Math 314/814: Matrix Theory **Dr. S. Cooper, Fall 2008**

Homework Solutions – Week of October 14

Note: You may not be able to complete the exercises from Section 4.4 until the week of Oct. 21.

Section 4.3:

(11) (a) The characteristic polynomial of A is

$$\begin{aligned}\det(A-\lambda I) &= \det\begin{bmatrix}(1-\lambda) & 0 & 0 & 0\\ 0 & (1-\lambda) & 0 & 0\\ 1 & 1 & (3-\lambda) & 0\\ -2 & 1 & 2 & (-1-\lambda)\end{bmatrix}\\ &= (1-\lambda)^2(3-\lambda)(-1-\lambda)\end{aligned}$$

(b) The eigenvalues of A are the roots of $\det(A-\lambda I)=0$: $\lambda_1=1, \lambda_2=3$ and $\lambda_3=-1$.

(c) To find a basis for E_1, we find the null space of $(A-1I)$:

$$[A-1I \mid \mathbf{0}] = \left[\begin{array}{cccc|c} 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ 1 & 1 & 2 & 0 & 0\\ -2 & 1 & 2 & -2 & 0\end{array}\right] \longrightarrow \left[\begin{array}{cccc|c} 1 & 1 & 2 & 0 & 0\\ 0 & 3 & 6 & -2 & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\end{array}\right].$$

Thus,

$$E_1 = span\left(\begin{bmatrix}-2/3\\ 2/3\\ 0\\ 1\end{bmatrix}, \begin{bmatrix}0\\ -2\\ 1\\ 0\end{bmatrix}\right).$$

A basis for E_1 is

$$\left\{\begin{bmatrix}-2/3\\ 2/3\\ 0\\ 1\end{bmatrix}, \begin{bmatrix}0\\ -2\\ 1\\ 0\end{bmatrix}\right\}.$$

We repeat the above process to find a basis for E_3, i.e. we find the null space of $(A-3I)$:

$$[A-3I \mid \mathbf{0}] = \begin{bmatrix} -2 & 0 & 0 & 0 & | & 0 \\ 0 & -2 & 0 & 0 & | & 0 \\ 1 & 1 & 0 & 0 & | & 0 \\ -2 & 1 & 2 & -4 & | & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 0 & 0 & | & 0 \\ 0 & 1 & 0 & 0 & | & 0 \\ 0 & 0 & 1 & -2 & | & 0 \\ 0 & 0 & 0 & 0 & | & 0 \end{bmatrix}.$$

Thus,

$$E_3 = span\left(\begin{bmatrix} 0 \\ 0 \\ 2 \\ 1 \end{bmatrix}\right).$$

A basis for E_3 is

$$\left\{\begin{bmatrix} 0 \\ 0 \\ 2 \\ 1 \end{bmatrix}\right\}.$$

Finally, to find a basis for E_{-1}, i.e. we find the null space of $(A-(-1)I)$:

$$[A+I \mid \mathbf{0}] = \begin{bmatrix} 2 & 0 & 0 & 0 & | & 0 \\ 0 & 2 & 0 & 0 & | & 0 \\ 1 & 1 & 4 & 0 & | & 0 \\ -2 & 1 & 2 & 0 & | & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 & 0 & | & 0 \\ 0 & 1 & 0 & 0 & | & 0 \\ 0 & 0 & 1 & 0 & | & 0 \\ 0 & 0 & 0 & 0 & | & 0 \end{bmatrix}.$$

Thus,

$$E_{-1} = span\left(\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}\right).$$

A basis for E_{-1} is

$$\left\{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}\right\}.$$

(d) The algebraic and geometric multiplicities for λ_2 and λ_3 are 1. The algebraic and geometric multiplicities for λ_1 are both 2.

(12) (a) The characteristic polynomial of A is

$$\begin{aligned}
\det(A - \lambda I) &= \det \begin{bmatrix} (4-\lambda) & 0 & 1 & 0 \\ 0 & (4-\lambda) & 1 & 1 \\ 0 & 0 & (1-\lambda) & 2 \\ 0 & 0 & 3 & (-\lambda) \end{bmatrix} \\
&= (4-\lambda) \det \begin{bmatrix} (4-\lambda) & 1 & 1 \\ 0 & (1-\lambda) & 2 \\ 0 & 3 & (-\lambda) \end{bmatrix} \\
&= (4-\lambda)(4-\lambda)[(1-\lambda)(-\lambda) - 6] \\
&= (4-\lambda)^2(\lambda - 3)(\lambda + 2)
\end{aligned}$$

(b) The eigenvalues of A are the roots of $\det(A - \lambda I) = 0$: $\lambda_1 = 4, \lambda_2 = 3$ and $\lambda_3 = -2$.

(c) To find a basis for E_4, we find the null space of $(A - 4I)$:

$$[A - 4I \mid \mathbf{0}] = \left[\begin{array}{cccc|c} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & -3 & 2 & 0 \\ 0 & 0 & 3 & -4 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cccc|c} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

Thus,

$$E_4 = span\left(\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right).$$

A basis for E_4 is

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}.$$

We repeat the above process to find a basis for E_3, i.e. we find the null space of $(A - 3I)$:

$$[A-3I \mid \mathbf{0}] = \begin{bmatrix} 1 & 0 & 1 & 0 & | & 0 \\ 0 & 1 & 1 & 1 & | & 0 \\ 0 & 0 & -2 & 2 & | & 0 \\ 0 & 0 & 3 & -3 & | & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 1 & 0 & | & 0 \\ 0 & 1 & 1 & 1 & | & 0 \\ 0 & 0 & 1 & -1 & | & 0 \\ 0 & 0 & 0 & 0 & | & 0 \end{bmatrix}.$$

Thus,

$$E_3 = span\left(\begin{bmatrix} -1 \\ -2 \\ 1 \\ 1 \end{bmatrix}\right).$$

A basis for E_3 is

$$\left\{\begin{bmatrix} -1 \\ -2 \\ 1 \\ 1 \end{bmatrix}\right\}.$$

Finally, to find a basis for E_{-2}, i.e. we find the null space of $(A-(-2)I)$:

$$[A+2I \mid \mathbf{0}] = \begin{bmatrix} 6 & 0 & 1 & 0 & | & 0 \\ 0 & 6 & 1 & 1 & | & 0 \\ 0 & 0 & 3 & 2 & | & 0 \\ 0 & 0 & 3 & 2 & | & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 6 & 0 & 1 & 0 & | & 0 \\ 0 & 6 & 1 & 1 & | & 0 \\ 0 & 0 & 3 & 2 & | & 0 \\ 0 & 0 & 0 & 0 & | & 0 \end{bmatrix}.$$

Thus,

$$E_{-2} = span\left(\begin{bmatrix} 1/9 \\ -1/18 \\ -2/3 \\ 1 \end{bmatrix}\right).$$

A basis for E_{-2} is

$$\left\{\begin{bmatrix} 1/9 \\ -1/18 \\ -2/3 \\ 1 \end{bmatrix}\right\}.$$

(d) The algebraic and geometric multiplicities for λ_2 and λ_3 are 1. The algebraic and geometric multiplicities for λ_1 are both 2.

(17) We first observe that $\mathbf{x} = \mathbf{v_1} - \mathbf{v_2} + 2\mathbf{v_3}$. So,

$$\begin{aligned} A^{20}\mathbf{x} &= \left(-\frac{1}{3}\right)^{20} \mathbf{v_1} - \left(\frac{1}{3}\right)^{20} \mathbf{v_2} + 2(1)^{20}\mathbf{v_3} \\ &= \begin{bmatrix} 2 \\ 2-(1/3)^{20} \\ 2 \end{bmatrix} \end{aligned}$$

(18) Since $\mathbf{x} = \mathbf{v_1} - \mathbf{v_2} + 2\mathbf{v_3}$, we have

$$A^k\mathbf{x} = \left(-\frac{1}{3}\right)^{k} \mathbf{v_1} - \left(\frac{1}{3}\right)^{k} \mathbf{v_2} + 2(1)^{k}\mathbf{v_3}.$$

Thus, as $k \to \infty$, we have

$$A^k\mathbf{x} \to 2\mathbf{v_3} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}.$$

(19) (a) By Theorem 3.4 and Theorem 4.10,

$$\begin{aligned} \det(A-\lambda I) &= \det(A-\lambda I)^T \\ &= \det[A^T - (\lambda I)^T] \\ &= \det(A^T - \lambda I^T) \\ &= \det(A^T - \lambda I) \end{aligned}$$

Since the roots of the characteristic equation give the eigenvalues of a matrix, A and A^T have the same eigenvalues.

(b) Let

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}.$$

Then

$$\det(A-\lambda I) = -\lambda(1-\lambda).$$

It's easy to verify that the eigenspace E_0 of A has basis

$$\left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}.$$

Now

$$A^T = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}.$$

By part (a), A^T also has the eigenvalue $\lambda = 0$. One can check that the eigenspace E_0 of A^T has basis

$$\left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}.$$

We see that even though A and A^T have the same eigenvalue $\lambda = 0$, the associated eigenspaces for the two matrices are not the same.

(21) Suppose $A^2 = A$. Let λ be an eigenvalue of A. Then there exists a non-zero vector $\mathbf{x}$ such that $A\mathbf{x} = \lambda\mathbf{x}$. Thus,

$$A\mathbf{x} = \lambda\mathbf{x} \implies A\mathbf{x} = A^2\mathbf{x} = \lambda A\mathbf{x} \implies \lambda = 0 \text{ or } \lambda = 1.$$

Thus, the only possible values for the eigenvalues of A are $\lambda = 0$ or $\lambda = 1$.

(24) (a) Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. One can check that the eigenvalues of A are $\lambda_1 = 1$ and $\lambda_2 = 0$. Also, the eigenvalues of B are $\beta_1 = 0$ and $\beta_2 = 1$.

We have

$$A + B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

The only eigenvalue of $A + B$ is $\lambda = 1$.

Observe that $\lambda_2 + \beta_1 = 0 + 0 = 0$ is not an eigenvalue of $A + B$.

(b) Let A and B be as in part (a). Then

$$AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

The only eigenvalue of AB is $\lambda = 0$.

Observe that $\lambda_1\beta_2 = (1)(1) = 1$ is not an eigenvalue of AB.

(c) We are told that $A\mathbf{x} = \lambda\mathbf{x}$ and $B\mathbf{x} = \mu\mathbf{x}$ for some non-zero vector $\mathbf{x}$. Thus,

$$(A + B)\mathbf{x} = A\mathbf{x} + B\mathbf{x} = \lambda\mathbf{x} + \mu\mathbf{x} = (\lambda + \mu)\mathbf{x}$$

which shows that $\lambda + \mu$ is an an eigenvalue of $A + B$.

Similarly,

$$(AB)\mathbf{x} = A(B\mathbf{x}) = A(\mu\mathbf{x}) = \mu(A\mathbf{x}) = \mu(\lambda\mathbf{x}) = \lambda\mu\mathbf{x}$$

which shows that $\lambda\mu$ is an eigenvalue of AB.

Section 4.4:

(1) We have

$$\det(A - \lambda I) = (4 - \lambda)(1 - \lambda) - 3 = \lambda^2 - 5\lambda + 1 \neq (1 - \lambda)^2 = \det(B - \lambda I).$$

That is, A and B have different characteristic polynomials. By Theorem 4.22, A and B are not similar.

(2) We have

$$\det(A - \lambda I) = (3 - \lambda)(7 - \lambda) - 5 = \lambda^2 - 10\lambda + 16$$

which does not equal

$$\det(B - \lambda I) = (2 - \lambda)(6 - \lambda) + 4 = \lambda^2 - 8\lambda + 16 = \det(B - \lambda I).$$

That is, A and B have different characteristic polynomials. By Theorem 4.22, A and B are not similar.

(5) The eigenvalues of A are $\lambda_1 = 4$ and $\lambda_2 = 3$.

E_4 has basis

$$\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}.$$

E_3 has basis

$$\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}.$$

(6) A has eigenvalues $\lambda_1 = 2, \lambda_2 = 0$, and $\lambda_3 = -1$.

E_2 has basis

$$\left\{ \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} \right\}.$$

E_0 has basis

$$\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \right\}.$$

E_{-1} has basis

$$\left\{ \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right\}.$$

(7) A has eigenvalues $\lambda_1 = 6$ and $\lambda_2 = -2$.

E_6 has basis

$$\left\{ \begin{bmatrix} 3 \\ 2 \\ 3 \end{bmatrix} \right\}.$$

E_{-2} has basis

$$\left\{ \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right\}.$$

(9) We first find the charateristic polynomial of A:

$$\det(A - \lambda I) = \det \begin{bmatrix} (-3-\lambda) & 4 \\ -1 & (1-\lambda) \end{bmatrix} = (-3-\lambda)(1-\lambda) + 4 = \lambda^2 + 2\lambda + 1.$$

The only root of $\det(A - \lambda I) = 0$ is $\lambda = -1$, and so the only eigenvalue of A is $\lambda = -1$.

To find E_{-1} we solve the system $(A - (-1)I)\mathbf{x} = \mathbf{0}$:

$$[(A+I) \mid \mathbf{0}] = \left[\begin{array}{cc|c} -2 & 4 & 0 \\ -1 & 2 & 0 \end{array} \right] \to \left[\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 0 & 0 \end{array} \right].$$

Solving the system, we see that

$$E_{-1} = span\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right).$$

So, a basis for E_{-1} is

$$\left\{\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right\}.$$

We see that the algebraic multiplicity of $\lambda = -1$ is 2, yet the geometric multiplicity of $\lambda = -1$ is $\dim(E_{-1}) = 1$. Thus, A is not diagonalizable (by Theorem 4.27).

(12) We begin by finding the characteristic polynomial of A. We have

$$\begin{aligned} \det(A - \lambda I) &= \det\begin{bmatrix} (1-\lambda) & 0 & 0 \\ 2 & (2-\lambda) & 1 \\ 3 & 0 & (1-\lambda) \end{bmatrix} \\ &= (1-\lambda)\det\begin{bmatrix} (2-\lambda) & 1 \\ 0 & (1-\lambda) \end{bmatrix} \\ &= (1-\lambda)^2(2-\lambda) \end{aligned}$$

The roots of $\det(A - \lambda I) = 0$ are the eigenvalues of A. Thus, A has eigenvalues $\lambda_1 = 1$ and $\lambda_2 = 2$.

To find E_1 we solve the system $(A - 1I)\mathbf{x} = \mathbf{0}$:

$$[(A - I) \mid \mathbf{0}] = \left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 2 & 1 & 1 & 0 \\ 3 & 0 & 0 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Solving the system, we see that

$$E_1 = span\left(\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}\right).$$

So, a basis for E_1 is

$$\left\{ \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \right\}.$$

We see that the algebraic multiplicity of $\lambda_1 = 1$ is 2, yet the geometric multiplicity of $\lambda_1 = 1$ is $\dim(E_1) = 1$. Thus, A is not diagonalizable (by Theorem 4.27).

(15) We begin by finding the characteristic polynomial of A. We have

$$\begin{aligned}
\det(A - \lambda I) &= \det \begin{bmatrix} (2-\lambda) & 0 & 0 & 4 \\ 0 & (2-\lambda) & 0 & 0 \\ 0 & 0 & (-2-\lambda) & 0 \\ 0 & 0 & 0 & (-2-\lambda) \end{bmatrix} \\
&= (2-\lambda)\det \begin{bmatrix} (2-\lambda) & 0 & 0 \\ 0 & (-2-\lambda) & 0 \\ 0 & 0 & (-2-\lambda) \end{bmatrix} \\
&= (2-\lambda)^2(-2-\lambda)^2
\end{aligned}$$

The roots of $\det(A - \lambda I) = 0$ are the eigenvalues of A. Thus, A has eigenvalues $\lambda_1 = 2$ and $\lambda_2 = -2$.

To find E_2 we solve the system $(A - 2I)\mathbf{x} = \mathbf{0}$:

$$[(A - 2I) \mid \mathbf{0}] = \left[\begin{array}{cccc|c} 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & -4 & 0 \end{array}\right] \to \left[\begin{array}{cccc|c} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

Solving the system, we see that

$$E_2 = span\left(\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right).$$

So, a basis for E_2 is

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}.$$

To find E_{-2} we solve the system $(A - (-2)I)\mathbf{x} = \mathbf{0}$:

$$[(A + 2I) \mid \mathbf{0}] = \begin{bmatrix} 4 & 0 & 0 & 4 & \mid & 0 \\ 0 & 4 & 0 & 0 & \mid & 0 \\ 0 & 0 & 0 & 0 & \mid & 0 \\ 0 & 0 & 0 & 0 & \mid & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 1 & \mid & 0 \\ 0 & 1 & 0 & 0 & \mid & 0 \\ 0 & 0 & 0 & 0 & \mid & 0 \\ 0 & 0 & 0 & 0 & \mid & 0 \end{bmatrix}.$$

Solving the system, we see that

$$E_{-2} = span \left(\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right).$$

So, a basis for E_{-2} is

$$\left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

Since the algebraic multiplicities equal the geometric multiplicities for each of the eigenvalues of A, we conclude that A is diagonalizable. That is, if we let

$$D = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix}$$

and

$$P = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

then $P^{-1}AP = D$.

(17) Let $A = \begin{bmatrix} -1 & 6 \\ 1 & 0 \end{bmatrix}$. We begin by finding the characteristic polynomial of A:

$$\det(A - \lambda I) = \det \begin{bmatrix} (-1-\lambda) & 6 \\ 1 & (-\lambda) \end{bmatrix} = (-1-\lambda)(-\lambda) - 6 = (\lambda+3)(\lambda-2).$$

We see that the eigenvalues of A are $\lambda_1 = -3$ and $\lambda_2 = 2$.

The eigenspace E_{-3} is the null space of $A + 3I$:

$$[A + 3I \mid \mathbf{0}] = \begin{bmatrix} 2 & 6 & \mid & 0 \\ 1 & 3 & \mid & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & \mid & 0 \\ 0 & 0 & \mid & 0 \end{bmatrix}.$$

We see that

$$E_{-3} = \mathit{span}\left(\begin{bmatrix} -3 \\ 1 \end{bmatrix}\right).$$

Thus, E_{-3} has basis

$$\left\{\begin{bmatrix} -3 \\ 1 \end{bmatrix}\right\}.$$

The eigenspace E_2 is the null space of $A - 2I$:

$$[A - 2I \mid \mathbf{0}] = \begin{bmatrix} -3 & 6 & \mid & 0 \\ 1 & -2 & \mid & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & \mid & 0 \\ 0 & 0 & \mid & 0 \end{bmatrix}.$$

We see that

$$E_2 = \mathit{span}\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right).$$

Thus, E_2 has basis

$$\left\{\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right\}.$$

We conclude that A is diagonalizable. That is, $A = PDP^{-1}$ where

$$P = \begin{bmatrix} -3 & 2 \\ 1 & 1 \end{bmatrix}$$

and

$$D = \begin{bmatrix} -3 & 0 \\ 0 & 2 \end{bmatrix}.$$

Thus,

$$\begin{aligned} A^{10} &= (PDP^{-1})^{10} = PD^{10}P^{-1} \\ &= \begin{bmatrix} -3 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} (-3)^{10} & 0 \\ 0 & 2^{10} \end{bmatrix} \begin{bmatrix} -1/5 & 2/5 \\ 1/5 & 3/5 \end{bmatrix} \\ &= \begin{bmatrix} -3 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 59049 & 0 \\ 0 & 1024 \end{bmatrix} \begin{bmatrix} -1/5 & 2/5 \\ 1/5 & 3/5 \end{bmatrix} \\ &= \begin{bmatrix} 35839 & -69630 \\ -11605 & 24234 \end{bmatrix} \end{aligned}$$

(18) Let $A = \begin{bmatrix} 4 & -3 \\ -1 & 2 \end{bmatrix}$. We begin by finding the characteristic polynomial of A:

$$\det(A - \lambda I) = \det \begin{bmatrix} (4-\lambda) & -3 \\ -1 & (2-\lambda) \end{bmatrix} = (4-\lambda)(2-\lambda) - 3 = (\lambda - 5)(\lambda - 1).$$

We see that the eigenvalues of A are $\lambda_1 = 5$ and $\lambda_2 = 1$.

The eigenspace E_5 is the null space of $A - 5I$:

$$[A - 5I \mid \mathbf{0}] = \left[\begin{array}{cc|c} -1 & -3 & 0 \\ -1 & -3 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 3 & 0 \\ 0 & 0 & 0 \end{array}\right].$$

We see that

$$E_5 = span\left(\begin{bmatrix} -3 \\ 1 \end{bmatrix}\right).$$

Thus, E_5 has basis

$$\left\{\begin{bmatrix} -3 \\ 1 \end{bmatrix}\right\}.$$

The eigenspace E_1 is the null space of $A - 1I$:

$$[A - I \mid \mathbf{0}] = \left[\begin{array}{cc|c} 3 & -3 & 0 \\ -1 & 1 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right].$$

We see that

$$E_1 = span\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right).$$

Thus, E_1 has basis

$$\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}.$$

We conclude that A is diagonalizable. That is, $A = PDP^{-1}$ where

$$P = \begin{bmatrix} -3 & 1 \\ 1 & 1 \end{bmatrix}$$

and

$$D = \begin{bmatrix} 5 & 0 \\ 0 & 1 \end{bmatrix}.$$

Since $A = PDP^{-1}$, we know that $A^{-1} = [PDP^{-1}]^{-1} = PD^{-1}P^{-1}$. So, $A^{-6} = (A^{-1})^6 = P(D^{-1})^6P^{-1}$.

Thus,

$$\begin{aligned} A^{-6} &= (PDP^{-1})^{-6} = P(D^{-1})^6P^{-1} \\ &= \begin{bmatrix} -3 & 1 \\ 1 & 1 \end{bmatrix}\left(\begin{bmatrix} 5 & 0 \\ 0 & 1 \end{bmatrix}\right)^{-1}\begin{bmatrix} -1/4 & 1/4 \\ 1/4 & 3/4 \end{bmatrix} \\ &= \begin{bmatrix} -3 & 1 \\ 1 & 1 \end{bmatrix}\left(\begin{bmatrix} 1/5 & 0 \\ 0 & 1 \end{bmatrix}\right)^{6}\begin{bmatrix} -1/4 & 1/4 \\ 1/4 & 3/4 \end{bmatrix} \\ &= \begin{bmatrix} -3 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} (1/5)^6 & 0 \\ 0 & 1^6 \end{bmatrix}\begin{bmatrix} -1/4 & 1/4 \\ 1/4 & 3/4 \end{bmatrix} \\ &= \begin{bmatrix} -3 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1/15625 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -1/4 & 1/4 \\ 1/4 & 3/4 \end{bmatrix} \\ &= \begin{bmatrix} 3907/15625 & 11718/15265 \\ 3906/15625 & 11719/15265 \end{bmatrix} \end{aligned}$$

(21) Since the given matrix A is triangular, its eigenvalues are the diagonal entries: $\lambda_1 = 1$ and $\lambda_2 = -1$.

To find E_1 we solve the system $(A - 1I)\mathbf{x} = \mathbf{0}$:

$$[(A-I) \mid \mathbf{0}] = \begin{bmatrix} 0 & 1 & 1 & | & 0 \\ 0 & -2 & 0 & | & 0 \\ 0 & 0 & -2 & | & 0 \end{bmatrix} \to \begin{bmatrix} 0 & 1 & 0 & | & 0 \\ 0 & 0 & 1 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}.$$

Solving the system, we see that

$$E_1 = span\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right).$$

So, a basis for E_1 is

$$\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right\}.$$

To find E_{-1} we solve the system $(A - (-1)I)\mathbf{x} = \mathbf{0}$:

$$[(A+I) \mid \mathbf{0}] = \begin{bmatrix} 2 & 1 & 1 & | & 0 \\ 0 & 0 & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}.$$

Solving the system, we see that

$$E_{-1} = span\left(\begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}\right).$$

So, a basis for E_{-1} is

$$\left\{\begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}\right\}.$$

Since the algebraic multiplicities equal the geometric multiplicities for each of the eigenvalues of A, we conclude that A is diagonalizable. That is, if we let

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

and

$$P = \begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

then $P^{-1}AP = D$. So,

$$\begin{aligned} A^{2002} &= PD^{2002}P^{-1} \\ &= P \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}^{2002} P^{-1} \\ &= P \begin{bmatrix} 1^{2002} & 0 & 0 \\ 0 & 1^{2002} & 0 \\ 0 & 0 & 1^{2002} \end{bmatrix} P^{-1} \\ &= PIP^{-1} \\ &= PP^{-1} \\ &= I \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

(34) Suppose A and B are invertible matrices. Let $P = B^{-1}$. Then

$$P^{-1}(AB)P = (B^{-1})^{-1}(AB)B^{-1} = (BA)(BB^{-1}) = BAI = BA.$$

So, by definition, AB and BA are similar matrices.

(37) We first concentrate on the matrix A. We have

$$\det(A-\lambda I) = \det \begin{bmatrix} (5-\lambda) & -3 \\ 4 & (-2-\lambda) \end{bmatrix} = (5-\lambda)(-2-\lambda)+12 = (\lambda-2)(\lambda-1).$$

The eigenvalues of A are $\lambda_1 = 2$ and $\lambda_2 = 1$. Since each eigenvalue has both algebraic multiplicity 1, each eigenvalue has geometric multiplicity 1 (by Theorem 4.26). We conclude that A is diagonalizable. So, A is similar to

$$D = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}.$$

We now repeat this argument on B. We have

$$\det(B-\lambda I) = \det\begin{bmatrix} (-1-\lambda) & 1 \\ -6 & (4-\lambda) \end{bmatrix} = (-1-\lambda)(4-\lambda)+6 = (\lambda-2)(\lambda-1).$$

The eigenvalues of B are $\lambda_1 = 2$ and $\lambda_2 = 1$. As was the case with A, we have that B is similar to

$$D = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}.$$

Since A and B are similar to the same diagonal matrix D, we must have that A and B are similar (by Theorem 4.21).

To find the desired matrix P we need to find the eigenspaces for A and B. Using the techniques from this section, one finds that A has eigenspaces

$$E_2 = span\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$$

and

$$E_1 = span\left(\begin{bmatrix} 3 \\ 4 \end{bmatrix}\right).$$

So, $A = SDS^{-1}$ where $S = \begin{bmatrix} 1 & 3 \\ 1 & 4 \end{bmatrix}$.

Similarly, B has eigenspaces

$$E_2' = span\left(\begin{bmatrix} 1 \\ 3 \end{bmatrix}\right)$$

and

$$E_1' = span\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right).$$

So, $B = QDQ^{-1}$ where $Q = \begin{bmatrix} 1 & 1 \\ 3 & 2 \end{bmatrix}$.

Rearranging the equation $A = SDS^{-1}$ to isolate for D we see that

$$D = S^{-1}AS.$$

Substituting this into the equation $B = QDQ^{-1}$, we have

$$B = QDQ^{-1} = QS^{-1}ASQ^{-1}.$$

Now let $P = SQ^{-1}$. This yields

$$B = P^{-1}AP.$$

We conclude that the desired matrix P is

$$\begin{aligned} P = SQ^{-1} &= \begin{bmatrix} 1 & 3 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} -2 & 1 \\ 3 & -1 \end{bmatrix} \\ &= \begin{bmatrix} 7 & -2 \\ 10 & -3 \end{bmatrix} \end{aligned}$$

(41) Let A be an $n \times n$ diagonalizable matrix. By Theorem 4.27, we know that the algebraic and geometric multiplicities for each eigenvalue of A are equal.

We want to show that A^T is also diagonalizable. This will follow from a series of observations.

By Section 4.3, exercise 19, we know that A and A^T have the same characteristic polynomial. This shows that A and A^T have the same eigenvalues with the same algebraic multiplicities. Let λ be an eigenvalue of A^T (and hence A). We need to show that the algebraic multiplicity of λ is equal to its geometric multiplicity. We have two eigenspaces associated to λ; one for A and one for A^T. Let E_λ denote the null space of $A - \lambda I$ and E'_λ denote the null space of $A^T - \lambda I$.

Observe that $(A - \lambda I)^T = A^T - \lambda I$. Thus, by Theorem 3.25,

$$Rank(A^T - \lambda I) = Rank(A - \lambda I)^T = Rank(A - \lambda I).$$

So, by the Rank Theorem,

$$\begin{aligned} n &= Rank(A^T - \lambda I) + Nullity(A^T - \lambda I) \\ n &= Rank(A - \lambda I) + Nullity(A - \lambda I) \end{aligned}$$

We conclude that $Nullity(A^T - \lambda I) = Nullity(A - \lambda I)$. That is, $\dim(E_\lambda) = \dim(E'_\lambda)$. This shows that the geometric multiplicities are equal for each shared eigenvalue of A and A^T. Thus, since the multiplicities are equal for A, the algebraic and geometric multiplicities for each eigenvalue of A^T must be equal. This shows that A^T is diagonalizable.