Math 314/814–006: Matrix Theory **Dr. S. Cooper, Fall 2008**

Name: ______________________________

Quiz 3 Solutions

Please write your solutions to the following exercises in the space provided. You should write legibly and fully explain your work.

Good Luck!

(1) Solve the system [8 pts]

$$\begin{aligned} w - x - y + 2z &= 1 \\ 2w - 2x - y + 3z &= 3 \\ -w + x - y &= -3 \end{aligned}$$

Solution: We row reduce the augmented matrix:

$$\left[\begin{array}{rrrr|r} 1 & -1 & -1 & 2 & 1 \\ 2 & -2 & -1 & 3 & 3 \\ -1 & 1 & -1 & 0 & -3 \end{array}\right] \xrightarrow{R_2 \to R_2 - 2R_1 \,\&\, R_3 \to R_3 + R_1} \left[\begin{array}{rrrr|r} 1 & -1 & -1 & 2 & 1 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & -2 & 2 & -2 \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 + 2R_2} \left[\begin{array}{rrrr|r} 1 & -1 & -1 & 2 & 1 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

We see that x and z are free variables. Let $x = s$ and $z = t$ for some $s, t \in \mathbb{R}$. Then

$$\begin{aligned} y - z = 1 &\implies y = z + 1 = t + 1 \\ w - x - y + 2z = 1 &\implies w = x + y - 2z + 1 = s + (t+1) - 2t + 1 = s - t + 2 \end{aligned}$$

So, the solution set for this system is

$$\left\{ \begin{bmatrix} s - t + 2 \\ s \\ t + 1 \\ t \end{bmatrix} : s, t \in \mathbb{R} \right\}.$$

(2) Let $A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix}$.

You can assume that A row reduces to $B = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

(a) Is B in reduced row echelon form? Briefly explain your answer. [3 pts]

Solution: B is not in reduced row echelon form since the leading entry in the third row is not the only nonzero entry in its column.

(b) What is the rank of A? [3 pts]

The rank of A equals the number of leading entries in B. Thus, $\text{rank}(A) = 3$.

(3) Let $\mathbf{u_1}, \ldots, \mathbf{u_k}$ be vectors in $\mathbb{R}^n$. Define $\text{span}(\mathbf{u_1}, \ldots, \mathbf{u_k})$. [3 pts]

Solution: $\text{Span}(\mathbf{u_1}, \ldots, \mathbf{u_k})$ is the set of all linear combinations of $\mathbf{u_1}, \ldots, \mathbf{u_k}$.

(4) Let $\mathbf{u}, \mathbf{v}, \mathbf{w}$ be in $\mathbb{R}^n$. Show that $\mathbf{w}$ is in $\text{span}(\mathbf{u}, \mathbf{u}+\mathbf{v}, \mathbf{u}+\mathbf{v}+\mathbf{w})$. [3 pts]

Solution: We have $\mathbf{w} = 0\mathbf{u} - 1(\mathbf{u}+\mathbf{v}) + 1(\mathbf{u}+\mathbf{v}+\mathbf{w})$. Since $\mathbf{w}$ is a linear combination of the given vectors, it is in the span of them.