Math 314/814 Exam 1 Review Exercises

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(1) Find all the polynomials $f(t) = at^2 + bt + c$ whose graphs run through the points $(1, 1)$ and $(3, 3)$, such that $f'(2) = 1$.

(2) Solve the system

$$\begin{aligned} w - x - y + 2z &= 1 \\ 2w - 2x - y + 3z &= 3 \\ -w + x - y &= -3 \end{aligned}$$

(3) Some parking meters in Milan, Italy, accept coins in the denominations of 20 cent, 50 cent, and 2 Euro. As an incentive program, the city administors offer a big reward (a brand new red Ferrari) to any meter maid who brings back exactly 1000 coins worth exactly 1000 Euro from the daily rounds. What are the odds of this reward being claimed anytime soon?

(4) Consider the linear system

$$\left\{ \begin{array}{ccccc} x_1 & + & 2x_2 & = & k \\ 4x_1 & + & hx_2 & = & 5 \end{array} \right\}$$

Determine h and k so that the solution set of the system:

(a) is empty;

(b) contains a unique solution;

(c) contains infinitely many solutions.

(5) Repeat exercise 4, but with the system

$$\left\{ \begin{array}{ccccc} -3x_1 & + & hx_2 & = & 1 \\ 6x_1 & + & kx_2 & = & -3 \end{array} \right\}$$

(6) Are the following transformations linear? If so, find the associated standard matrix.

(a) $T : \mathbb{R} \to \mathbb{R}$, where $T(r) = \frac{4}{3}\pi r^3$ (i.e. $T(r)$ gives the volume of a ball of radius r)

(b) $T : \mathbb{R}^4 \to \mathbb{R}$ which is defined as the projection onto the 2nd component:

$$T \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_2$$

(c) $T : \mathbb{R}^3 \to \mathbb{R}^2$ defined by $T \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \sin(x_2) \\ x_1 - x_3 \end{bmatrix}$

(d) $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by counter-clockwise rotation by θ radians:

$$T \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} (\cos\theta)x_1 - (\sin\theta)x_2 \\ (\sin\theta)x_1 + (\cos\theta)x_2 \end{bmatrix}$$

(7) Suppose $T : \mathbb{R}^2 \to \mathbb{R}^4$ is a linear transformation such that:

$$T \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 3 \end{bmatrix} \text{ and } T \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 1 \\ 4 \\ 4 \end{bmatrix}. \text{ Find } T \begin{bmatrix} 2 \\ -3 \end{bmatrix}.$$

(8) Suppose that S is a linear transformation with standard matrix $A = \begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & 4 \end{bmatrix}$.

(a) What is the domain and codomain of S?

(b) Find $S \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $S \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ and $S \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

(c) Find a general formula for S.

(9) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation defined by the projection onto the $y-z$-plane. Find the standard matrix of T.

(10) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation defined by the rotation of a vector through an angle of $\pi/2$, counterclockwise as viewed from the positive z-axis.

Find the standard matrix of T.

(11) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation with standard matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{bmatrix}$. Is T invertible? If so, find the standard matrix of T^{-1}.

(12) Compute the matrix operations:

(a) $\begin{bmatrix} 1 & -1 \\ 0 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 1 & 0 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} - 2 \begin{bmatrix} 7 & 3 & 1 \\ 5 & 3 & -1 \end{bmatrix}$

(13) Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be the linear map with standard matrix $[T] = A = \begin{bmatrix} 1 & 2 & 0 \\ 1 & -1 & 3 \end{bmatrix}$ and $S : \mathbb{R}^2 \to \mathbb{R}^2$ with standard matrix $[S] = B = \begin{bmatrix} 3 & -1 \\ 1 & -1 \end{bmatrix}$. Find the standard matrix of $S \circ T : \mathbb{R}^3 \to \mathbb{R}^2$.

(14) Are the following formulas true or false in general for $n \times n$ invertible matrices A and B?

(a) A^2 is invertible and $(A^2)^{-1} = (A^{-1})^2$

(b) $(A - B)(A + B) = A^2 - B^2$

(c) $ABA^{-1} = B$

(15) Is $V = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : x \geq 0 \right\}$ a subspace of $\mathbb{R}^2$?

(16) Consider two subspaces V and W of $\mathbb{R}^n$. Define $V + W$ to be the set of all vectors of the form $\mathbf{v} + \mathbf{w}$, where $\mathbf{v}$ is in V and $\mathbf{w}$ is in W. Is $V + W$ a subspace of $\mathbb{R}^n$?

(17) Determine if the following sets of vectors are linearly independent or dependent.

(a) $\begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix}$

(b) $\begin{bmatrix} 2 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 8 \\ 5 \\ 10 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ -1 \end{bmatrix}$

(c) Any 5 vectors in $\mathbb{R}^4$

(18) Find a basis for $col(A), null(A)$ and $row(A)$ of

$$A = \begin{bmatrix} 1 & 2 & 2 & -5 & 6 \\ -1 & -2 & -1 & 1 & -1 \\ 4 & 8 & 5 & -8 & 9 \\ 3 & 6 & 1 & 5 & -7 \end{bmatrix}$$

(19) For which values of k do the following vectors form a basis for $\mathbb{R}^3$?

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ k \\ k^2 \end{bmatrix}$$

(20) Let A be an $n \times n$ matrix. What are all the equivalent formulations of the statement: "A is invertible"?

(21) Consider a 3×5 matrix A. What are the possible values for $\dim(\text{null}(A))$? Explain carefully.

(22) Let V be the subspace of $\mathbb{R}^3$ with basis $\mathcal{B} = \left\{ \begin{bmatrix} 8 \\ 4 \\ -1 \end{bmatrix}, \begin{bmatrix} 5 \\ 2 \\ -1 \end{bmatrix} \right\}$.

(a) Let $\mathbf{x} = \begin{bmatrix} 1 \\ -2 \\ -2 \end{bmatrix}$. Find $[\mathbf{x}]_{\mathcal{B}}$.

(b) If $[\mathbf{y}]_{\mathcal{B}} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$, find $\mathbf{y}$.

(23) Are the following statements true or false? If true, then explain why. If false, then give a specific counter-example.

(a) If vectors $\mathbf{u}, \mathbf{v}$ and $\mathbf{w}$ are in a subspace V of $\mathbb{R}^n$, then the vector $2\mathbf{u}-3\mathbf{v}+4\mathbf{w}$ must be in V as well.

(b) If the null space of a 5×4 matrix A consists of the zero vector alone, and if $AB = AC$ for two 4×5 matrices B and C, then $B = C$.

(c) If A is any $n \times n$ matrix such that $A^2 = A$, then the column space of A and the null space of A have only the zero vector in common.

(d) There exists an invertible 2×2 matrix A such that $A^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$.

(e) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are linearly independent, then so are $\mathbf{u} - \mathbf{v}, \mathbf{v} - \mathbf{w}, \mathbf{u} - \mathbf{w}$.

(f) Let $S = span(\mathbf{v_1}, \dots, \mathbf{v_k})$ and suppose $\mathbf{v_1}$ is a linear combination of $\mathbf{v_2}, \dots, \mathbf{v_k}$. Then $S = span(\mathbf{v_2}, \dots, \mathbf{V_K})$.

(g) Let A be an $n \times n$ matrix. Then the columns of A are linearly independent if and only if the columns of A^T are linearly independent.

(24) Is $\mathbf{v} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$ in $span \left(\begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix} \right)$?

(25) Be prepared to prove one of the three statements:

(a) If A is any $m \times n$ matrix, then AA^T is symmetric.

(b) Let S be a subspace of $\mathbb{R}^n$ with basis $\mathcal{B} = \{\mathbf{v_1}, \dots, \mathbf{v_k}\}$. Then for every $\mathbf{v}$ in S there is exactly one way to write $\mathbf{v}$ as a linear combination of the vectors in $\mathcal{B}$.

(c) If AS and B are two $n \times n$ invertible matrices, then AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$.

(26) Be prepared to state any definition which was given in class!

Answers

(1) $f(t) = at^2 + (1 - 4a)t + 3a$ for an arbitrary constant a.

(2) This is Example 2.11 in your text (pp 74 - 75).

(3) There are no integer solutions to this problem. No one will ever claim the prize.

(4) (a) $h = 8$ and $k \neq 5/4$

(b) $h \neq 8$ and k any real value

(c) $h = 8$ and $k = 5/4$

(5) (a) $2h + k = 0$

(b) $2h + k \neq 0$

(c) the system cannot infinitely many solutions

(6) (a) Not linear

(b) Linear: $[T] = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix}$

(c) Not linear

(d) Linear: $[T] = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

(7) $\begin{bmatrix} -16 \\ -5 \\ -12 \\ -6 \end{bmatrix}$

(8) (a) A has size 2×3, so the domain is $\mathbb{R}^3$ and the codomain is $\mathbb{R}^2$

(b) $S\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $S\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and $S\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$

(c) $S\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = A\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 + 3x_2 + 2x_3 \\ x_2 + 4x_3 \end{bmatrix}$

(9) $[T] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(10) $[T] = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(11) Yes, T is invertible; $[T^{-1}] = A^{-1} = \begin{bmatrix} 3 & -3 & 1 \\ -3 & 5 & -2 \\ 1 & -2 & 1 \end{bmatrix}$

(12) (a) $\begin{bmatrix} 2 & 2 \\ 2 & 0 \\ 7 & 4 \end{bmatrix}$

(b) Undefined

(c) $\begin{bmatrix} -13 & -4 & 1 \\ -6 & -1 & 8 \end{bmatrix}$

(13) $BA = \begin{bmatrix} 2 & 7 & -3 \\ 0 & 3 & -3 \end{bmatrix}$

(14) (a) True

(b) False

(c) False

(15) No

(16) Yes

(17) (a) Linearly dependent

(b) Linearly independent

(c) Linearly dependent

(18) A basis for the null space is $\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 4 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \\ -5 \\ 0 \\ 1 \end{bmatrix} \right\}$.

A basis for the column space is $\left\{ \begin{bmatrix} 1 \\ -1 \\ 4 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 5 \\ 1 \end{bmatrix} \right\}$.

A basis for the row space is

$$\{[1\ \ 2\ \ 2\ \ -5\ \ 6], [0\ \ 0\ \ 1\ \ -4\ \ 5]\}.$$

(19) All values of k such that $k \neq -1, 1$

(20) This is the *Fundamental Theorem for Invertible Matrices* from class.

(21) The dimension of the null space is 2, 3, 4, or 5 by the Rank Theorem

(22) (a) $\begin{bmatrix} -3 \\ 5 \end{bmatrix}$

(b) $\begin{bmatrix} -9 \\ -6 \\ 1 \end{bmatrix}$

(23) (a) True; (b) True; (c) True; (d) False; (e) False; (F) True; (g) True

(24) No

(25) These were proven in class.