Math 314/814: Matrix Theory **Dr. S. Cooper, Fall 2008**

Homework Solutions – Week of September 23

Note: Exercises Section 3.6: # 1, 4, 5, 9, 12, 13, 15, 17, 18, 19, 20, 21, 23, 24, 31, 37 should be completed during the week of September 30.

Section 3.5:

(35) We row reduce:

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix}.$$

Since $RREF(A)$ has two leading entries, we see that $rank(A) = 2$. By the Rank Theorem,

$$nullity(A) = 3 - rank(A) = 3 - 2 = 1.$$

(39) A complete solution to this exercise can be found at the back of the text (page 683).

(41) Since A is 3×5, we have that $rank(A) \leq min(3,5) = 3$. By the Rank Theorem,

$$nullity(A) = 5 - rank(A).$$

So, $nullity(A)$ is 2, 3, 4, or 5.

(42) Since A is 4×2, we have that $rank(A) \leq min(4,2) = 2$. By the Rank Theorem,

$$nullity(A) = 2 - rank(A).$$

So, $nullity(A)$ is 0, 1, or 2.

(46) We form a matrix whose columns are the given vectors and row reduce:

$$\begin{bmatrix} 1 & -1 & 1 \\ -1 & 5 & -3 \\ 3 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Since the rank of this matrix is 2 and not 3, the Fundamental Theorem of Invertible Matrices says that the given vectors do not form a basis for $\mathbb{R}^3$.

(49) A complete solution to this exercise can be found at the back of the text (page 683).

Section 3.6:

(1) We have

$$T_A(\mathbf{u}) = A\mathbf{u} = \begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 11 \end{bmatrix}$$

and

$$T_A(\mathbf{v}) = A\mathbf{v} = \begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 8 \\ 1 \end{bmatrix}.$$

(4) Let $\mathbf{u} = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$ be two vectors in $\mathbb{R}^2$. Then

$$\begin{aligned} T(\mathbf{u}+\mathbf{v}) &= T\begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \end{bmatrix} \\ &= \begin{bmatrix} -(y_1 + y_2) \\ (x_1 + x_2) + 2(y_1 + y_2) \\ 3(x_1 + x_2) - 4(y_1 + y_2) \end{bmatrix} \\ &= \begin{bmatrix} -y_1 \\ x_1 + 2y_1 \\ 3x_1 - 4y_1 \end{bmatrix} + \begin{bmatrix} -y_2 \\ x_2 + 2y_2 \\ 3x_2 - 4y_2 \end{bmatrix} \\ &= T\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + T\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} \\ &= T(\mathbf{u}) + T(\mathbf{v}) \end{aligned}$$

Now let $c \in \mathbb{R}$ and $\mathbf{u} = \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2$. Then

$$\begin{aligned} T(c\mathbf{u}) &= T\begin{bmatrix} cx \\ cy \end{bmatrix} \\ &= \begin{bmatrix} -cy \\ cx + 2cy \\ 3cx - 4cy \end{bmatrix} \\ &= c\begin{bmatrix} -y \\ x + 2y \\ 3x - 4y \end{bmatrix} \\ &= cT\begin{bmatrix} x \\ y \end{bmatrix} \\ &= cT(\mathbf{u}) \end{aligned}$$

(5) Let $\mathbf{u} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}, \mathbf{v} = \begin{bmatrix} d \\ e \\ f \end{bmatrix}$ be two vectors in $\mathbb{R}^3$. Then

$$\begin{aligned} T(\mathbf{u} + \mathbf{v}) &= T\begin{bmatrix} a + d \\ b + e \\ c + f \end{bmatrix} \\ &= \begin{bmatrix} (a + d) - (b + e) + (c + f) \\ 2(a + d) + (b + e) - 3(c + f) \end{bmatrix} \\ &= \begin{bmatrix} a - b + c \\ 2a + b - 3c \end{bmatrix} + \begin{bmatrix} d - e + f \\ 2d + e - 3f \end{bmatrix} \\ &= T\begin{bmatrix} a \\ b \\ c \end{bmatrix} + T\begin{bmatrix} d \\ e \\ f \end{bmatrix} \\ &= T(\mathbf{u}) + T(\mathbf{v}) \end{aligned}$$

Now let $c \in \mathbb{R}$ and $\mathbf{u} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3$. Then

$$\begin{aligned} T(c\mathbf{u}) &= T\begin{bmatrix} cx \\ cy \\ cz \end{bmatrix} \\ &= \begin{bmatrix} cx - cy + cz \\ 2cx + cy - 3cz \end{bmatrix} \\ &= c\begin{bmatrix} x - y + z \\ 2x + y - 3z \end{bmatrix} \\ &= cT\begin{bmatrix} x \\ y \\ z \end{bmatrix} \\ &= cT(\mathbf{u}) \end{aligned}$$

(9) Let $\mathbf{u} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Then

$$T(\mathbf{u}) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

and

$$T(\mathbf{v}) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

But,

$$T(\mathbf{u} + \mathbf{v}) = T\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \neq T(\mathbf{u}) + T(\mathbf{v}) = \begin{bmatrix} 0 \\ 2 \end{bmatrix}.$$

(12) We calculate

$$T\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}$$

and

$$T\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ -4 \end{bmatrix}.$$

So, the standard matrix is

$$[T] = \begin{bmatrix} 0 & -1 \\ 1 & 2 \\ 3 & -4 \end{bmatrix}.$$

(13) We find

$$T\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, T\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, T\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}.$$

So, the standard matrix is

$$[T] = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & -3 \end{bmatrix}.$$

(15) In general,

$$F\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -x \\ y \end{bmatrix}.$$

So,

$$F\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, F\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Thus, if F were linear, the standard matrix would be

$$[F] = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

We show that F is a matrix transformation by verifying that $F(\mathbf{v}) = [F]\mathbf{v}$:

$$F\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -x \\ y \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}.$$

(17) In general,

$$D\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x \\ 3y \end{bmatrix}.$$

So,

$$D\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}, D\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}.$$

Thus, if D were linear, the standard matrix would be

$$[D] = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}.$$

We show that D is a matrix transformation by verifying that $D(\mathbf{v}) = [D]\mathbf{v}$:

$$D\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x \\ 3y \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}.$$

(18) First note that $\mathbf{d} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is a direction vector for the line $y = x$. Observe that

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = x + y$$

and

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 2.$$

So, by Example 3.59,

$$P\begin{bmatrix} x \\ y \end{bmatrix} = \frac{x+y}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} (x+y)/2 \\ (x+y)/2 \end{bmatrix}.$$

So,

$$P\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}, P\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}.$$

Thus, if P were linear, the standard matrix would be

$$[P] = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}.$$

We show that P is a matrix transformation by verifying that $P(\mathbf{v}) = [P]\mathbf{v}$:

$$P\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} (x+y)/2 \\ (x+y)/2 \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}.$$

(19) Each transformation is defined by matrix multiplication:

$$T_{A_1}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} kx \\ y \end{bmatrix},$$

$$T_{A_2}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ ky \end{bmatrix},$$

$$T_{A_3}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ x \end{bmatrix},$$

$$T_{A_4}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x+ky \\ y \end{bmatrix},$$

and

$$T_{A_5}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ kx+y \end{bmatrix}.$$

Geometric descriptions and pictures to illustrate the results can be found at the back of the text (page 683).

(20) We let $\theta = 120°$ in Example 3.58. This gives the standard matrix

$$[R_{120}] = \begin{bmatrix} \cos 120° & -\sin 120° \\ \sin 120° & \cos 120° \end{bmatrix} = \begin{bmatrix} -1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{bmatrix}.$$

(21) Note that a clockwise rotation through 30° about the origin is the inverse of a 30° counterclockwise rotation. Thus, the standard matrix is the inverse of $[R_\theta]$ from Example 3.58 where $\theta = 30°$. That is, by Theorem 3.33,

$$[R_{-30}] = [(R_{30})^{-1}] = [R_{30}]^{-1} = \begin{bmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{bmatrix}^{-1} = \begin{bmatrix} \sqrt{3}/2 & 1/2 \\ -1/2 & \sqrt{3}/2 \end{bmatrix}.$$

(23) First note that $\mathbf{d} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is a direction vector for the line $y = -x$. Observe that

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = x - y$$

and

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 2.$$

So, by Example 3.59,

$$P \begin{bmatrix} x \\ y \end{bmatrix} = \frac{x - y}{2} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} (x-y)/2 \\ -(x-y)/2 \end{bmatrix}.$$

So,

$$P \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/2 \\ -1/2 \end{bmatrix}, P \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1/2 \\ 1/2 \end{bmatrix}.$$

Thus, the standard matrix is

$$[P] = \begin{bmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{bmatrix}.$$

(24) We have

$$T \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, T \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

So the standard matrix is

$$[T] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

(31) (a) We calculate

$$\begin{aligned}(S \circ T)\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= S\left(T\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) \\ &= S\begin{bmatrix} x_1 + 2x_2 \\ -3x_1 + x_2 \end{bmatrix} \\ &= \begin{bmatrix} (x_1 + 2x_2) + 3(-3x_1 + x_2) \\ (x_1 + 2x_2) - (-3x_1 + x_2) \end{bmatrix} \\ &= \begin{bmatrix} -8x_1 + 5x_2 \\ 4x_1 + x_2 \end{bmatrix}.\end{aligned}$$

(b) We first find

$$[S] = \begin{bmatrix} 1 & 3 \\ 1 & -1 \end{bmatrix}$$

and

$$[T] = \begin{bmatrix} 1 & 2 \\ -3 & 1 \end{bmatrix}.$$

So,

$$\begin{aligned}(S \circ T)\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= [S][T]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 3 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ &= \begin{bmatrix} -8 & 5 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ &= \begin{bmatrix} -8x_1 + 5x_2 \\ 4x_1 + x_2 \end{bmatrix}\end{aligned}$$

which equals the answer in part (a).

(37) Let T be the linear transformation given by reflection in the y-axis and S be the linear transformation which is clockwise rotation through 30°. We want to

find the standard matrix $[S \circ T]$. The standard matrices $[T]$ and $[S]$ were found in Exercises # 15 and 21, respectively. Therefore, by Theorem 3.32,

$$[S \circ T] = [S][T] = \begin{bmatrix} \sqrt{3}/2 & 1/2 \\ -1/2 & \sqrt{3}/2 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -\sqrt{3}/2 & 1/2 \\ 1/2 & \sqrt{3}/2 \end{bmatrix}.$$