Math 314/814–002: Matrix Theory **Dr. S. Cooper, Fall 2008**

Name: ______________________________

Quiz 14 Solutions

Please write your solutions to the following exercises in the space provided. You should write legibly and fully explain your work.

Good Luck!

(1) Let $T : \mathcal{P}_2 \to \mathbb{R}^2$ be the linear transformation defined by

$$T(a + bx + cx^2) = \begin{bmatrix} a - b \\ b + c \end{bmatrix}.$$

(a) Find a basis for the kernel of T. [5 pts]

Solution:

$$\begin{aligned} ker(T) &= \{a + bx + cx^2 : T(a + bx + cx^2) = \mathbf{0}\} \\ &= \left\{ a + bx + cx^2 : \begin{bmatrix} a - b \\ b + c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\} \\ &= \{a + bx + cx^2 : a = b, c = -b\} \\ &= \{b + bx - bx^2\} \\ &= Span(1 + x - x^2) \end{aligned}$$

Thus, a basis for $ker(T)$ is $\{1 + x - x^2\}$.

(b) Without finding the range of T, find $rank(T)$. [3 pts]

Solution: By part (a), we see that $nullity(T) = 1$. So, by the Rank Theorem,

$$\dim(\mathcal{P}_2) = 3 = Rank(T) + Nullity(T) = Rank(T) + 1 \implies Rank(T) = 3 - 1 = 2.$$

(c) Is $\mathbf{v} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ in the range of T? If so, find $p(x)$ such that $T(p(x)) = \mathbf{v}$. [2 pts]

Solution: Yes, $\mathbf{v}$ is in the range of T. Indeed, $T(x^2) = \mathbf{v}$.

(2) Let $T : V \to W$ be a linear transformation.

(a) Complete the definition: T is *one-to-one* if [2 pts]

Solution: $T(\mathbf{u}) = T(\mathbf{v}) \implies \mathbf{u} = \mathbf{v}$ for all $\mathbf{u}, \mathbf{v}$ in V.

(b) Suppose $T : \mathcal{P}_1 \to \mathbb{R}^3$ is defined by

$$T(a+bx) = \begin{bmatrix} 2a \\ a-b \\ 0 \end{bmatrix}.$$

Is T 1-1? Is T onto? Be sure to justify your answers. [8 pts]

Solution: We have

$$\begin{aligned} ker(T) &= \{a+bx : T(a+bx) = \mathbf{0}\} \\ &= \left\{ a+bx : \begin{bmatrix} 2a \\ a-b \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\} \\ &= \{a+bx : a = 0 = b\} \\ &= \{\mathbf{0}\} \end{aligned}$$

Since $ker(T) = \{\mathbf{0}\}$ (where $\mathbf{0} = 0 + 0x$), we conclude that T is 1-1.

T is not onto since $range(T) \neq \mathbb{R}^3$. To see this, note that

$$Range(T) = \left\{ \begin{bmatrix} s \\ t \\ 0 \end{bmatrix} : s, t \in \mathbb{R} \right\}.$$

So any vector in $\mathbb{R}^3$ whose third component is not zero will never be in $range(T)$.