbf{e_1}, \mathbf{e_2}\}$ are the standard bases of $\mathcal{P}_2$ and $\mathbb{R}^2$, respectively. Since A is not a square matrix, it is not invertible. Therefore, by Theorem 6.28, the transformation T is not invertible.