Math 314/814: Matrix Theory **Dr. S. Cooper, Fall 2008**

Homework Solutions – Week of October 23

Section 4.6:

(7) We know that $\lambda = 1$ is an eigenvalue of the given matrix P and that L has 2 equal columns which is the probability vector given by the eigenvector with eigenvalue 1 . We need to find E_1:

$$[P - I \mid \mathbf{0}] = \left[\begin{array}{cc|c} -2/3 & 1/6 & 0 \\ 2/3 & -1/6 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 2/3 & -1/6 & 0 \\ 0 & 0 & 0 \end{array}\right].$$

So, if $\mathbf{x}$ is in the null space of $P - I$, then

$$\mathbf{x} = \begin{bmatrix} 1/4t \\ t \end{bmatrix}$$

for any $t \in \mathbb{R}$. Letting $t = 4$, we see that E_1 has basis

$$\left\{ \begin{bmatrix} 1 \\ 4 \end{bmatrix} \right\}.$$

We now turn this into a probability vector

$$\begin{bmatrix} \frac{1}{1+4} \\ \frac{4}{1+4} \end{bmatrix} = \begin{bmatrix} 1/5 \\ 4/5 \end{bmatrix}.$$

Therefore,

$$L = \begin{bmatrix} 1/5 & 1/5 \\ 4/5 & 4/5 \end{bmatrix}.$$

(8) We know that $\lambda = 1$ is an eigenvalue of the given matrix P and that L has 3 equal columns which is the probability vector given by the eigenvector with eigenvalue 1 . We need to find E_1:

$$[P - I \mid \mathbf{0}] = \left[\begin{array}{ccc|c} -1/2 & 1/3 & 1/6 & 0 \\ 1/2 & -1/2 & 1/3 & 0 \\ 0 & 1/6 & -1/2 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1/2 & -1/2 & 1/3 & 0 \\ 0 & 1/6 & -1/2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

So, if $\mathbf{x}$ is in the null space of $P - I$, then

$$\mathbf{x} = \begin{bmatrix} 7/3t \\ 3t \\ t \end{bmatrix}$$

for any $t \in \mathbb{R}$. Letting $t = 3$, we see that E_1 has basis

$$\left\{ \begin{bmatrix} 7 \\ 9 \\ 3 \end{bmatrix} \right\}.$$

We now turn this into a probability vector

$$\begin{bmatrix} \frac{7}{7+9+3} \\ \frac{9}{7+9+3} \\ \frac{3}{7+9+3} \end{bmatrix} = \begin{bmatrix} 7/19 \\ 9/19 \\ 3/19 \end{bmatrix}.$$

Therefore,

$$L = \begin{bmatrix} 7/19 & 7/19 & 7/19 \\ 9/19 & 9/19 & 9/19 \\ 3/19 & 3/19 & 3/19 \end{bmatrix}.$$

(57) Let $A = \begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix}$. Then

$$A \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix}.$$

We concentrate on A! We begin by finding the eigenvalues of A:

$$\det(A - \lambda I) = \det \begin{bmatrix} (1-\lambda) & 3 \\ 2 & (2-\lambda) \end{bmatrix} = (1-\lambda)(2-\lambda) - 6 = (\lambda - 4)(\lambda + 1).$$

We conclude that A has eigenvalues $\lambda_1 = 4$ and $\lambda_2 = -1$.

To find the eigenspace E_4 we find the null space of $A - 4I$:

$$[(A - 4I) \mid \mathbf{0}] = \left[\begin{array}{rr|r} -3 & 3 & 0 \\ 2 & -2 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{rr|r} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right].$$

We see that

$$E_4 = span\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right).$$

Thus, a basis for E_4 is

$$\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}.$$

To find the eigenspace E_{-1} we find the null space of $A + I$:

$$[(A+I) \mid \mathbf{0}] = \begin{bmatrix} 2 & 3 & | & 0 \\ 2 & 3 & | & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 2 & 3 & | & 0 \\ 0 & 0 & | & 0 \end{bmatrix}.$$

We see that

$$E_{-1} = span\left(\begin{bmatrix} 3 \\ -2 \end{bmatrix}\right).$$

Thus, a basis for E_{-1} is

$$\left\{\begin{bmatrix} 3 \\ -2 \end{bmatrix}\right\}.$$

We see that A is diagonalizable! That is, let

$$P = \begin{bmatrix} 1 & 3 \\ 1 & -2 \end{bmatrix}$$

and

$$D = \begin{bmatrix} 4 & 0 \\ 0 & -1 \end{bmatrix}.$$

Then $P^{-1}AP = D$ or, equivalently, $P^{-1}A = DP^{-1}$.

To solve our system of differential equations we now consider the functions u and v which satisfy the equations

$$\begin{bmatrix} u \\ v \end{bmatrix} = P^{-1}\begin{bmatrix} x \\ y \end{bmatrix}.$$

Then

$$\begin{aligned}
\begin{bmatrix} u' \\ v' \end{bmatrix} &= P^{-1}\begin{bmatrix} x' \\ y' \end{bmatrix} \\
&= P^{-1}A\begin{bmatrix} x \\ y \end{bmatrix} \\
&= DP^{-1}\begin{bmatrix} x \\ y \end{bmatrix} \\
&= D\begin{bmatrix} u \\ v \end{bmatrix}
\end{aligned}$$

That is,

$$\begin{bmatrix} u' \\ v' \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 4u \\ -v \end{bmatrix}.$$

In other words,

$$\begin{aligned}
u' &= 4u \\
v' &= -v
\end{aligned}$$

We conclude that

$$\begin{aligned}
u &= C_1e^{4t} \\
v &= C_2e^{-t}
\end{aligned}$$

for some scalars $C_1, C_2 \in \mathbb{R}$. Thus

$$\begin{aligned}
\begin{bmatrix} x \\ y \end{bmatrix} &= P\begin{bmatrix} u \\ v \end{bmatrix} \\
&= \begin{bmatrix} 1 & 3 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} C_1e^{4t} \\ C_2e^{-t} \end{bmatrix} \\
&= \begin{bmatrix} C_1e^{4t} + 3C_2e^{-t} \\ C_1e^{4t} - 2C_2e^{-t} \end{bmatrix}
\end{aligned}$$

This tells us that

$$\begin{aligned}
x &= C_1e^{4t} + 3C_2e^{-t} \\
y &= C_1e^{4t} - 2C_2e^{-t}
\end{aligned}$$

To complete the exercise, we need to solve the system with the initial conditions $x(0) = 0$ and $y(0) = 5$. This gives us the system

$$\begin{aligned} x(0) = 0 &= C_1 + 3C_2 \\ y(0) = 5 &= C_1 - 2C_2 \end{aligned}$$

We row reduce the augmented matrix:

$$\begin{bmatrix} 1 & 3 & | & 0 \\ 1 & -2 & | & 5 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 3 & | & 0 \\ 0 & 1 & | & -1 \end{bmatrix}.$$

We see that $C_1 = 3$ and $C_2 = -1$. Thus,

$$\begin{aligned} x &= 3e^{4t} - 3e^{-t} \\ y &= 3e^{4t} + 2e^{-t} \end{aligned}$$

Section 5.1:

(1) We check that the involved dot products are all 0:

$$\begin{aligned} \begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix} &= (-3)(2) + (1)(4) + (2)(1) = 0 \\ \begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} &= (-3)(1) + (1)(-1) + (2)(2) = 0 \\ \begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} &= (2)(1) + (4)(-1) + (1)(2) = 0 \end{aligned}$$

We conclude that the given set of vectors is orthogonal.

(5) We check that the involved dot products are all 0:

$$\begin{bmatrix} 2 \\ 3 \\ -1 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} -2 \\ 1 \\ -1 \\ 0 \end{bmatrix} = (2)(-2) + (3)(1) + (-1)(-1) + (4)(0) = 0$$
$$\begin{bmatrix} 2 \\ 3 \\ -1 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} -4 \\ -6 \\ 2 \\ 7 \end{bmatrix} = (2)(-4) + (3)(-6) + (-1)(2) + (4)(7) = 0$$
$$\begin{bmatrix} -2 \\ 1 \\ -1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} -4 \\ -6 \\ 2 \\ 7 \end{bmatrix} = (-2)(-4) + (1)(-6) + (-1)(2) + (0)(7) = 0$$

We conclude that the given set of vectors is orthogonal.

(9) We have

$$\begin{aligned} \mathbf{v_1} \cdot \mathbf{v_2} &= (1)(1) + (0)(2) + (-1)(1) = 0 \\ \mathbf{v_1} \cdot \mathbf{v_3} &= (1)(1) + (0)(-1) + (-1)(1) = 0 \\ \mathbf{v_2} \cdot \mathbf{v_3} &= (1)(1) + (2)(-1) + (1)(1) = 0 \end{aligned}$$

We conclude that the set $\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$ is an orthogonal set of vectors. By Theorem 5.1, $\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$ is a linearly independent set. By the Fundamental Theorem of Invertible Matrices, any set of 3 linearly independent vectors forms a basis for $\mathbb{R}^3$. Thus, $\mathcal{B} = \{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$ is a basis for $\mathbb{R}^3$.

By Theorem 5.2,

$$\mathbf{w} = c_1\mathbf{v_1} + c_2\mathbf{v_2} + c_3\mathbf{v_3}$$

where

$$c_i = \frac{\mathbf{w} \cdot \mathbf{v_i}}{\mathbf{v_i} \cdot \mathbf{v_i}}$$

for $i = 1, 2, 3$.

We calculate

$$\begin{aligned}
\mathbf{w} \cdot \mathbf{v_1} &= (1)(1) + (1)(0) + (1)(-1) = 0 \\
\mathbf{w} \cdot \mathbf{v_2} &= (1)(1) + (1)(2) + (1)(1) = 4 \\
\mathbf{w} \cdot \mathbf{v_3} &= (1)(1) + (1)(-1) + (1)(1) = 1 \\
\mathbf{v_1} \cdot \mathbf{v_1} &= (1)(1) + (0)(0) + (-1)(-1) = 2 \\
\mathbf{v_2} \cdot \mathbf{v_2} &= (1)(1) + (2)(2) + (1)(1) = 6 \\
\mathbf{v_3} \cdot \mathbf{v_3} &= (1)(1) + (-1)(-1) + (1)(1) = 3
\end{aligned}$$

So,

$$\begin{aligned}
c_1 &= \frac{0}{2} = 0 \\
c_2 &= \frac{4}{6} = \frac{2}{3} \\
c_3 &= \frac{1}{3}
\end{aligned}$$

That is,

$$\mathbf{w} = 0\mathbf{v_1} + \frac{2}{3}\mathbf{v_2} + \frac{1}{3}\mathbf{v_3}.$$

By definition,

$$[\mathbf{w}]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 2/3 \\ 1/3 \end{bmatrix}.$$

(13) Call the given vectors $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}$, respectively.

We calculate

$$\begin{aligned}
\mathbf{v_1} \cdot \mathbf{v_2} &= (1/3)(2/3) + (2/3)(-1/3) + (2/3)(0) = 0 \\
\mathbf{v_1} \cdot \mathbf{v_3} &= (1/3)(1) + (2/3)(2) + (2/3)(-5/2) = 0 \\
\mathbf{v_2} \cdot \mathbf{v_3} &= (2/3)(1) + (-1/3)(2) + (0)(-5/2) = 0
\end{aligned}$$

Thus, the given set of vectors is orthogonal.

Now, we calculate

$$\begin{aligned}
\mathbf{v_1} \cdot \mathbf{v_1} &= (1/3)(1/3) + (2/3)(2/3) + (2/3)(2/3) = 1 \\
\mathbf{v_2} \cdot \mathbf{v_2} &= (2/3)(2/3) + (-1/3)(-1/3) + (0)(0) = 5/9 \neq 1 \\
\mathbf{v_3} \cdot \mathbf{v_3} &= (1)(1) + (2)(2) + (-5/2)(-5/2) = 45/4 \neq 1
\end{aligned}$$

Since $\mathbf{v_2} \cdot \mathbf{v_2} \neq 1$ and $\mathbf{v_3} \cdot \mathbf{v_3} \neq 1$, the given set is not orthonormal. To create an orthonormal set of vectors from those given, we need to normalize $\mathbf{v_2}$ and $\mathbf{v_3}$. Let $\mathbf{u_2}$ and $\mathbf{u_3}$ be the vectors:

$$\mathbf{u_2} := \frac{1}{||\mathbf{v_2}||}\mathbf{v_2} = \frac{1}{\sqrt{5/9}}\mathbf{v_2} = \frac{3}{\sqrt{5}}\mathbf{v_2} = \begin{bmatrix} 2/\sqrt{5} \\ -1/\sqrt{5} \\ 0 \end{bmatrix}$$

and

$$\mathbf{u_3} := \frac{1}{||\mathbf{v_3}||}\mathbf{v_3} = \frac{1}{\sqrt{45/4}}\mathbf{v_3} = \frac{2}{3\sqrt{5}}\mathbf{v_3} = \begin{bmatrix} 2/(3\sqrt{5}) \\ 4/(3\sqrt{5}) \\ -5/(3\sqrt{5}) \end{bmatrix}$$

Thus, the vectors $\mathbf{v_1}, \mathbf{u_2}, \mathbf{u_3}$ is an orthonormal set of vectors.

(37) (a) Since $\{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}\}$ is an orthonormal basis for $\mathbb{R}^n$, Theorem 5.3 says that we have the linear combinations

$$\begin{aligned}
\mathbf{x} &= (\mathbf{x} \cdot \mathbf{v_1})\mathbf{v_1} + \cdots + (\mathbf{x} \cdot \mathbf{v_n})\mathbf{v_n} \\
\mathbf{y} &= (\mathbf{y} \cdot \mathbf{v_1})\mathbf{v_1} + \cdots + (\mathbf{y} \cdot \mathbf{v_n})\mathbf{v_n}
\end{aligned}$$

So,

$$\mathbf{x} \cdot \mathbf{y} = [(\mathbf{x} \cdot \mathbf{v_1})\mathbf{v_1} + \cdots + (\mathbf{x} \cdot \mathbf{v_n})\mathbf{v_n}] \cdot [(\mathbf{y} \cdot \mathbf{v_1})\mathbf{v_1} + \cdots + (\mathbf{y} \cdot \mathbf{v_n})\mathbf{v_n}]$$

By distributivity, a term of $\mathbf{x} \cdot \mathbf{y}$ looks like

$$[(\mathbf{x} \cdot \mathbf{v_j})(\mathbf{y} \cdot \mathbf{v_i})]\mathbf{v_j} \cdot \mathbf{v_i}.$$

But, the vectors $\mathbf{v_1}, \ldots, \mathbf{v_n}$ are orthonormal, and so if $j \neq i$ then $\mathbf{v_j} \cdot \mathbf{v_i} = 0$ and if $j = i$ then $\mathbf{v_j} \cdot \mathbf{v_i} = 1$. This shows that

$$\mathbf{x} \cdot \mathbf{y} = (\mathbf{x} \cdot \mathbf{v_1})(\mathbf{y} \cdot \mathbf{v_1}) + \cdots + (\mathbf{x} \cdot \mathbf{v_n})(\mathbf{y} \cdot \mathbf{v_n}).$$

(b) By Theorem 5.3,

$$\begin{aligned} \mathbf{x} &= (\mathbf{x}\cdot\mathbf{v_1})\mathbf{v_1} + \cdots + (\mathbf{x}\cdot\mathbf{v_n})\mathbf{v_n} \\ \mathbf{y} &= (\mathbf{y}\cdot\mathbf{v_1})\mathbf{v_1} + \cdots + (\mathbf{y}\cdot\mathbf{v_n})\mathbf{v_n} \end{aligned}$$

That is,

$$[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} \mathbf{x}\cdot\mathbf{v_1} \\ \mathbf{x}\cdot\mathbf{v_2} \\ \vdots \\ \mathbf{x}\cdot\mathbf{v_n} \end{bmatrix}$$

and

$$[\mathbf{y}]_{\mathcal{B}} = \begin{bmatrix} \mathbf{y}\cdot\mathbf{v_1} \\ \mathbf{y}\cdot\mathbf{v_2} \\ \vdots \\ \mathbf{y}\cdot\mathbf{v_n} \end{bmatrix}.$$

We see that

$$[\mathbf{x}]_{\mathcal{B}}\cdot[\mathbf{y}]_{\mathcal{B}} = (\mathbf{x}\cdot\mathbf{v_1})(\mathbf{y}\cdot\mathbf{v_1}) + \cdots + (\mathbf{x}\cdot\mathbf{v_n})(\mathbf{y}\cdot\mathbf{v_n}) = \mathbf{x}\cdot\mathbf{y}.$$