Problem Set 4
Due: Thursday, February 17

Work all of the following problems. Remember, you are encouraged to work together on Problem Sets, but each student must turn in his or her own write-up. Be sure to adhere to the Rules and Expectations outlined in the Course Information Sheet.

1 Traditional Problems

1. (Gallian, Chapter 4 Exercises, #35) Determine the subgroup lattice for $\mathbb{Z}_{p^n}$, where p is a prime and n is some positive integer.
2. (Gallian, Chapter 4 Exercises, #36) Prove that a finite group is the union of proper subgroups if and only if the group is not cyclic.
3. (Gallian, Supplementary Exercises for Chapters 1–4, #34) Suppose that G is a group that has exactly one nontrivial proper subgroup. Prove that G is cyclic and $|G| = p^2$, where p is prime.
4. (Gallian, Supplementary Exercises for Chapters 1–4, #38) If p is an odd prime, prove that there is no group that has exactly p elements of order p.

2 Computer Problems

As outlined on Problem Set 0, please intersperse your GAP commands and output with your explanations. You should create a log file as described in Chapter −1 of the lab manual. If you type up your solutions, you can cut and paste from this log file into your solution file; please use a different font so it is easy to tell what is what. If you hand-write your solutions, you should still print out your log file; then physically cut and paste it into your solutions.

1. Let $G = \langle a \rangle$ be the cyclic group of order 30 generated by the element a. Since G is cyclic of order 30, we know that every subgroup of G cyclic, that there is a subgroup of G of order d if and only if d divides 30, and, when d divides 30, the subgroup of G having order d is unique. Use GAP to find a generator for the smallest subgroup H of G containing:
   - (a) a^4 and a^6
   - (b) a^{10} and a^2
   - (c) a^{15} and a^2
   - (d) a^9 and a^{12}
   - (e) a^8 and a^{12}
2. Fill in the blank in the following conjecture: If $G = \langle a \rangle$ is a cyclic group of order n, then the smallest subgroup containing the elements a^i and a^j is $\langle a^t \rangle$, where $t =$ ____________. You do not need to prove your conjecture. (Do more examples if you need to.)

3. Test your conjecture from Computer Problem (2) by repeating Computer Problem (1) with $n = 60$.

Hints for the Computer Problems: The command

```
c30 := CyclicGroup(IsPermGroup,30);
```

sets up the cyclic group of order 30 as all powers of the 30-cycle $(1, 2, \ldots, 30)$. The command

```
a := c30.1;
```

tells GAP to assign the name a to this 30-cycle. The command

```
h := Subgroup(c30,[a^4,a^6]);
```

sets up h to be the smallest subgroup of $c30$ containing the elements a^4 and a^6. Once you've done this, the command

```
Size(h);
```

will return $|h|$, and the command

```
h = Subgroup(c30,[a^3]);
```

will return either "true" or "false", depending on whether h is the same subgroup as $\langle a^3 \rangle$. (Note the difference between "=" and ":="!)