Math 314/814–002: Matrix Theory **Dr. S. Cooper, Fall 2008**

Name: ____

Quiz 12 Solutions

This is a take-home quiz. You are allowed to use your class notes and text, but no other resources (including books, internet, or people). This is due in class on Thursday, November 20. No late submissions will be accepted.

Please write up your solutions to the following exercises. You should write legibly and fully explain your work. Staple your pages together with this page as the cover – remember to write your full name at the top.

Exercises:

- Section 7.3: # 30
- Let $V = \mathbb{R}^3$ with the usual addition but scalar multiplication defined by

$$c\begin{bmatrix} x \\ y \\ z \end{bmatrix} := \begin{bmatrix} cx \\ y \\ cz \end{bmatrix}.$$

Determine if V is a vector space with these operations. If it is not, list all of the axioms that fail to hold with specific examples supporting your solution.

- Let $W = \{p(x) = a + bx + cx^2 \in \mathcal{P}_2 : c = -a\}$. Is W a subspace of $\mathcal{P}_2$? Be sure to completely verify your answer using Theorem 6.2, giving a specific counterexample if you conclude that W is not a subspace.

Solutions:

- Section 7.3: #30

(a) Substituting the data points into the equation $L = a + bF$ we obtain the linear system of equations

$$\begin{aligned} a + 2b &= 7.4 \\ a + 4b &= 9.6 \\ a + 6b &= 11.5 \\ a + 8b &= 13.6 \end{aligned}$$

This is equivalent to the system $A\mathbf{x} = \mathbf{b}$ where

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 4 \\ 1 & 6 \\ 1 & 8 \end{bmatrix}$$

and

$$\mathbf{b} = \begin{bmatrix} 7.4 \\ 9.6 \\ 11.5 \\ 13.6 \end{bmatrix}.$$

We need to find $\overline{\mathbf{x}}$ where $A^T A\overline{\mathbf{x}} = A^T\mathbf{b}$. We have

$$A^T A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 4 & 6 & 8 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 4 \\ 1 & 6 \\ 1 & 8 \end{bmatrix} = \begin{bmatrix} 4 & 20 \\ 20 & 120 \end{bmatrix}.$$

Since $\det(A^T A) \neq 0$, we see that $A^T A$ is invertible. So, the least squares solution is

$$\begin{aligned}
\bar{\mathbf{x}} &= \begin{bmatrix} a \\ b \end{bmatrix} \\
&= (A^T A)^{-1} A^T \mathbf{b} \\
&= \begin{bmatrix} 3/2 & -1/4 \\ -1/4 & 1/20 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 4 & 6 & 8 \end{bmatrix} \begin{bmatrix} 7.4 \\ 9.6 \\ 11.5 \\ 13.6 \end{bmatrix} \\
&= \begin{bmatrix} 5.39999999999999946 \\ 1.02499999999999991 \end{bmatrix}
\end{aligned}$$

The least squares approximating line for these data is (after rounding)

$$L = 5.4 + 1.02F.$$

The constant a represents the length of the spring before any force is applied by attaching various weights.

(b) If a weight of 5 ounces is attached to the spring, then the length of the spring is approximately

$$L = 5.4 + 1.02(5) = 5.4 + 5.1 = 10.5 \quad in.$$

• The set $\mathbb{R}^3$ with the defined operations is not a vector space. Axiom 8 is the only one that fails to hold. For example,

$$(2+3)\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = 5\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \\ 15 \end{bmatrix} \neq 2\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + 3\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 6 \end{bmatrix} + \begin{bmatrix} 3 \\ 2 \\ 9 \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \\ 15 \end{bmatrix}.$$

• W is a subspace of $\mathcal{P}_2$. We apply Theorem 6.2 to verify this.

Let $f(x) = a + bx + cx^2$ and $g(x) = s + tx + ux^2$ be two polynomials in W. Then $c = -a$ and $u = -s$. So,

$$\begin{aligned}
f(x) + g(x) &= (a+s) + (b+t)x + (c+u)x^2 \\
&= (a+s) + (b+t)x + (-a-s)x^2 \\
&= (a+s) + (b+t)x - (a+s)x^2.
\end{aligned}$$

Since the coefficient of x^2 of $f(x)+g(x)$ is minus the constant coefficient, we see that $f(x)+g(x)$ is also in W. Thus, W is closed under vector addition.

Let $f(x) = a + bx + cx^2$ be in W and α be a scalar. Then

$$\alpha f(x) = \alpha(a + bx + cx^2) = \alpha(a + bx - ax^2) = \alpha a + \alpha bx - \alpha ax^2.$$

Since the coefficient of x^2 in $\alpha f(x)$ is minus the constant coefficient, we have that $\alpha f(x)$ is in W. This shows that W is closed under scalar multiplication.