Math 314/814: Matrix Theory **Dr. S. Cooper, Fall 2008**

Homework Solutions – Week of September 16

Notes: The following remarks should be kept in mind when working on this homework set.

(a) Exercises Section 3.3: # 25, 27, 29, 33, 34, 35, 39 should be omitted from the course syllabus.

(b) Exercises Section 3.4: # 15, 17, 18, 21, 22, 28, 29, 31 should be completed during the week of September 23.

Section 3.3:

(3) $\det(A) = (3)(8) - (6)(4) = 0$ and so the given matrix A is not invertible.

(8) $\det(A) = (2.54)(0.8) - (8.128)(0.25) = 0$ and so the given matrix A is not invertible.

(12) Let A be the coefficient matrix of the system. So,

$$A = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}.$$

Let $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Note that $\det(A) = (1)(1) - (-1)(2) = 3 \neq 0$. Thus A is invertible and

$$A\mathbf{x} = \mathbf{b} \implies \mathbf{x} = A^{-1}\mathbf{b} = \frac{1}{3}\begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

(13) (a) We compute

$$A^{-1} = \frac{1}{2}\begin{bmatrix} 6 & -2 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ -1 & 1/2 \end{bmatrix}.$$

Thus,

$$A\mathbf{x} = \mathbf{b_1} \implies \mathbf{x} = A^{-1}\mathbf{b_1} = \begin{bmatrix} 3 & -1 \\ -1 & 1/2 \end{bmatrix}\begin{bmatrix} 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 4 \\ -1/2 \end{bmatrix}$$

and

$$A\mathbf{x} = \mathbf{b_2} \implies \mathbf{x} = A^{-1}\mathbf{b_2} = \begin{bmatrix} 3 & -1 \\ -1 & 1/2 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -5 \\ 2 \end{bmatrix}$$

and

$$A\mathbf{x} = \mathbf{b_3} \implies \mathbf{x} = A^{-1}\mathbf{b_3} = \begin{bmatrix} 3 & -1 \\ -1 & 1/2 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 6 \\ -2 \end{bmatrix}.$$

(b) We row reduce the multi-augmented matrix:

$$\begin{bmatrix} 1 & 2 & | & 3 & -1 & 2 \\ 2 & 6 & | & 5 & 2 & 0 \end{bmatrix} \xrightarrow{R_2 \to R_2 - 2R_1} \begin{bmatrix} 1 & 2 & | & 3 & -1 & 2 \\ 0 & 2 & | & -1 & 4 & -4 \end{bmatrix}$$

$$\xrightarrow{R_2 \to (1/2)R_2} \begin{bmatrix} 1 & 2 & | & 3 & -1 & 2 \\ 0 & 1 & | & -1/2 & 2 & -2 \end{bmatrix}$$

$$\xrightarrow{R_1 \to R_1 - 2R_2} \begin{bmatrix} 1 & 0 & | & 4 & -5 & 6 \\ 0 & 1 & | & -1/2 & 2 & -2 \end{bmatrix}$$

The solutions to the three systems are the 3rd, 4th, and 5th column vectors, respectively, of this last matrix.

(c) The method in part (b) uses fewer multiplications.

(17) (a) Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. One can check that

$$(AB)^{-1} = \begin{bmatrix} -7/2 & 3/2 \\ 3/2 & -1/2 \end{bmatrix} \neq A^{-1}B^{-1} = \begin{bmatrix} -2 & 3 \\ 3/2 & 1 \end{bmatrix}.$$

(b) We have

$$
\begin{aligned}
A^{-1}B^{-1} = (AB)^{-1} &\Leftrightarrow A^{-1}B^{-1} = B^{-1}A^{-1} \\
&\Leftrightarrow (AA^{-1})B^{-1} = AB^{-1}A^{-1} \\
&\Leftrightarrow IB^{-1} = AB^{-1}A^{-1} \\
&\Leftrightarrow B^{-1} = AB^{-1}A^{-1} \\
&\Leftrightarrow BB^{-1} = BAB^{-1}A^{-1} \\
&\Leftrightarrow I = BAB^{-1}A^{-1} \\
&\Leftrightarrow IA = BAB^{-1}(A^{-1}A) \\
&\Leftrightarrow A = BAB^{-1}I \\
&\Leftrightarrow A = BAB^{-1} \\
&\Leftrightarrow AB = BA(B^{-1}B) \\
&\Leftrightarrow AB = BAI \\
&\Leftrightarrow AB = BA
\end{aligned}
$$

So, $(AB)^{-1} = A^{-1}B^{-1}$ if and only if $AB = BA$ (i.e. if and only if A and B commute).

(19) Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. One can check that

$$
(A+B)^{-1} = \begin{bmatrix} 5 & -3 \\ -3 & 2 \end{bmatrix} \neq A^{-1} + B^{-1} = \begin{bmatrix} -1 & 0 \\ 3/2 & 1/2 \end{bmatrix}.
$$

(22)

$$
\begin{aligned}
(A^{-1}X)^{-1} = A(B^{-2}A)^{-1} \quad &\Leftrightarrow \quad X^{-1}(A^{-1})^{-1} = AA^{-1}(B^{-1}B^{-1})^{-1} \\
&\Leftrightarrow \quad X^{-1}A = (AA^{-1})(B^{-1})^{-1}(B^{-1})^{-1} \\
&\Leftrightarrow \quad X^{-1}A = I(BB) \\
&\Leftrightarrow \quad X^{-1}A = B^2 \\
&\Leftrightarrow \quad (XX^{-1})A = XB^2 \\
&\Leftrightarrow \quad IA = XB^2 \\
&\Leftrightarrow \quad A = XB^2 \\
&\Leftrightarrow \quad A(B^2)^{-1} = X[B^2(B^2)^{-1}] \\
&\Leftrightarrow \quad AB^{-2} = XI \\
&\Leftrightarrow \quad AB^{-2} = X
\end{aligned}
$$

(23)

$$\begin{aligned}
ABXA^{-1}B^{-1} = I + A \;&\Leftrightarrow\; ABXA^{-1}(B^{-1}B) = (I + A)B \\
&\Leftrightarrow\; ABXA^{-1}I = (I + A)B \\
&\Leftrightarrow\; ABXA^{-1} = (I + A)B \\
&\Leftrightarrow\; ABX(A^{-1}A) = (I + A)BA \\
&\Leftrightarrow\; ABXI = (I + A)BA \\
&\Leftrightarrow\; ABX = (I + A)BA \\
&\Leftrightarrow\; (A^{-1}A)BX = A^{-1}(I + A)BA \\
&\Leftrightarrow\; IBX = A^{-1}(I + A)BA \\
&\Leftrightarrow\; BX = A^{-1}(I + A)BA \\
&\Leftrightarrow\; (B^{-1}B)X = B^{-1}A^{-1}(I + A)BA \\
&\Leftrightarrow\; IX = B^{-1}A^{-1}(I + A)BA \\
&\Leftrightarrow\; X = (AB)^{-1}(I + A)BA \\
&\Leftrightarrow\; X = [(AB)^{-1}I + (AB)^{-1}A]BA \\
&\Leftrightarrow\; X = (AB)^{-1}BA + (AB)^{-1}ABA \\
&\Leftrightarrow\; X = (AB)^{-1}BA + B^{-1}(A^{-1}A)BA \\
&\Leftrightarrow\; X = (AB)^{-1}BA + (B^{-1}IB)A \\
&\Leftrightarrow\; X = (AB)^{-1}BA + (B^{-1}B)A \\
&\Leftrightarrow\; X = (AB)^{-1}BA + IA \\
&\Leftrightarrow\; X = (AB)^{-1}BA + A
\end{aligned}$$

(43) (a) A complete solution to this exercise can be found at the back of the text (page 681).

(b) Let $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. Then A is not invertible. Let $B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. Then $BA = CA = A$, but $B \neq C$.

(49) Let A be the given matrix. We row reduce

$$[A \mid I] = \left[\begin{array}{rr|rr} -2 & 4 & 1 & 0 \\ 3 & -1 & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{rr|rr} 1 & 0 & 1/10 & 2/5 \\ 0 & 1 & 3/10 & 1/5 \end{array}\right] = [I \mid A^{-1}].$$

Thus,

$$A^{-1} = \begin{bmatrix} 1/10 & 2/5 \\ 3/10 & 1/5 \end{bmatrix}.$$

(53) Let A be the given matrix. We row reduce

$$[A \mid I] = \left[\begin{array}{rrr|rrr} 1 & -1 & 2 & 1 & 0 & 0 \\ 3 & 1 & 2 & 0 & 1 & 0 \\ 2 & 3 & -1 & 0 & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{rrr|rrr} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & -3/4 & 1/4 & 0 \\ 0 & 0 & 0 & 7/20 & -1/4 & 1/5 \end{array}\right].$$

We see that the rank of A is 2, and hence A is not row equivalent to the identity matrix. That is, by the Fundamental Theorem of Invertible Matrices, A is not invertible.

Section 3.5:

(4)

$$S = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 : xy \geq 0 \right\}$$

is not a subspace of $\mathbb{R}^2$. To see this, note that $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is in S since $(1)(1) \geq 0$ and $\begin{bmatrix} -1/2 \\ -10 \end{bmatrix}$ is in S since $(-1/2)(-10) \geq 0$. However,

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} -1/2 \\ -10 \end{bmatrix} = \begin{bmatrix} 1/2 \\ -9 \end{bmatrix}$$

is not in S since $(1/2)(-9) < 0$.

(5)

$$S = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 : x = y = z \right\}$$

is a subspace of $\mathbb{R}^3$. We verify the definition of subspace:

(a) The zero vector $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ is in S since $0 = 0 = 0$.

(b) Let $\mathbf{x}$ and $\mathbf{y}$ be two vectors in S. Then there exist scalars a and b such that $\mathbf{x} = \begin{bmatrix} a \\ a \\ a \end{bmatrix}$ and $\mathbf{y} = \begin{bmatrix} b \\ b \\ b \end{bmatrix}$. Observe that

$$\mathbf{x} + \mathbf{y} = \begin{bmatrix} a+b \\ a+b \\ a+b \end{bmatrix}.$$

Since $a + b = a + b = a + b$, we have that $\mathbf{x} + \mathbf{y}$ is a vector in S.

(c) Let $\mathbf{x}$ be a vector in S and let $d \in \mathbb{R}$. Then there exists a scalar a such that $\mathbf{x} = \begin{bmatrix} a \\ a \\ a \end{bmatrix}$. Notice that

$$c\mathbf{x} = \begin{bmatrix} ca \\ ca \\ ca \end{bmatrix}.$$

Since $ca = ca = ca$, we have that $c\mathbf{x}$ is a vector in S.

(7)

$$S = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 : x - y + z = 1 \right\}$$

is not a subspace of $\mathbb{R}^3$. To see this, note that $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is in S since $1 - 0 + 0 = 1$.

However, $5 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \\ 0 \end{bmatrix}$ is not in S since $5 - 0 + 0 = 5 \neq 1$.

(11) $\mathbf{b}$ is in $col(A)$ if it is a linear combination of the columns of A. To see this we form an augmented matrix and row reduce:

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 3 \\ 1 & 1 & 1 & 2 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & 3 \\ 0 & 1 & 2 & -1 \end{array}\right]$$

We see that the associated linear system is consistent. Thus, $\mathbf{b}$ is in $col(A)$.

$\mathbf{w}$ is in $row(A)$ if it is a linear combination of the rows of A. To see if this is the case, we form the augmented matrix and row reduce

$$\left[\begin{array}{ccc} 1 & 0 & -1 \\ 1 & 1 & 1 \\ - & - & - \\ -1 & 1 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc} 1 & 0 & -1 \\ 0 & 1 & 2 \\ - & - & - \\ 0 & 0 & -2 \end{array}\right].$$

Since we do not obtain a row of zeros at the bottom, this system is inconsistent and so $\mathbf{w}$ is not in $row(A)$.

(15) Since

$$A\mathbf{v} = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \neq \mathbf{0}$$

we conclude that $\mathbf{v}$ is not in null(A).

(17) To find bases for $row(A)$ and $col(A)$, we row reduce A:

$$A \longrightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix} = B.$$

A basis for $row(A)$ is obtained by simply selecting the nonzero rows of B. Thus a basis for $row(A)$ is $\{[1 \quad 0 \quad -1], [0 \quad 1 \quad 2]\}$.

A basis for $col(A)$ is obtained by selecting the original columns of A which correspond to columns of B that have leading entries. Thus, a basis for $col(A)$ is $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$.

To find a basis for $null(A)$, we need to solve the homogeneous system $A\mathbf{x} = \mathbf{0}$. We row reduce the associated augmented matrix

$$[A \mid \mathbf{0}] \longrightarrow \left[\begin{array}{rrr|r} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right]$$

We let $x_3 = t \in \mathbb{R}$. Then we solve

$$\begin{aligned} x_2 &= -2t \\ x_1 &= t \end{aligned}$$

So,

$$null(A) = \left\{ t \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} : t \in \mathbb{R} \right\}.$$

A basis for $null(A)$ is then $\left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \right\}$.

(18) To find bases for $row(A)$ and $col(A)$, we row reduce A:

$$A \longrightarrow \begin{bmatrix} 1 & 0 & -7/2 \\ 0 & 1 & 1/2 \\ 0 & 0 & 0 \end{bmatrix} = B.$$

A basis for $row(A)$ is obtained by simply selecting the nonzero rows of B. Thus a basis for $row(A)$ is $\{[1 \quad 0 \quad -7/2], [0 \quad 1 \quad 1/2]\}$.

A basis for $col(A)$ is obtained by selecting the original columns of A which correspond to columns of B that have leading entries. Thus, a basis for $col(A)$ is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \right\}$.

To find a basis for $null(A)$, we need to solve the homogeneous system $A\mathbf{x} = \mathbf{0}$. We row reduce the associated augmented matrix

$$[A \mid \mathbf{0}] \longrightarrow \left[\begin{array}{rrr|r} 1 & 0 & -7/2 & 0 \\ 0 & 1 & 1/2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

We let $x_3 = t \in \mathbb{R}$. Then we solve

$$\begin{aligned} x_2 &= (-1/2)t \\ x_1 &= (7/2)x_3 = (7/2)t \end{aligned}$$

So,

$$null(A) = \left\{ t \begin{bmatrix} 7/2 \\ -1/2 \\ 1 \end{bmatrix} : t \in \mathbb{R} \right\}.$$

A basis for $null(A)$ is then $\left\{ \begin{bmatrix} 7/2 \\ -1/2 \\ 1 \end{bmatrix} \right\}$.

(21) Note that $row(A) = col(A^T)$ and $col(A) = row(A^T)$. We row reduce A^T:

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \\ -1 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

A basis for $row(A)$ is then obtained by finding a basis for $col(A^T)$. Thus, using A^T, we find that a basis for $row(A)$ is $\{[1 \quad 0 \quad -1], [1 \quad 1 \quad 1]\}$.

A basis for $col(A)$ is obtained by finding a basis for $row(A^T)$. So, using A^T, we find that a basis for $col(A)$ is $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$.

(22) Note that $row(A) = col(A^T)$ and $col(A) = row(A^T)$. We row reduce A^T:

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 2 & -1 \\ -3 & 1 & -4 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}.$$

A basis for $row(A)$ is then obtained by finding a basis for $col(A^T)$. Thus, using A^T, we find that a basis for $row(A)$ is $\{[1 \quad 1 \quad -3], [0 \quad 2 \quad 1]\}$.

A basis for $col(A)$ is obtained by finding a basis for $row(A^T)$. So, using A^T, we find that a basis for $col(A)$ is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right\}$.

(28) A basis for the span of the given vectors is obtained by finding a basis for $col(A)$ where A is the matrix whose columns are the given vectors. We row reduce A:

$$A = \begin{bmatrix} 1 & 1 & 0 & 2 \\ -1 & 2 & 1 & 1 \\ 1 & 0 & 1 & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 0 & 2 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 4 & 3 \end{bmatrix} = B.$$

Thus, a basis for the span of the given vectors is comprised of the original column vectors of A which correspond to the columns of B with leading entries.

$$\left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

(29) A basis for the span of the given vectors is obtained by finding a basis for $row(A)$ where A is the matrix whose rows are the given vectors. We row reduce A:

$$A = \begin{bmatrix} 2 & -3 & 1 \\ 1 & -1 & 0 \\ 4 & -4 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = B.$$

Thus, a basis for the span of the given vectors is comprised of the nonzero row vectors of B.

$$\{[1 \quad 0 \quad 0], [0 \quad 1 \quad 0], [0 \quad 0 \quad 1]\}.$$

(31) We continue with the same notation from (29). Since A and B are row equivalent, we have $row(A) = row(B)$. Thus, we can obtain a basis for A by selecting the rows of A which correspond to the rows of B selected for the basis in (29). Thus, another basis for the given vectors is

$$\{[2 \quad -3 \quad 1], [1 \quad -1 \quad 0], [4 \quad -4 \quad 1]\}.$$