Math 314/814: Matrix Theory **Dr. S. Cooper, Fall 2008**

Homework Solutions – Week of September 2

Section 2.2:

(3) A complete solution to this exercise can be found at the back of your text (page 674).

(5) This matrix is not in row echelon form because the row of zeros occurs above a row of non-zeros.

(7) This matrix is not in row echelon form because the first non-zero entry of row 2 occurs immediately below the first non-zero entry of row 1.

(13) (a)

$$\begin{bmatrix} 3 & -2 & -1 \\ 2 & -1 & -1 \\ 4 & -3 & -1 \end{bmatrix} \xrightarrow{R_1 \to 1/3 R_2} \begin{bmatrix} 1 & -2/3 & -1/3 \\ 2 & -1 & -1 \\ 4 & -3 & -1 \end{bmatrix}$$

$$\xrightarrow{R_2 \to R_2 - 2R_1 \& R_3 \to R_3 - 4R_1} \begin{bmatrix} 1 & -2/3 & -1/3 \\ 0 & 1/3 & -1/3 \\ 0 & -1/3 & 1/3 \end{bmatrix}$$

$$\xrightarrow{R_3 \to R_3 + R_2} \begin{bmatrix} 1 & -2/3 & -1/3 \\ 0 & 1/3 & -1/3 \\ 0 & 0 & 0 \end{bmatrix}$$

This last matrix is in row echelon form. Note that the row echelon form of a matrix is not unique, and so your answer may differ from that here.

(b) To reduce the given matrix into reduced row echelon form we continue elementary row operations on the resulting matrix from (a):

$$\begin{bmatrix} 1 & -2/3 & -1/3 \\ 0 & 1/3 & -1/3 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1 \to R_1 + 2R_2} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1/3 & -1/3 \\ 0 & 0 & 0 \end{bmatrix}$$
$$\xrightarrow{R_2 \to 3R_2} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

This matrix is the unique matrix that is the reduced row echelon form of the given matrix.

(14) (a)

$$\begin{bmatrix} -2 & -4 & 7 \\ -3 & -6 & 10 \\ 1 & 2 & -3 \end{bmatrix} \xrightarrow{R_3 \leftrightarrow R_1} \begin{bmatrix} 1 & 2 & -3 \\ -3 & -6 & 10 \\ -2 & -4 & 7 \end{bmatrix}$$
$$\xrightarrow{R_2 \to R_2 + 3R_1 \& R_3 \to R_3 + 2R_1} \begin{bmatrix} 1 & 2 & -3 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$
$$\xrightarrow{R_3 \to R_3 - R_2} \begin{bmatrix} 1 & 2 & -3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

This is a row echelon form of the given matrix. As in exercise (13a), the matrix is not unique.

(b) We only have to perform one row operation from the matrix in part (a) to reduce the given matrix to reduced row echelon form:

$$\begin{bmatrix} 1 & 2 & -3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1 \to R_1 + 3R_2} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

(17) A complete answer to this exercise can be found at the back of your text (page

674). Another possible sequence of elementary row operations is:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}$$

$$\xrightarrow{R_2 \to R_2 - 1/3 R_1} \begin{bmatrix} 3 & 4 \\ 0 & 2/3 \end{bmatrix}$$

$$\xrightarrow{R_2 \to 3R_2} \begin{bmatrix} 3 & 4 \\ 0 & 2 \end{bmatrix}$$

$$\xrightarrow{R_1 \to R_1 - 5/2 R_2} \begin{bmatrix} 3 & -1 \\ 0 & 2 \end{bmatrix}$$

$$\xrightarrow{R_2 \to 1/2 R_2} \begin{bmatrix} 3 & -1 \\ 0 & 1 \end{bmatrix}$$

$$\xrightarrow{R_2 \to R_2 + R_1} \begin{bmatrix} 3 & -1 \\ 3 & 0 \end{bmatrix}$$

$$\xrightarrow{R_2 \to 1/3 R_2} \begin{bmatrix} 3 & -1 \\ 1 & 0 \end{bmatrix} = B$$

Thus, A and B are row equivalent.

(23) To fond the rank of a matrix you need to reduce it and then count the number of leading entries. There are solutions to the matrices in Exercises 1, 3, 5, 7 at the back of your text (page 674). The ranks of the remaining matrices are:

(Exercise 2) 2

(Exercise 4) 0

(Exercsie 6) 3

(Exercise 8) 3

(25) We row-reduce the augmented matrix of the system and then use back-substitution:

$$\left[\begin{array}{rrrcr} 1 & 2 & -3 & | & 9 \\ 2 & -1 & 1 & | & 0 \\ 4 & -1 & 1 & | & 4 \end{array}\right] \xrightarrow{R_2 \to R_2 - 2R_1 \& R_3 \to R_3 - 4R_1} \left[\begin{array}{rrrcr} 1 & 2 & -3 & | & 9 \\ 0 & -5 & 7 & | & -18 \\ 0 & -9 & 13 & | & -32 \end{array}\right]$$

$$\xrightarrow{R_2 \to -1/5 R_2} \left[\begin{array}{rrrcr} 1 & 2 & -3 & | & 9 \\ 0 & 1 & -7/5 & | & 18/5 \\ 0 & -9 & 13 & | & -32 \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 + 9R_2} \left[\begin{array}{rrrcr} 1 & 2 & -3 & | & 9 \\ 0 & 1 & -7/5 & | & 18/5 \\ 0 & 0 & 2/5 & | & 2/5 \end{array}\right]$$

$$\xrightarrow{R_3 \to 5/2 R_3} \left[\begin{array}{rrrcr} 1 & 2 & -3 & | & 9 \\ 0 & 1 & -7/5 & | & 18/5 \\ 0 & 0 & 1 & | & 1 \end{array}\right]$$

Thus

$$\begin{aligned} x_3 &= 1 \\ \Longrightarrow x_2 &= 7/5(1) + 18/5 = 5 \\ \Longrightarrow x_1 &= -2(5) + 3(1) + 9 = 2 \end{aligned}$$

The solution set to the system is

$$\left\{ \begin{bmatrix} 2 \\ 5 \\ 1 \end{bmatrix} \right\}.$$

(26) We first row reduce the augmented matrix of the system:

$$\left[\begin{array}{rrr|r} 1 & -1 & 1 & 0 \\ -1 & 3 & 1 & 5 \\ 3 & 1 & 7 & 2 \end{array}\right] \xrightarrow{R_2 \to R_2 + R_1 \& R_3 \to R_3 - 3R_1} \left[\begin{array}{rrr|r} 1 & -1 & 1 & 0 \\ 0 & 2 & 2 & 5 \\ 0 & 4 & 4 & 2 \end{array}\right]$$

$$\xrightarrow{R_2 \to 1/2 R_2 \& R_3 \to 1/4 R_3} \left[\begin{array}{rrr|r} 1 & -1 & 1 & 0 \\ 0 & 1 & 1 & 5/2 \\ 0 & 1 & 1 & 1/2 \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 - R_2} \left[\begin{array}{rrr|r} 1 & -1 & 1 & 0 \\ 0 & 1 & 1 & 5/2 \\ 0 & 0 & 0 & -2 \end{array}\right]$$

The last row corresponds to the equation

$$0x + 0y + 0z = -2$$

which has no solution. Considering the last row, we see that this system has no solution.

(33) We again first begin by row reducing the augmented matrix.

$$\left[\begin{array}{rrrr|r} 1 & 1 & 2 & 1 & 1 \\ 1 & -1 & -1 & 1 & 0 \\ 0 & 1 & 1 & 0 & -1 \\ 1 & 1 & 0 & 1 & 2 \end{array}\right] \xrightarrow{R_2 \to R_2 - R_1 \& R_4 \to R_4 - R_1} \left[\begin{array}{rrrr|r} 1 & 1 & 2 & 1 & 1 \\ 0 & -2 & -3 & 0 & -1 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & -2 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{rrrr|r} 1 & 1 & 2 & 1 & 1 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & -2 & -3 & 0 & -1 \\ 0 & 0 & -2 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 + 2R_2} \left[\begin{array}{rrrr|r} 1 & 1 & 2 & 1 & 1 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & -1 & 0 & -3 \\ 0 & 0 & -2 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_3 \to -R_3 \& R_4 \to -R_4} \left[\begin{array}{rrrr|r} 1 & 1 & 2 & 1 & 1 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 2 & 0 & -1 \end{array}\right]$$

$$\xrightarrow{R_4 \to R_4 - 2R_3} \left[\begin{array}{rrrr|r} 1 & 1 & 2 & 1 & 1 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 0 & -7 \end{array}\right]$$

As with exercise (26), the bottom row shows that this system has no solution.

(35) It is easy to see, upon inspection, that after interchanging rows 1 and 2 there will be a pivot in each column corresponding to a variable. Thus, there is one unique solution to the given system.

(36) The bottom two rows of the augmented matrix correspond to

$$\begin{aligned} x_1 + x_2 - 3x_3 + x_4 &= -1 \\ 2x_1 + 4x_2 - 6x_3 + 2x_4 &= 0 \end{aligned}$$

Observe that each coefficient infront of the variables in the second equation is twice the corresponding coefficient in the fist equation. However, the constant term in the second equation is not twice the constant term in the first. We see that there can be no solution to this system.

(37) This is a homoegenous system of 3 equations in 4 unknowns. Since $3 < 4$, Theorem 2.3 says that the system has infinitely many solutions.

(43) We row reduce the augmented matrix of the system:

$$\left[\begin{array}{ccc|c} 1 & 1 & k & 1 \\ 1 & k & 1 & 1 \\ k & 1 & 1 & -2 \end{array}\right] \xrightarrow{R_2 \to R_2 - R_1 \& R_3 \to R_3 - kR_1} \left[\begin{array}{ccc|c} 1 & 1 & k & 1 \\ 0 & (k-1) & (1-k) & 0 \\ 0 & (1-k) & (1-k^2) & -2-k \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 + R_2} \left[\begin{array}{ccc|c} 1 & 1 & k & 1 \\ 0 & (k-1) & (1-k) & 0 \\ 0 & 0 & (2-k-k^2) & -2-k \end{array}\right]$$

Note that $2 - k - k^2 = 0$ if and only if $(k-1)(k+2) = 0$.

(a) There is no solution if $2 - k - k^2 = 0$ and $-2 - k \neq 0$. Thus, there is no solution if $k = 1$.

(b) There is a unique solution if every variable corresponds to a pivot. Thus, there is a unique solution if $2 - k - k^2 \neq 0$, i.e. if $k \neq 1$ and $k \neq -2$.

(c) There are infinitely many solutions if the system is consistent and free variables exits. So, there are infinitely many solutions if $2 - k - k^2 = 0$ and $k \neq 1$. i.e. if $k = -2$.

(49) If $\mathbf{x} = \mathbf{p} + s\mathbf{u}$ and $\mathbf{x} = \mathbf{q} + t\mathbf{v}$, then

$$\mathbf{q} - \mathbf{p} = s\mathbf{u} - t\mathbf{v}.$$

Thus, we must have

$$s \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - t \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} -4 \\ 0 \\ -1 \end{bmatrix}.$$

This gives the linear system of equations:

$$\begin{aligned} s - 2t &= -4 \\ -3t &= 0 \\ s - t &= -1 \end{aligned}$$

Equations 2 and 3 together yield $t = 0$ and $s = -1$. However, $s = -1$ and $t = 0$ is not a solution to the first equation. Thus, the given lines do not intersect.

(50) Let $\mathbf{q}$ be the vector from the origin to Q. As in exercise (49), we must have $\mathbf{q} - \mathbf{p} = s\mathbf{u} - t\mathbf{v}$. That is,

$$\begin{bmatrix} a-1 \\ b-2 \\ c-3 \end{bmatrix} = \begin{bmatrix} s-2t \\ s-t \\ -s \end{bmatrix}.$$

This gives us the linear system

$$\begin{aligned} a - 1 &= s - 2t \\ b - 2 &= s - t \\ c - 3 &= -s \end{aligned}$$

We solve the system by row reducing the augmented matrix.

$$\left[\begin{array}{cc|c} 1 & -2 & a-1 \\ 1 & -1 & b-2 \\ -1 & 0 & c-3 \end{array}\right] \xrightarrow{R_2 \to R_2 - R_1 \& R_3 \to R_3 + R_1} \left[\begin{array}{cc|c} 1 & -2 & a-1 \\ 0 & 1 & b-a-1 \\ 0 & -2 & a+c-4 \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 + 2R_2} \left[\begin{array}{cc|c} 1 & -2 & a-1 \\ 0 & 1 & b-a-1 \\ 0 & 0 & -a+2b+c-6 \end{array}\right]$$

We see that there will be an intersection as long as $-a + 2b + c - 6 = 0$. Thus, the points $Q = (a, b, c)$ that give an intersection are:

$$\{(a, b, c) \mid a, b, c \in \mathbb{R} \ \& \ a - 2b - c = -6\}.$$

Section 2.3:

(1) Let

$$A = \begin{bmatrix} \mathbf{u_1} & \mathbf{u_2} \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix}.$$

The linear system whose augmented matrix is $[A \mid \mathbf{v}]$ row reduces as:

$$\left[\begin{array}{cc|c} 1 & 2 & 1 \\ -1 & -1 & 2 \end{array}\right] \xrightarrow{R_2 \to R_2 + R_1} \left[\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & 1 & 3 \end{array}\right].$$

Since the system is consistent, Thereom 2.4 gives that $\mathbf{v}$ is a linear combination of $\mathbf{u_1}$ and $\mathbf{u_2}$.

(2) Let

$$A = \begin{bmatrix} \mathbf{u_1} & \mathbf{u_2} \end{bmatrix} = \begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix}.$$

The linear system whose augmented matrix is $[A \mid \mathbf{v}]$ row reduces as:

$$\left[\begin{array}{cc|c} 4 & -2 & 2 \\ -2 & 1 & 1 \end{array}\right] \xrightarrow{R_2 \to R_2 + 1/2R_1} \left[\begin{array}{cc|c} 4 & -2 & 2 \\ 0 & 0 & 2 \end{array}\right].$$

Since the system is inconsistent, Thereom 2.4 gives that $\mathbf{v}$ is not a linear combination of $\mathbf{u_1}$ and $\mathbf{u_2}$.

(3) Let

$$A = \begin{bmatrix} \mathbf{u_1} & \mathbf{u_2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

The linear system whose augmented matrix is $[A \mid \mathbf{v}]$ row reduces as:

$$\left[\begin{array}{cc|c} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 0 & 1 & 3 \end{array}\right] \xrightarrow{R_2 \to R_2 - R_1} \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 3 \end{array}\right] \xrightarrow{R_3 \to R_3 - R_2} \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{array}\right].$$

Since the system is inconsistent, Thereom 2.4 gives that $\mathbf{v}$ is not a linear combination of $\mathbf{u_1}$ and $\mathbf{u_2}$.

(7) We row reduce the augmented matrix $[A \mid \mathbf{b}]$:

$$\left[\begin{array}{cc|c} 1 & 2 & 5 \\ 3 & 4 & 6 \end{array}\right] \xrightarrow{R_2 \to R_2 - 3R_1} \left[\begin{array}{cc|c} 1 & 2 & 5 \\ 0 & -2 & -9 \end{array}\right] \xrightarrow{R_2 \to -R_2} \left[\begin{array}{cc|c} 1 & 2 & 5 \\ 0 & 2 & 9 \end{array}\right].$$

We see that corresponding system is consistent, and so Theorem 2.4 says that $\mathbf{b}$ is in the span of the columns of the matrix A.

(9) Let $\mathbf{u_1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\mathbf{u_2} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

First observe that any vector in $\mathrm{span}(\mathbf{u_1}, \mathbf{u_2})$ is a vector in $\mathbb{R}^2$.

Now let $\mathbf{x} = \begin{bmatrix} a \\ b \end{bmatrix} \in \mathbb{R}^2$. We need to show that we can write this vector as a linear combination of $\mathbf{u_1}$ and $\mathbf{u_2}$. We are looking for constants $c_1, c_2 \in \mathbb{R}$ such that $\mathbf{x} = c_1\mathbf{u_1} + c_2\mathbf{u_2}$, i.e.

$$\begin{aligned} a &= c_1 + c_2 \\ b &= c_1 - c_2 \end{aligned}$$

We row reduce the augmented matrix to find c_1 and c_2:

$$\left[\begin{array}{cc|c} 1 & 1 & a \\ 1 & -1 & b \end{array}\right] \xrightarrow{R_2 \to R_2 - R_1} \left[\begin{array}{cc|c} 1 & 1 & a \\ 0 & -2 & b-a \end{array}\right] \xrightarrow{R_2 \to -1/2R_2} \left[\begin{array}{cc|c} 1 & 1 & a \\ 0 & 1 & (a-b)/2 \end{array}\right].$$

Using back-substitution, we solve $c_1 = (a+b)/2$ and $c_2 = (a-b)/2$.

Since the system is consistent, we see that $\mathbf{x}$ is in $\mathrm{span}(\mathbf{u_1}, \mathbf{u_2})$.

(15) (a) $\mathrm{Span}\left(\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}\right) = \left\{ c_1 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix} \mid c_1, c_2 \in \mathbb{R} \right\}$. This is the plane through the origin with direction vectors $\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}$.

(b) To describe the span of the given vectors algebraically, we find the general equation $ax + by + cz = 0$ of the plane. To find this, observe that the points $(1,2,0)$ and $(3,2,-1)$ lie on the plane. Thus, when we substitute these points into the general equation, we obtain the linear system

$$\begin{aligned}(1)a+(2)b+(0)c &= 0\\(3)a+(2)b+(-1)c &= 0\end{aligned}$$

We solve this system for a, b, c:

$$\left[\begin{array}{ccc|c}1 & 2 & 0 & 0\\3 & 2 & -1 & 0\end{array}\right] \xrightarrow{R_2 \to R_2 - 3R_1} \left[\begin{array}{ccc|c}1 & 2 & 0 & 0\\0 & -4 & -1 & 0\end{array}\right]$$

Thus, $c=-4b$ and $a=-2b$. One solution to this system is $a=2, b=-1, c=4$.

Thus, the general equation of the plane is $2x-y+4z=0$.

(17) Substituting the given points in the equation for x, y, z, we obtain the linear system

$$\begin{aligned}a+0b+3c &= 0\\-a+b-3c &= 0\\0a+0b+0c &= 0\end{aligned}$$

We row reduce the augmented matrix:

$$\left[\begin{array}{ccc|c}1 & 0 & 3 & 0\\-1 & 1 & -3 & 0\\0 & 0 & 0 & 0\end{array}\right] \xrightarrow{R_2 \to R_2 + R_1} \left[\begin{array}{ccc|c}1 & 0 & 3 & 0\\0 & 1 & 0 & 0\\0 & 0 & 0 & 0\end{array}\right]$$

Setting $c=t$ for $t \in \mathbb{R}$ yields the solution set

$$\left\{ t \begin{bmatrix}-3\\0\\1\end{bmatrix} : t \in \mathbb{R} \right\}.$$

We see that there are infinitely many solutions for a, b, c. One solution is $a=-3, b=0, c=1$.

(19) A complete solution to this exercise can be found at the back of your text (page 675).

(23) We solve the linear system whose augmented matrix is $[A \mid \mathbf{0}]$, where the matrix A has the given vectors as its columns.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 2 & -1 & 0 \\ 1 & 3 & 2 & 0 \end{array}\right] \xrightarrow{R_2 \to R_2 - R_1 \& R_3 \to R_3 - R_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 2 & 1 & 0 \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 - 2R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 5 & 0 \end{array}\right]$$

Since there are no free variables, we see that there is only the trivial solution. So, by Thereom 2.6, the given vectors are linearly independent.

(24) We repeat the process of Exercise 23. We have::

$$\left[\begin{array}{ccc|c} 2 & 3 & 1 & 0 \\ 2 & 1 & -5 & 0 \\ 1 & 2 & 2 & 0 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_3} \left[\begin{array}{ccc|c} 1 & 2 & 2 & 0 \\ 2 & 1 & -5 & 0 \\ 2 & 3 & 1 & 0 \end{array}\right]$$

$$\xrightarrow{R_2 \to R_2 - 2R_1 \& R_3 \to R_3 - 2R_1} \left[\begin{array}{ccc|c} 1 & 2 & 2 & 0 \\ 0 & -3 & -9 & 0 \\ 0 & -1 & -3 & 0 \end{array}\right]$$

$$\xrightarrow{R_2 \to -1/3 R_2} \left[\begin{array}{ccc|c} 1 & 2 & 2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & -1 & -3 & 0 \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 + R_2} \left[\begin{array}{ccc|c} 1 & 2 & 2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{R_1 \to R_1 - 2R_2} \left[\begin{array}{ccc|c} 1 & 0 & -4 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Since there is a free variable, and hence a nontrivial solution to the system, Theorem 2.6 says that the given vectors are linearly dependent. Looking at the relationship among the columns of the reduced row echelon form above, we see the dependece relation

$$\begin{bmatrix} 1 \\ -5 \\ 2 \end{bmatrix} = -4 \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} + 3 \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}.$$

(29) We again repeat the process of Exercise 23. We have::

$$\left[\begin{array}{cccc|c} 1 & -1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 1 & 0 \end{array}\right] \xrightarrow{R_2 \to R_2 + R_1 \& R_3 \to R_3 - R_1} \left[\begin{array}{cccc|c} 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 1 & 0 \end{array}\right]$$

$$\xrightarrow{row\ interchanges} \left[\begin{array}{cccc|c} 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 - R_2} \left[\begin{array}{cccc|c} 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right]$$

$$\xrightarrow{R_4 \to R_4 + R_3} \left[\begin{array}{cccc|c} 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 3 & 0 \end{array}\right]$$

Since this system has only the trivial solution, Theorem 2.6 gives that the vectors in question are linearly independent.

(33) We start with the matrix whose rows are the given vectors and row reduce

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & -1 & 2 \end{bmatrix} \xrightarrow{R_2 \to R_2 - R_1 \& R_3 \to R_3 - R_1} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & -2 & 1 \end{bmatrix}$$

$$\xrightarrow{R_3 \to R_3 + 2R_2} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 5 \end{bmatrix}$$

This shows that the rank of the initial matrix is 3. By Theorem 2.7, this implies that the given vectors are linearly independent.

(34) We start with the matrix whose rows are the given vectors and row reduce

$$\begin{bmatrix} 2 & 2 & 1 \\ 3 & 1 & 2 \\ 1 & -5 & 2 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_3} \begin{bmatrix} 1 & -5 & 2 \\ 3 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$$

$$\xrightarrow{R_2 \to R_2 - 3R_1 \& R_3 \to R_3 - 2R_1} \begin{bmatrix} 1 & -5 & 2 \\ 0 & 16 & -4 \\ 0 & 12 & -3 \end{bmatrix}$$

$$\xrightarrow{R_2 \to (1/4)R_2 \& R_3 \to (1/3)R_3} \begin{bmatrix} 1 & -5 & 2 \\ 0 & 4 & -1 \\ 0 & 4 & -1 \end{bmatrix}$$

$$\xrightarrow{R_3 \to R_3 - R_2} \begin{bmatrix} 1 & -5 & 2 \\ 0 & 4 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

This shows that the rank of the initial matrix is 2. By Theorem 2.7, this implies that the given vectors are linearly dependent.

(42) (a) If the columns are linearly independent as vectors in $\mathbb{R}^n$, then Thereom 2.6 says that the linear system whose augmented matrix is $[A \mid \mathbf{0}]$ has only the trivial solution. This means that the system yields no free variables. So, by Theorem 2.2, $\text{rank}(A) = n$.

(b) We know that $\text{rank}(A) \leq n$. By Theorem 2.7, if the rows are lnearly independent as vectors in $\mathbb{R}^n$, the rank of A must equal n.

(43) (a) Yes, $\mathbf{u}+\mathbf{v}, \mathbf{v}+\mathbf{w}, \mathbf{u}+\mathbf{w}$ will be linearly independent. To see this, consider the equation

$$\mathbf{0} = c_1(\mathbf{u}+\mathbf{v}) + c_2(\mathbf{v}+\mathbf{w}) + c_3(\mathbf{u}+\mathbf{w}).$$

Simplifying the equation, we consider

$$\mathbf{0} = (c_1+c_3)\mathbf{u} + (c_1+c_2)\mathbf{v} + (c_2+c_3)\mathbf{w}.$$

Since $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are linearly independent, we must have the linear system

$$\begin{aligned} c_1 + c_3 &= 0 \\ c_1 + c_2 &= 0 \\ c_2 + c_3 &= 0 \end{aligned}$$

Working with the augmented matrix, we solve for c_1, c_2 and c_3:

$$\begin{bmatrix} 1 & 0 & 1 & | & 0 \\ 1 & 1 & 0 & | & 0 \\ 0 & 1 & 1 & | & 0 \end{bmatrix} \xrightarrow{R_2 \to R_2 - R_1} \begin{bmatrix} 1 & 0 & 1 & | & 0 \\ 0 & 1 & -1 & | & 0 \\ 0 & 1 & 1 & | & 0 \end{bmatrix} \xrightarrow{R_3 \to R_3 - R_2} \begin{bmatrix} 1 & 0 & 1 & | & 0 \\ 0 & 1 & -1 & | & 0 \\ 0 & 0 & 2 & | & 0 \end{bmatrix}$$

This yields $c_1 = c_2 = c_3 = 0$, showing that the vectors $\mathbf{u}+\mathbf{v}, \mathbf{v}+\mathbf{w}, \mathbf{u}+\mathbf{w}$ must be linearly independent.

(b) No, the vectors $\mathbf{u}-\mathbf{v}, \mathbf{v}-\mathbf{w}, \mathbf{u}-\mathbf{w}$ will not be linearly independent. Observe that we have the dependence relation

$$\mathbf{u} - \mathbf{w} = (\mathbf{u}-\mathbf{v}) + (\mathbf{v}-\mathbf{w}).$$