Problem Set 2
Due: Thursday, February 3

Work all of the following problems. Remember, you are encouraged to work together on Problem Sets, but each student must turn in his or her own write-up. Be sure to adhere to the Rules and Expectations outlined in the Course Information Sheet.

1 Traditional Problems

1. (Gallian, Chapter 2 Exercises, #26) Prove that if $(ab)^2 = a^2b^2$ in a group G, then $ab = ba$.

2. (Gallian, Chapter 2 Exercises, #39) Let

$$G = \left\{ \begin{bmatrix} a & a \\ a & a \end{bmatrix} \mid a \in \mathbb{R}, a \neq 0 \right\}.$$

   Show that G is a group under matrix multiplication. (Notice that each element of G has an inverse even though the matrices have 0 determinant!)

3. Let G be a group. For elements $a, b \in G$, define $a \sim b$ if there is some $x \in G$ with $a = xbx^{-1}$. Prove that $\sim$ is an equivalence relation on G.

4. (Gallian, Chapter 3 Exercises, #18) Let H and K be subgroups of the group G. Prove that $H \cap K$ is a subgroup of G.

5. (Gallian, Chapter 3 Exercises, #60) Let G be a finite group with more than one element. Show that G has an element of prime order.

6. (Gallian, Supplementary Exercises for Chapters 1–4, #2) Let G be a group and let H be a subgroup of G. For any fixed $x \in G$, define

$$xHx^{-1} = \{xhx^{-1} \mid h \in H\}.$$

   Define

$$N(H) = \{x \in G \mid xHx^{-1} = H\}.$$

   Prove that $N(H)$ (called the *normalizer* of H) is a subgroup of G.

2 Computer Problems

As outlined on Problem Set 0, please intersperse your GAP commands and output with your explanations. You should create a log file as described in Chapter -1 of the lab manual. If you type up your solutions, you can cut and paste from this log file into your solution file; please use a different font so it is easy to tell what is what. If you hand-write your solutions, you should still print out your log file; then physically cut and paste it into your solutions.

1. Use the GAP commands `Size` and `ulist` (see Chapter 2 of the lab manual) to determine the order of the group $U(p^a)$ for various odd primes p and positive integers a. For example taking $p = 3$ and $a = 1$ means you compute the order of $U(3)$, and taking $p = 5$ and $a = 3$ means you compute the order of $U(125)$. Do enough examples so that you feel comfortable making a conjecture about the order of $U(p^a)$ where p is an odd prime and a is a positive integer. Carefully state your conjecture and do a few more examples to make sure you believe it. (You do not need to prove your conjecture.) Then use GAP to decide whether your conjecture applies also to the group $U(2^a)$ where a is a positive integer. Explain.

2. Let r and s be relatively prime positive integers, i.e., let r and s be positive integers with $\gcd(r, s) = 1$. Use GAP to help you make a conjecture about the order of the group $U(rs)$ in terms of the orders of the groups $U(r)$ and $U(s)$. Again, do enough examples so that you feel comfortable making a conjecture, carefully state your conjecture, and then do a few more examples to make sure you believe it. (You do not need to prove your conjecture.)

3. Use the GAP commands `ulist` and `cyclic` (see Chapter 3 of the lab manual) to determine whether the group $U(n)$ is cyclic for various values of n of the form p^a where p is an odd prime and a is a positive integer. As usual, do enough examples so that you feel comfortable making a conjecture, carefully state your conjecture, and then do a few more examples to make sure you believe it. (You do not need to prove your conjecture.) Then use GAP to decide whether your conjecture applies also to the group $U(2^a)$ where a is a positive integer. Explain.