Math 314/814–002: Matrix Theory **Dr. S. Cooper, Fall 2008**

Name: ______________________________

Quiz 6 Solutions

This is a take-home quiz. You are allowed to use your class notes and text, but no other resources (including books, internet, or people). This is due in class on Thursday, October 2. No late submissions will be accepted.

Please write up your solutions to the following exercises. You should write legibly and fully explain your work. Staple your pages together with this page as the cover – remember to write your full name at the top.

Exercises:

- Section 3.5: # 20, 50
- Could a 6×9 matrix have a 2-dimensional null space? Fully explain your conclusion.

Solutions:

Section 3.5:

(20) We begin by row-reducing the matrix A:

$$\begin{bmatrix} 1 & -2 & 1 & 4 & 4 \\ -1 & 2 & 1 & 2 & 3 \\ 2 & -4 & 0 & 2 & 1 \end{bmatrix} \xrightarrow{R_2 \to R_2 + R_1 \& R_3 \to R_3 - 2R_1} \begin{bmatrix} 1 & -2 & 1 & 4 & 4 \\ 0 & 0 & 2 & 6 & 7 \\ 0 & 0 & -2 & -6 & -7 \end{bmatrix}$$

$$\xrightarrow{R_3 \to R_3 + R_2} \begin{bmatrix} 1 & -2 & 1 & 4 & 4 \\ 0 & 0 & 2 & 6 & 7 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

- We see that a basis for $row(A)$ is

$$\{[1 \ \ -2 \ \ 1 \ \ 4 \ \ 4], [0 \ \ 0 \ \ 2 \ \ 6 \ \ 7]\}.$$

• A basis for $col(A)$ is found by selecting the original columns of A that correspond to pivot columns in its row echelon form. So, a basis for $col(A)$ is

$$\left\{ \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

• To find a basis for $null(A)$, we need to solve the system $A\mathbf{x} = \mathbf{0}$. We have 3 free variables: let $x_2 = m, x_4 = s, x_5 = t$ for some parameters $m, s, t \in \mathbb{R}$. Then we apply back-substitution:

$$\begin{aligned} & 2x_3 = -6x_4 - 7x_5 \\ \Longrightarrow \quad & x_3 = -3s - (7/2)m \end{aligned}$$

and

$$\begin{aligned} & x_1 = 2x_2 - x_3 - 4x_4 - 4x_5 \\ \Longrightarrow \quad & x_1 = 2m - (-3s - (7/2)m) - 4s - 4t = (11/2)m - s - 4t \end{aligned}$$

Thus, the null space of A is

$$null(A) = \left\{ \begin{bmatrix} (11/2)m - s - 4t \\ m \\ -3s - (7/2)m \\ s \\ t \end{bmatrix} = m \begin{bmatrix} 11/2 \\ 1 \\ -7/2 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -1 \\ 0 \\ -3 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -4 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} : m, s, t \in \mathbb{R} \right\}.$$

Since the vectors $\begin{bmatrix} 11/2 \\ 1 \\ -7/2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ span $null(A)$ and are linearly independent, they form a basis for $null(A)$.

(50) We need to write $\mathbf{w}$ as a linear combination of the given vectors. To do this we

row-reduce the associated augmented matrix:

$$\left[\begin{array}{cc|c} 3 & 5 & 1 \\ 1 & 1 & 3 \\ 4 & 6 & 4 \end{array}\right] \xrightarrow{R_1 \to R_2} \left[\begin{array}{cc|c} 1 & 1 & 3 \\ 3 & 5 & 1 \\ 4 & 6 & 4 \end{array}\right]$$

$$\xrightarrow{R_2 \to R_2 - 3R_1 \& R_3 \to R_3 - 4R_1} \left[\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & 2 & -8 \\ 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 - R_2} \left[\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & 1 & -4 \\ 0 & 0 & 0 \end{array}\right]$$

This system is consistent, and so $\mathbf{w}$ is a linear combination of the basis vectors in $\mathcal{B}$. Solving for the constants in our linear combination we see that

$$\mathbf{w} = 7 \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix} - 4 \begin{bmatrix} 5 \\ 1 \\ 6 \end{bmatrix}.$$

Thus,

$$[\mathbf{w}]_{\mathcal{B}} = \begin{bmatrix} 7 \\ -4 \end{bmatrix}.$$

• *Question:* Could a 6×9 matrix have a 2-dimensional null space? Fully explain your conclusion.

No such matrix can exist. To see this, suppose A is a 6×9 matrix with $nullity(A) = 2$. Then, by the Rank Theorem,

$$rank(A) = 9 - 2 = 7.$$

But then $\dim(col(A)) = rank(A) = 7$. However, each column of A is in $\mathbb{R}^6$ and so the dimension of $col(A)$ cannot exceed 6. (Another way to see this is to note that $rank(A) \leq \min(6, 9) = 6$.)