

Using Math to Understand Our World
Project 5
Building Up Savings And Debt

Note: You will have to hand in answers to all numbered questions in the “Project Description”. See the “What to Hand In” sheet for additional materials to submit.

1 Introduction

During the January Workshop, oh so long ago, we talked about compound interest. Recall that if you deposit a principal P into a savings account with an interest rate of $r\%$ compounded n times a year, the amount of money you have after the first compounding period is

$$P + \frac{r}{n}P = P \left(1 + \frac{r}{n}\right).$$

After two compounding periods it is

$$P \left(1 + \frac{r}{n}\right) + P \left(1 + \frac{r}{n}\right) \frac{r}{n};$$

that is, the previous balance, $P(1 + \frac{r}{n})$, plus the interest on that balance, $P(1 + \frac{r}{n})\frac{r}{n}$. We could also write this as

$$P \left(1 + \frac{r}{n}\right) \left(1 + \frac{r}{n}\right) = P \left(1 + \frac{r}{n}\right)^2.$$

What is the balance after k compounding periods?

Now suppose, as is often the case, that each compounding period we make a deposit of amount v to the account (v for value). This is often how retirement funds work - you start with an initial investment and then have a certain amount deducted from your paycheck each month to add to it. Of course, the deposits really may not happen as often as the compounding, but the math is easier if they do, so let's go with that for now. So suppose we start with an initial investment P , an interest rate of r , and suppose we add v to the account at the end of each compounding period. Then at the end of the first compounding period we have a balance of

$$P + \frac{r}{n}P + v = P \left(1 + \frac{r}{n}\right) + v$$

(the principal plus the interest plus v) . It's a little bit easier if we let s stand for $1 + \frac{r}{n}$. Then we would write our first balance as $Ps + v$.

1. Show that at the end of the second compounding period, the balance is $(Ps + v)s + v$. Multiply this out so that you have no parentheses remaining. Be sure to show all the steps.
2. What is the balance after the third compounding period? Again multiply it out. Be sure to show all the steps.
3. Do you see a pattern starting to emerge? Can you use this pattern to predict the balance after 10 compounding periods? After 100 compounding periods (you don't have to write it all out)? After k compounding periods?

2 Finding A Usable Formula

The formula you got above is not very usable. I mean, if you wanted to compute your balance at the end of 40 years, it would be a real pain. But there is a way to simplify this!

2.1 The Geometric Series

The sum

$$1 + z + z^2 + z^3 + \dots + z^k$$

is part of what is known as the *geometric series*, which is very famous in mathematics. The entire geometric series is

$$1 + z + z^2 + z^3 + \dots + z^k + z^{k+1} + \dots$$

where we just keep going on and on forever, so you have a sum of an infinite number of things. You might think that such a sum would be infinite, but, as long as $z < 1$, it's not! For example, use your calculator to help you decide what the geometric series would be if $z = 1/2$; that is, what is

$$1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^4 + \dots?$$

The sum of $1 + z + z^2 + z^3 + \dots + z^k$ is sometimes called the *finite geometric series*.

There is actually a very simple formula for the sum of the geometric series and for the sum of a finite geometric series. The formula is tricky to find, especially for the infinite series, but let's see if we can figure it out for the finite series. First multiply the entire finite series by $(z - 1)$; i.e., consider

$$(1 + z + z^2 + z^3 + \dots + z^k) \cdot (z - 1).$$

Multiply it out; that is, multiply each term in the series by z and then multiply each term by -1 . Is there some cancelling you can do? What happens? Can you use this to get a simple formula for the finite geometric series itself?

1. Show your formula and its step by step derivation. The final form for your formula should be $1 + z + z^2 + \dots + z^k =$ a nice expression.
2. Use your formula to compute the following

$$1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^4 + \dots + \left(\frac{1}{2}\right)^{16}.$$

3. What is

$$1 + \frac{1}{3} + \left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^3 + \left(\frac{1}{3}\right)^4 + \dots + \left(\frac{1}{3}\right)^{10}?$$

Verify this with your calculator.

4. This is not immediately applicable to our project, but can you make a guess as to the formula for the infinite geometric series

$$1 + z + z^2 + z^3 + \dots + z^k + z^{k+1} + \dots$$

in the case when $z < 1$? (Why does z have to be less than 1?) Try out your formula for

$$1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^4 + \dots$$

How does this compare to your answer in number 2? Are the answers exactly the same? a little bit different? a lot different? Explain.

The infinite geometric series is only tangentially related to our problem of computing balance, but it is so famous that I couldn't resist showing it to you. And it's kind of cool that you can add up an infinite number of numbers and get a finite answer. Now you can answer a famous puzzle that Archimedes wondered about. Suppose a frog starts at point A and hops 1 unit to the right. Then she hops half a unit to the right, then she hops a fourth of a unit to the right. Each hop is half the distance of the previous hop. How far does she get in the end (or would she get if she lived forever)?

OK, so now we will stick to the finite geometric series for the rest of the project.

2.2 Putting it all Together

1. Use your formula for the finite geometric series to write simple formulas for the expressions in problem 3 from Part 1 (the *Introduction*).
2. Suppose you start an account with an initial deposit of \$1000 and add \$100 a month where the interest is compounded monthly at an interest rate of 8% (quite obtainable from a stock portfolio, or at least it used to be). How much money will you have after 20 years? How much money have you put into the account over that time? What if you hadn't made the monthly deposits, but just made the initial deposit and let the interest accumulate? That is, in this case, how much money would you have after 20 years and how much money would you have made? Try not to round too much. When you have large exponents, a little rounding can make a big difference.

3. Suppose you began the savings plan above when you were 25 (again paying in \$100 a month). What would your balance be when you retired at age 65? How much money would you have paid in during that time?
4. Suppose you began the savings plan above for your child at birth. What would the balance be when your child retired at age 65? How much money would you have paid in?
5. In each of the scenarios 2-4, how much would you have to deposit each month to have a million dollars in the end?

You can read about these and other formulas by going to

www.moneychimp.com.

Start by clicking on “compound interest”.

3 Paying Out

Once you reach retirement age you often stop making payments into your retirement account and start withdrawing money at regular intervals to support yourself. At the same time, of course, you’ll continue to earn interest on whatever remains in the account. It’s important to figure out how much you can take out of the account and still have enough to last for the rest of your life. Suppose you start with a balance of R when you retire, and your interest rate is still r , and you take out an amount w at the beginning of each compounding period.

1. Find a formula that shows how much money you have left in the account after k compounding periods. Your work will be very similar to what you did above. Start by writing down the balance after one compounding period, two compounding periods, three compounding periods. Multiply things out and see if you can spot a pattern. You should get a sum $z + z^2 + z^3 + \dots + z^n$ in your formula which you can simplify as you did above. (You might find it useful to think of $z + z^2 + z^3 + \dots + z^n$ as $z(1 + z + z^2 + \dots + z^{n-1})$.)
2. If you start with \$600,000 and you take out \$48,000 a year and the interest rate is 6%, compounded quarterly, how long will your retirement savings last? If you need it to last 30 years, how much can you take out each quarter? each year?

4 Credit Card Debt

Now let’s talk about credit card debt. Suppose you currently have a balance of P dollars on your credit card and you pay off w dollars at the beginning of each month. Suppose the interest rate is r , usually compounded monthly.

1. Find a formula that tells you the balance on your card after m months provided you don’t charge anything else to the card in the meantime. This will be very similar to what you did above.

2. If your credit card balance is \$10,000 and your interest rate is 14% (compounded monthly) and you pay off \$200 a month, how long will it take you to pay it off (provided you don't charge anything else in the meantime)? How much money will you pay in total? What if you pay only \$100 a month? What if you continue to charge to your card, say at a rate of \$50 a month? (P.S. if your balance is \$10,000 and the interest rate is 14% compounded monthly, how much interest does the credit card company charge you the first month?)

5 The National Debt

It's quite complicated to discuss national debt correctly. The government borrows from many many different lenders, each with a different interest rate. But you can imagine how quickly debt can grow when the government does not pay off at least the accrued interest each month. You can check the current national debt at

www.brillig.com/debt_clock/

6 Using Up Oil Reserves

In the January Workshop we learned that world oil consumption is currently growing at a rate of 2.3% per year. In 2000, world oil consumption was about 27,740,000,000 barrels. The total amount believed to be in the earth is about 1027 billion barrels.

1. At this rate, how long will it be until we run out of oil?

7 How Many Grains Of Rice?

Remember the Chinese emperor who was so taken with the game of chess that he offered the inventor anything he wanted? The inventor asked for one grain of rice on the first square of a chessboard, two on the second, 4 on the third, 8 on the 4th, and so on, doubling the amount each square. A chess board has 64 squares.

1. Can you use the ideas in this project to develop a simple formula that will tell us how many grains of rice the emperor of China would have had to have given the inventor of chess (show your work)? What if he had only had to go up to square 32 instead?

Albert Einstein called compound interest the 8th Wonder of the World.