Math 314/814–006: Matrix Theory **Dr. S. Cooper, Fall 2008**

Name: ____

Quiz 10 Solutions

This is a take-home quiz. You are allowed to use your class notes and text, but no other resources (including books, internet, or people). This is due in class on Thursday, November 6. No late submissions will be accepted.

Please write up your solutions to the following exercises. You should write legibly and fully explain your work. Staple your pages together with this page as the cover – remember to write your full name at the top.

Exercises:

• Solve the system of differential equations

$$\begin{aligned} x_1'(t) &= (7/2)x_1(t) - 3x_2(t) \\ x_2'(t) &= (3/2)x_1(t) - x_2(t) \end{aligned}$$

with the initial conditions $x_1(0) = 4$ and $x_2(0) = 3$.

• Section 5.1: # 10, 12

• Let Q be an orthogonal matrix. Prove that $\det(Q) = \pm 1$.

Solutions:

• Let A be the matrix

$$A = \begin{bmatrix} 7/2 & -3 \\ 3/2 & -1 \end{bmatrix}$$

Also, define the vectors

$$\mathbf{x}(\mathbf{t}) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

and

$$\mathbf{x}'(\mathbf{t}) = \begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix}.$$

We diagonalize the matrix A:

$$\det(A-\lambda I) = \det\begin{bmatrix} (7/2-\lambda) & -3 \\ 3/2 & (-1-\lambda) \end{bmatrix} = \lambda^2 - (5/2)\lambda + 1 = (\lambda-2)(\lambda-1/2).$$

We see that A has eigenvalues $\lambda_1 = 2$ and $\lambda_2 = 1/2$.

To find a basis for the eigenspace E_2, we solve the system $(A-2I)\mathbf{x} = \mathbf{0}$:

$$[A-2I \mid \mathbf{0}] = \left[\begin{array}{cc|c} 3/2 & -3 & 0 \\ 3/2 & -3 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 3/2 & -3 & 0 \\ 0 & 0 & 0 \end{array}\right].$$

We see that E_2 has basis

$$\left\{\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right\}.$$

To find a basis for the eigenspace $E_{1/2}$, we solve the system $[A-(1/2)I]\mathbf{x} = \mathbf{0}$:

$$[A-(1/2)I \mid \mathbf{0}] = \left[\begin{array}{cc|c} 3 & -3 & 0 \\ 3/2 & -3/2 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right].$$

We see that $E_{1/2}$ has basis

$$\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}.$$

We conclude that A is diagonalizable. That is, $D = P^{-1}AP$, where

$$P = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

and

$$D = \begin{bmatrix} 2 & 0 \\ 0 & 1/2 \end{bmatrix}.$$

To solve the system, we let

$$\mathbf{u}(\mathbf{t}) = \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} = P^{-1}\mathbf{x}(\mathbf{t}).$$

Then, as shown in class,

$$\mathbf{u}'(\mathbf{t}) = P^{-1}\mathbf{x}'(\mathbf{t}) = P^{-1}A\mathbf{x}(\mathbf{t}) = DP^{-1}\mathbf{x}(\mathbf{t}) = D\mathbf{u}(\mathbf{t}).$$

That is,

$$\begin{bmatrix} u_1'(t) \\ u_2'(t) \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 1/2 \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} = \begin{bmatrix} 2u_1(t) \\ (1/2)u_2(t) \end{bmatrix}.$$

We conclude that

$$\begin{aligned} u_1(t) &= C_1 e^{2t} \\ u_2(t) &= C_2 e^{(1/2)t} \end{aligned}$$

for some constants C_1, C_2. Therefore,

$$\mathbf{x(t)} = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = P\mathbf{u(t)} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} C_1 e^{2t} \\ C_2 e^{(1/2)t} \end{bmatrix} = \begin{bmatrix} 2C_1 e^{2t} + C_2 e^{(1/2)t} \\ C_1 e^{2t} + C_2 e^{(1/2)t} \end{bmatrix}.$$

So,

$$\begin{aligned} x_1(t) &= 2C_1 e^{2t} + C_2 e^{(1/2)t} \\ x_2(t) &= C_1 e^{2t} + C_2 e^{(1/2)t} \end{aligned}$$

To complete the exercise, we need to solve the system with the given initial conditions. Substituting $t = 0$ into x_1 and x_2, we have

$$\begin{aligned} 4 &= 2C_1 + C_2 \\ 3 &= C_1 + C_2 \end{aligned}$$

Solving this system, we see that $C_1 = 1$ and $C_2 = 2$. This yields

$$\begin{aligned} x_1(t) &= 2e^{2t} + 2e^{(1/2)t} \\ x_2(t) &= e^{2t} + 2e^{(1/2)t} \end{aligned}$$

Section 5.1:

(10) We first calculate the relevant dot products:

$$\begin{aligned} \mathbf{v_1} \cdot \mathbf{v_2} &= (1)(1) + (1)(-1) + (1)(0) = 0 \\ \mathbf{v_1} \cdot \mathbf{v_3} &= (1)(1) + (1)(1) + (1)(-2) = 0 \\ \mathbf{v_2} \cdot \mathbf{v_3} &= (1)(1) + (-1)(1) + (0)(-2) = 0 \end{aligned}$$

Thus, $\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$ is an orthogonal set. So, by Theorem 5.1, the set $\mathcal{B} = \{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$ is linearly independent. Since any three linearly independent vectors in $\mathbb{R}^3$ form a basis for $\mathbb{R}^3$, the set $\mathcal{B} = \{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$ is therefore a basis for $\mathbb{R}^3$.

To write $\mathbf{w}$ as a linear combination of the vectors in $\mathcal{B}$, we need to find the following dot products:

$$\begin{aligned}
\mathbf{w} \cdot \mathbf{v_1} &= (1)(1) + (2)(1) + (3)(1) = 6 \\
\mathbf{w} \cdot \mathbf{v_2} &= (1)(1) + (2)(-1) + (3)(0) = -1 \\
\mathbf{w} \cdot \mathbf{v_3} &= (1)(1) + (2)(1) + (3)(-2) = -3 \\
\mathbf{v_1} \cdot \mathbf{v_1} &= (1)(1) + (1)(1) + (1)(1) = 3 \\
\mathbf{v_2} \cdot \mathbf{v_2} &= (1)(1) + (-1)(-1) + (0)(0) = 2 \\
\mathbf{v_3} \cdot \mathbf{v_3} &= (1)(1) + (1)(1) + (-2)(-2) = 6
\end{aligned}$$

Thus, by Theorem 5.2,

$$\begin{aligned}
\mathbf{w} &= \left(\frac{\mathbf{w} \cdot \mathbf{v_1}}{\mathbf{v_1} \cdot \mathbf{v_1}}\right)\mathbf{v_1} + \left(\frac{\mathbf{w} \cdot \mathbf{v_2}}{\mathbf{v_2} \cdot \mathbf{v_2}}\right)\mathbf{v_2} + \left(\frac{\mathbf{w} \cdot \mathbf{v_3}}{\mathbf{v_3} \cdot \mathbf{v_3}}\right)\mathbf{v_3} \\
&= 2\mathbf{v_1} - \frac{1}{2}\mathbf{v_2} - \frac{1}{2}\mathbf{v_3}
\end{aligned}$$

So,

$$[\mathbf{w}]_{\mathcal{B}} = \begin{bmatrix} 2 \\ -1/2 \\ -1/2 \end{bmatrix}.$$

(12) Let $\mathbf{u_1} = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}$ and $\mathbf{u_2} = \begin{bmatrix} 1/2 \\ -1/2 \end{bmatrix}$. We calculate the lengths of these two vectors:

$$\begin{aligned}
||\mathbf{u_1}|| &= \sqrt{\mathbf{u_1} \cdot \mathbf{u_1}} = \sqrt{(1/2)^2 + (1/2)^2} = \sqrt{1/2} = 1/\sqrt{2} \\
||\mathbf{u_2}|| &= \sqrt{\mathbf{u_2} \cdot \mathbf{u_2}} = \sqrt{(1/2)^2 + (-1/2)^2} = \sqrt{1/2} = 1/\sqrt{2}
\end{aligned}$$

Since these vectors do not have unit length, $\{\mathbf{u_1}, \mathbf{u_2}\}$ is not an orthonormal set of vectors. We normalize the vectors as follows.

Let

$$\mathbf{w_1} := \frac{1}{||\mathbf{u_1}||}\mathbf{u_1} = \sqrt{2}\mathbf{u_1} = \begin{bmatrix} \sqrt{2}/2 \\ \sqrt{2}/2 \end{bmatrix}$$

and

$$\mathbf{w_2} := \frac{1}{||\mathbf{u_2}||}\mathbf{u_2} = \sqrt{2}\mathbf{u_2} = \begin{bmatrix} \sqrt{2}/2 \\ -\sqrt{2}/2 \end{bmatrix}.$$

Then the set $\{\mathbf{w_1}, \mathbf{w_2}\}$ is an orthonormal set of vectors.

• Let Q be an orthogonal matrix. Then, by Theorem 5.5, Q is invertible and $Q^{-1} = Q^T$. So,

$$\det(Q) = \det(Q^T) = \det(Q^{-1}) = \frac{1}{\det(Q)}$$

which implies that

$$[\det(Q)]^2 = 1.$$

We conclude that $\det(Q) = \pm 1$.