Math 314/814: Matrix Theory **Dr. S. Cooper, Fall 2008**

Homework Solutions – Week of September 9

Section 2.4:

(2) Let x_1 be the number of bacteria in Strand I, x_2 be the number of bacteria in Strand II, and x_3 be the number of bacteria in Strand III. We need to solve the linear system

$$\begin{aligned} x_1 + 2x_2 &= 400 \\ 2x_1 + x_2 + 3x_3 &= 500 \\ x_1 + x_2 + x_3 &= 600 \end{aligned}$$

We perform Gaussian elimination:

$$\begin{bmatrix} 1 & 2 & 0 & | & 400 \\ 2 & 1 & 3 & | & 500 \\ 1 & 1 & 1 & | & 600 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 0 & | & 400 \\ 0 & 1 & -1 & | & 100 \\ 0 & 0 & 0 & | & 300 \end{bmatrix}.$$

The bottom row indicates that this system is inconsistent. No bacteria of any strand can coexist in the tube and consume all of the food.

(3) Let x_1 equal the number of small arrangements, x_2 be the number of medium arrangements, and x_3 be the number of large arrangements. Then we have to solve the system

$$\begin{aligned} x_1 + 2x_2 + 4x_3 &= 24 \\ 3x_1 + 4x_2 + 8x_3 &= 50 \\ 3x_1 + 6x_2 + 6x_3 &= 48 \end{aligned}$$

We row reduce the augmented matrix:

$$\begin{bmatrix} 1 & 2 & 4 & | & 24 \\ 3 & 4 & 8 & | & 50 \\ 3 & 6 & 6 & | & 48 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 4 & | & 24 \\ 0 & 1 & 2 & | & 11 \\ 0 & 0 & 1 & | & 4 \end{bmatrix}.$$

This gives us

$$\begin{aligned} x_3 &= 4 \\ x_2 &= -2x_3 + 11 = -8 + 11 = 3 \\ x_1 &= -2x_2 - 4x_3 + 24 = -6 - 16 + 24 = 2 \end{aligned}$$

Thus, 2 small arrangements, 3 medium arrangements, and 4 large arrangements were made.

(9) We need to find non-negative integers w, x, y, z such that

$$w\mathrm{C_4H_{10}} + x\mathrm{O_2} \longrightarrow y\mathrm{CO_2} + z\mathrm{H_2O}$$

is a balanced equation. This leads to the system

$$\begin{aligned} 4w &= y \\ 10w &= 2z \\ 2x &= 2y + z \end{aligned}$$

After re-arranging the equations to get a homogeneous system, we row reduce the augmented matrix

$$\left[\begin{array}{rrrr|r} 4 & 0 & -1 & 0 & 0 \\ 10 & 0 & 0 & -2 & 0 \\ 0 & 2 & -2 & -1 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{rrrr|r} 1 & 0 & -1/4 & 0 & 0 \\ 0 & 1 & -1 & -1/2 & 0 \\ 0 & 0 & 10/4 & -2 & 0 \end{array}\right].$$

We see that z is a free variable. Let $z = t$ where $t \in \mathbb{R}$. Then

$$\begin{aligned} (10/4)y &= 2z \implies y = (4/5)t \\ x &= y + (1/2)z \implies x = (4/5)t + (1/2)t = (13/10)t \\ w &= (1/4)y \implies w = (2/10)t \end{aligned}$$

Since w, x, y, z need to be non-negative integers we let $t = 10$. Then $w = 2, x = 13, y = 8, z = 10$. So the balanced equation is

$$2\mathrm{C_4H_{10}} + 13\mathrm{O_2} \longrightarrow 8\mathrm{CO_2} + 10\mathrm{H_2O}$$

(15) (a) Keeping in mind that in-flow must equal out-flow at A, B and C, we obtain the system

$$\begin{aligned} f_1 + f_2 &= 20 \\ f_2 - f_3 &= -10 \\ f_1 + f_3 &= 30 \end{aligned}$$

We row reduce the sugmented matrix:

$$\begin{bmatrix} 1 & 1 & 0 & | & 20 \\ 0 & 1 & -1 & | & -10 \\ 1 & 0 & 1 & | & 30 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 0 & | & 20 \\ 0 & 1 & -1 & | & -10 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}.$$

We see that f_3 corresponds to a free variable. Let $f_3 = t$ where $t \in \mathbb{R}$. Then

$$\begin{aligned} f_3 &= t \\ f_2 &= f_3 - 10 = t - 10 \\ f_1 &= -f_2 + 20 = 30 - t \end{aligned}$$

Note that each f_i must be greater than or equal to 0. Thus, t is a positive integer where $10 \leq t \leq 30$.

(b) If $f_2 = 5$, then $t = 15$. Thus, $f_1 = 15 = f_3$.

(c) An answer for this part can be found at the back of the text (page 676).

(d) An answer for this part can be found at the back of the text (page 676).

(16) (a) In-flow must equal out-flow at A, B, C and D. So, we obtain the system

$$\begin{aligned} f_1 + f_2 &= 20 \\ f_1 + f_3 &= 25 \\ f_2 + f_4 &= 25 \\ f_3 + f_4 &= 30 \end{aligned}$$

We row reduce the augmented matrix:

$$\begin{bmatrix} 1 & 1 & 0 & 0 & | & 20 \\ 1 & 0 & 1 & 0 & | & 25 \\ 0 & 1 & 0 & 1 & | & 25 \\ 0 & 0 & 1 & 1 & | & 30 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 0 & 0 & | & 20 \\ 0 & 1 & -1 & 0 & | & -5 \\ 0 & 0 & 1 & 1 & | & 30 \\ 0 & 0 & 0 & 0 & | & 0 \end{bmatrix}.$$

We see that f_4 corresponds to a free variable. Let $f_4 = t$ where $t \in \mathbb{R}$. Then

$$\begin{aligned} f_3 &= 30 - t \\ f_2 &= f_3 - 5 = 30 - t - 5 = 25 - t \\ f_1 &= -f_2 + 20 = t - 25 + 20 = t - 5 \end{aligned}$$

Here, we need t to be an integer such that each $f_i \geq 0$. Thus, t is an integer such that $5 \leq t \leq 25$.

(b) If $f_4 = 10$, then $t = 10$. So,

$$\begin{aligned} f_1 &= 5 \\ f_2 &= 15 \\ f_3 &= 20 \\ f_4 &= 10 \end{aligned}$$

where each number represents vehicles per minute.

(c) By part (a), $5 \leq t \leq 25$. Thus

$$\begin{aligned} 0 \leq & \; f_1 \leq 20 \\ 0 \leq & \; f_2 \leq 20 \\ 5 \leq & \; f_3 \leq 25 \\ 5 \leq & \; f_4 \leq 25 \end{aligned}$$

(d) If all the directions were reversed then the solution would remain the same. This is because the in-flow and out-flow relations would remain the same.

(34) Let x_i be the measure of type i in a bundle, for $i = 1, 2, 3$. We need to solve the system

$$\begin{aligned} 3x_1 + 2x_2 + x_3 &= 39 \\ 2x_1 + 3x_2 + x_2 &= 34 \\ x_1 + 2x_2 + 3x_3 &= 26 \end{aligned}$$

We form the augmented matrix and row reduce:

$$\begin{bmatrix} 1 & 2 & 3 & | & 26 \\ 2 & 3 & 1 & | & 34 \\ 3 & 2 & 1 & | & 39 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 3 & | & 26 \\ 0 & 1 & 5 & | & 18 \\ 0 & 0 & 12 & | & 33 \end{bmatrix}$$

This gives us the solution

$$\left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 111/12 \\ 51/12 \\ 33/12 \end{bmatrix} \right\}.$$

(39 b) We plug each point into the equation $y = ax^2 + bx + c$ to obtain the system

$$\begin{aligned} 9a - 3b + c &= 1 \\ 4a - 2b + c &= 2 \\ a - b + c &= 5 \end{aligned}$$

We use Gaussian elimiation to solve the system:

$$\begin{bmatrix} 1 & -1 & 1 & | & 5 \\ 4 & -2 & 1 & | & 2 \\ 9 & -3 & 1 & | & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -1 & 1 & | & 5 \\ 0 & 1 & -3/2 & | & -9 \\ 0 & 0 & 1 & | & 10 \end{bmatrix}.$$

We compute the solution set to be:

$$\left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \\ 10 \end{bmatrix} \right\}.$$

Therefore, $y = x^2 + 6x + 10$.

(42) Multiplying both sides of the given equation by $x^3 + 2x^2 + x = x(x+1)^2$ we obtain

$$x^2 - 3x + 3 = A(x^2 + 2x + 1) + B(x^2 + x) + Cx$$

or, equivalently,

$$x^2 - 3x + 3 = (A + B)x^2 + (2A + B + C)x + A.$$

Comparing coefficients, we have the system

$$\begin{aligned} A + B &= 1 \\ 2A + B + C &= -3 \\ A &= 3 \end{aligned}$$

We row reduce the augmented matrix:

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 2 & 1 & 1 & -3 \\ 1 & 0 & 0 & 3 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 1 & -9 \\ 0 & 0 & 1 & -7 \end{array}\right].$$

This yields the solution

$$\left\{ \begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ -7 \end{bmatrix} \right\}.$$

So,

$$\frac{x^2 - 3x + 3}{x^3 + 2x^2 + x} = \frac{3}{x} - \frac{2}{x+1} - \frac{7}{(x+1)^2}.$$

Section 3.1:

(3) $B - C$ cannot be computed since B is 2×3 but C is 3×2

(5) An answer to this exercise can be found at the back of your text (page 680).

(7) An answer to this exercise can be found at the back of your text (page 680).

(13) An answer to this exercise can be found at the back of your text (page 680).

(14)

$$\begin{aligned} DA - AD &= \begin{bmatrix} 0 & -3 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ -1 & 5 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ -1 & 5 \end{bmatrix} \begin{bmatrix} 0 & -3 \\ -2 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 3 & -15 \\ -7 & 5 \end{bmatrix} - \begin{bmatrix} 0 & -9 \\ -10 & 8 \end{bmatrix} \\ &= \begin{bmatrix} 3 & -6 \\ 3 & -3 \end{bmatrix} \end{aligned}$$

(17) An answer to this exercise can be found at the back of the text (page 680).

(18) Let $B = \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix}$ and $C = \begin{bmatrix} c_1 & c_2 \\ c_3 & c_4 \end{bmatrix}$. We want

$$AB = \begin{bmatrix} 2b_1 + b_3 & 2b_2 + b_4 \\ 6b_1 + 3b_3 & 6b_2 + 3b_4 \end{bmatrix} = \begin{bmatrix} 2c_1 + c_3 & 2c_2 + c_4 \\ 6c_1 + 3c_3 & 6c_2 + 3c_4 \end{bmatrix} = AC$$

where $b_i \neq c_i$ for any $i = 1, 2, 3, 4$. Letting $b_1 = b_2 = 1, b_3 = b_4 = 0$ and $c_1 = c_2 = 0, c_3 = c_4 = 2$ would satisfy the conditions we are looking for. Thus,

$$B = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \quad \& \quad C = \begin{bmatrix} 0 & 0 \\ 2 & 2 \end{bmatrix}.$$

(19) A solution to this exercise can be found at the back of your text (page 680).

(20) Let $C = \begin{bmatrix} 0.75 \\ 1.00 \end{bmatrix}$. Then

$$AC = \begin{bmatrix} 200 & 75 \\ 150 & 100 \\ 100 & 125 \end{bmatrix} \begin{bmatrix} 0.75 \\ 1.00 \end{bmatrix} = \begin{bmatrix} 225.00 \\ 212.50 \\ 200.00 \end{bmatrix}.$$

Row i is the total cost to distribute product i. For example, the cost to distribute product 1 is

$$200(0.75) + 75(1.00) = 225.00$$

which is the first row of AC.

(21) A solution to this exercise can be found at the back of your text (page 680).

(22)

$$\begin{bmatrix} -1 & 0 & 2 \\ 1 & -1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix}$$

(23) A solution to this exercise can be found at the back of the text (page 680).

(29) A solution to this exercise can be found at the back of the text (page 680).

(39) Answers for parts (a) and (c) can be found at the back of the text (page 680). Answers for parts (b) and (d) are:

(b) $A = \begin{bmatrix} 0 & 1 & 2 & 3 \\ -1 & 0 & 1 & 2 \\ -2 & -1 & 0 & 1 \\ -3 & -2 & -1 & 0 \end{bmatrix}$

(d) $A = \begin{bmatrix} \sqrt{2}/2 & 1 & \sqrt{2}/2 & 0 \\ 1 & \sqrt{2}/2 & 0 & -\sqrt{2}/2 \\ \sqrt{2}/2 & 0 & -\sqrt{2}/2 & -1 \\ 0 & -\sqrt{2}/2 & -1 & -\sqrt{2}/2 \end{bmatrix}$

Section 3.2

(3) We have

$$2(A + 2B) = 3X \implies X = \frac{2}{3}(A + 2B) = \begin{bmatrix} -2/3 & 4/3 \\ 10/3 & 4 \end{bmatrix}.$$

(4) We have

$$2(A - B + X) = 3(X - A) \implies 2A - 2B + 2X = 3X - 3A \implies X = 5A - 2B.$$

So,

$$X = 5 \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} - 2 \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 7 & 10 \\ 13 & 18 \end{bmatrix}.$$

(7) We want to find scalars $c_1, c_2, c_3 \in \mathbb{R}$ such that

$$B = c_1 A_1 + c_2 A_2 + c_3 A_3.$$

That is,

$$\begin{aligned}
\begin{bmatrix} 3 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} &= \begin{bmatrix} c_1 & 0 & -c_1 \\ 0 & c_1 & 0 \end{bmatrix} + \begin{bmatrix} -c_2 & 2c_2 & 0 \\ 0 & c_2 & 0 \end{bmatrix} + \begin{bmatrix} c_3 & c_3 & c_3 \\ 0 & 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} c_1 - c_2 + c_3 & 2c_2 + c_3 & -c_1 + c_3 \\ 0 & c_1 + c_2 & 0 \end{bmatrix}.
\end{aligned}$$

Thus, we have to solve the system

$$\begin{aligned}
c_1 - c_2 + c_3 &= 3 \\
2c_2 + c_3 &= 1 \\
-c_1 + c_3 &= 1 \\
c_1 + c_2 &= 1
\end{aligned}$$

We row reduce the associated augmented matrix

$$\left[\begin{array}{rrr|r} 1 & -1 & 1 & 3 \\ 0 & 2 & 1 & 1 \\ -1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{rrr|r} 1 & -1 & 1 & 3 \\ 0 & 1 & -2 & -4 \\ 0 & 0 & 1 & 5/9 \\ 0 & 0 & 1 & 3/2 \end{array}\right].$$

We see that this system is inconsistent (the bottom two rows give two different values for c_3). Thus, B is not a linear combination of A_1, A_2, A_3.

(11)

$$\begin{aligned}
Span(A_1, A_2, A_3) &= \left\{ c_1 \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} + c_2 \begin{bmatrix} -1 & 2 & 0 \\ 0 & 1 & 0 \end{bmatrix} + c_3 \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} : c_1, c_2, c_3 \in \mathbb{R} \right\} \\
&= \left\{ \begin{bmatrix} c_1 - c_2 + c_3 & 2c_2 + c_3 & -c_1 + c_3 \\ 0 & c_1 + c_2 & 0 \end{bmatrix} : c_1, c_2, c_3 \in \mathbb{R} \right\}
\end{aligned}$$

To find the general form of the span of these matrices, we need to solve the linear system

$$\begin{aligned} c_1 - c_2 + c_3 &= w \\ 2c_2 + c_3 &= x \\ -c_1 + c_3 &= y \\ c_1 + c_2 &= z \end{aligned}$$

for some $w, x, y, z \in \mathbb{R}$. We row reduce the augmented matrix of this system:

$$\begin{bmatrix} 1 & -1 & 1 & | & w \\ 0 & 2 & 1 & | & x \\ -1 & 0 & 1 & | & y \\ 1 & 1 & 0 & | & z \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -1 & 1 & | & w \\ 0 & 1 & -2 & | & -w-y \\ 0 & 0 & 1 & | & x/5 + 2/5w + 2/5y \\ 0 & 0 & 0 & | & z - 1/5w + 4/5y - 3/5x \end{bmatrix}.$$

For the system to be consistent we must have

$$z - 1/5w + 4/5y - 3/5x = 0 \implies w = -3x + 4y + 5z.$$

So, the general form of the span(A_1, A_2, A_3) is

$$span(A_1, A_2, A_3) = \left\{ \begin{bmatrix} -3x + 4y + 5z & x & y \\ 0 & z & 0 \end{bmatrix} : x, y, z \in \mathbb{R} \right\}.$$

(15) Suppose we have the linear combination

$$c_1 \begin{bmatrix} 0 & 1 \\ 5 & 2 \\ -1 & 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 & 0 \\ 2 & 3 \\ 1 & 1 \end{bmatrix} + c_3 \begin{bmatrix} -2 & -1 \\ 0 & 1 \\ 0 & 2 \end{bmatrix} + c_4 \begin{bmatrix} -1 & -3 \\ 1 & 9 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

for some constants $c_1, c_2, c_3, c_4 \in \mathbb{R}$. Comparing entries of the matrices in this equation, we obtain the system

$$\begin{aligned} c_2 - 2c_3 - c_4 &= 0 \\ c_1 - c_3 - 3c_4 &= 0 \\ 5c_1 + 2c_2 + c_4 &= 0 \\ 2c_1 + 3c_2 + c_3 + 9c_4 &= 0 \\ -c_1 + c_2 + 4c_4 &= 0 \\ c_2 + 2c_3 + 5c_4 &= 0 \end{aligned}$$

We row reduce the augmented matrix:

$$\begin{bmatrix} 0 & 1 & -2 & -1 & | & 0 \\ 1 & 0 & -1 & -3 & | & 0 \\ 5 & 2 & 0 & 1 & | & 0 \\ 2 & 3 & 1 & 9 & | & 0 \\ -1 & 1 & 0 & 4 & | & 0 \\ 0 & 1 & 2 & 5 & | & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -1 & -3 & | & 0 \\ 0 & 1 & -2 & -1 & | & 0 \\ 0 & 0 & 1 & 2 & | & 0 \\ 0 & 0 & 0 & -1 & | & 0 \\ 0 & 0 & 0 & 0 & | & 0 \\ 0 & 0 & 0 & 0 & | & 0 \end{bmatrix}.$$

Since there are no free variables, we conclude that the system has only the trivial solution $c_1 = c_2 = c_3 = c_4 = 0$. So, by definition, the given matrices are linearly independent.

(22) Suppose that $AB = BA$. Then

$$(A - B)(A + B) = A^2 + AB - BA - B^2 = A^2 + AB - AB - B^2 = A^2 - B^2.$$

Now assume that $(A - B)(A + B) = A^2 - B^2$. Then

$$(A-B)(A+B) = A^2+AB-BA-B^2 = A^2-B^2 \implies AB-BA = 0 \implies AB = BA,$$

where 0 denotes the zero matrix.

(23) We want

$$AB = \begin{bmatrix} a+c & b+d \\ c & d \end{bmatrix} = \begin{bmatrix} a & a+b \\ c & c+d \end{bmatrix} = BA.$$

Thus, we must have $a + c = a, b + d = a + b, d = c + d$. This implies that $c = 0$ and $a = d$.

(26) We want

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} a & 0 \\ c & 0 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

and

$$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & b \\ 0 & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Thus, comparing the matrix entries, we see that $b = 0$ and $c = 0$.

(27) If B commuted with every 2×2 matrix, then in particular B would commute with $\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$. So,

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} a+b & 0 \\ c+d & 0 \end{bmatrix} = \begin{bmatrix} a & b \\ a & b \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

and

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} a & a \\ c & c \end{bmatrix} = \begin{bmatrix} a+c & b+d \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Therefore, we must have $b = 0$ and $c = 0$.

Letting $b = c = 0$, we want

$$\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} \begin{bmatrix} s & t \\ u & v \end{bmatrix} = \begin{bmatrix} as & at \\ du & dv \end{bmatrix} = \begin{bmatrix} as & dt \\ au & dv \end{bmatrix} = \begin{bmatrix} s & t \\ u & v \end{bmatrix} \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}.$$

Comparing entries, we conclude that $au = du$ for all u and so $a = d$.