Math 314/814 Exam 2 Review Exercises

Dr. S. Cooper — Fall 2008

(1) Find $\det \begin{bmatrix} 2 & -3 & 0 & 1 \\ 5 & 4 & 2 & 0 \\ 1 & -1 & 0 & 3 \\ -2 & 1 & 0 & 0 \end{bmatrix}$ by hand.

(2) Assume that $\det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = 7$. What is $\det \begin{bmatrix} g & h & i \\ 2d+a & 2e+b & 2f+c \\ a & b & c \end{bmatrix}$?

(3) Use the adjoint to compute the inverse of $A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 2 & 4 \\ 1 & 3 & -3 \end{bmatrix}$.

(4) Let $A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$. Show that $\mathbf{u} = \begin{bmatrix} 6 \\ -5 \end{bmatrix}$ is an eigenvector of A. What is the corresponding eigenvalue?

(5) Let $A = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}$.

- (a) Find all the eigenvalues of A by hand.
- (b) For each eigenvalue you found, find a basis for the corresponding eigenspace by hand.
- (c) Is A diagonalizable? If so, find an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$.

(6) Is $A = \begin{bmatrix} 2 & 4 & 3 \\ -4 & -6 & -3 \\ 3 & 3 & 1 \end{bmatrix}$ diagonalizable?

(7) Find $\begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}^{10}$ without using a calculator.

(8) Let $P = \begin{bmatrix} 0.2 & 0.3 & 0.4 \\ 0.6 & 0.1 & 0.4 \\ 0.2 & 0.6 & 0.2 \end{bmatrix}$.

- (a) Is P a regular transition matrix?
- (b) Find the long range transition matrix L of P.

(9) Solve the system of differential equations

$$\begin{aligned} x_1' &= x_1 + x_2 \\ x_2' &= x_1 - x_2 \end{aligned}$$

where $x_1(0) = 1$ and $x_2(0) = 0$

(10) Let $\mathbf{v_1} = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{v_2} = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$ and $\mathbf{v_3} = \begin{bmatrix} -1/2 \\ -2 \\ 7/2 \end{bmatrix}$. Show that $\mathcal{B} = \{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$ is an orthogonal basis for $\mathbb{R}^3$ and find $[\mathbf{w}]_{\mathcal{B}}$ if $\mathbf{w} = \begin{bmatrix} 6 \\ 1 \\ -8 \end{bmatrix}$

(11) Is $Q = \begin{bmatrix} 3/\sqrt{11} & -1/\sqrt{6} & -1/\sqrt{66} \\ 1/\sqrt{11} & 2/\sqrt{6} & -4/\sqrt{66} \\ 1/\sqrt{11} & 1/\sqrt{6} & 7/\sqrt{66} \end{bmatrix}$ orthogonal? If so, find its inverse.

(12) Let $W = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : x = t, y = -t, z = 3t \right\}$. Find $W^{\perp}$ and give a basis for $W^{\perp}$.

(13) Find the orthogonal decomposition of $\mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ with respect to

$$W = span\left(\begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} \right).$$

(14) Find an orthogonal basis for the column space of the matrix $A = \begin{bmatrix} 3 & -5 & 1 \\ 1 & 1 & 1 \\ -1 & 5 & -2 \\ 3 & -7 & 8 \end{bmatrix}$

(15) Find all least squares solutions to the inconsistent system

$$\begin{aligned} x + y &= -3 \\ x + y &= -1 \\ x + w &= 0 \\ x + w &= 2 \\ x + z &= 5 \\ x + z &= 1 \end{aligned}$$

(16) A square matrix A is skew-symmetric if $A^T = -A$. Is the set of all 2×2 skew-symmetric matrices, with the usual matrix addition and scalar multiplication, a vector space?

(17) Is W a subspace of V? If W is a subspace, verify this using the definition and find $\dim(W)$. If W is not a subspace, give an explicit example showing how it fails to be one.

(a) $V = \mathbb{R}^3$ and $W = \left\{ \begin{bmatrix} a \\ b \\ |a| \end{bmatrix} \right\}$

(b) $V = \mathcal{P}_2$ and $W = \{p(x) \in \mathcal{P}_2 : xp'(x) = p(x)\}$

(c) $V = M_{22}$ and $W = \left\{ A \in M_{22} : \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} A = A \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \right\}$

(18) Show that $\mathcal{B} = \{1+x, x+x^2, 1+x^2\}$ is a basis for $\mathcal{P}_2$.

(19) The sets $\mathcal{B} = \{1, x, x^2\}$ and $\mathcal{C} = \{1+x, x+x^2, 1+x^2\}$ are two bases for $\mathcal{P}_2$. Find $P_{\mathcal{C}\leftarrow\mathcal{B}}$ and use this to find $[1+2x-x^2]_{\mathcal{C}}$.

(20) Are the following transformations linear? Be sure to support your answer.

(a) $T : \mathcal{P}_2 \longrightarrow \mathcal{P}_2$ where $T(a+bx+cx^2) = a + b(x+1) + c(x+1)^2$

(b) $T : \mathbb{R}^2 \longrightarrow \mathcal{P}_1$ such that $T\begin{bmatrix} 1 \\ 2 \end{bmatrix} = 1 + 2x, T\begin{bmatrix} 3 \\ 4 \end{bmatrix} = 5 - x$ and $T\begin{bmatrix} -5 \\ -6 \end{bmatrix} = -9 + 5x$

(21) Consider the linear transformations $T : \mathbb{R}^2 \longrightarrow \mathcal{P}_1$ and $S : \mathcal{P}_1 \longrightarrow \mathcal{P}_2$ such that

$$T\begin{bmatrix} a \\ b \end{bmatrix} = a + (a+b)x$$

and

$$S(p(x)) = xp(x).$$

Find $(S \circ T)\begin{bmatrix} 3 \\ -2 \end{bmatrix}$.

(22) Are the following statements true or false? Be sure to justify your answers with complete explanations or counterexamples.

(a) Let A be an $n \times n$ matrix. A real number λ is an eigenvalue of A if and only if $rank(A - \lambda I_n) < n$.

(b) Suppose A is a 4×4 matrix and $RREF(A - 7I_4) = I_4$. Then 7 is an eigenvalue of A.

(c) If A is a 3×3 matrix and $\det(A) = 3$, then the equation $A\mathbf{x} = \mathbf{0}$ has a non-trivial solution.

(d) If W is a subspace of $\mathbb{R}^3$ spanned by $\mathbf{w_1}$ and $\mathbf{w_2}$ and $\mathbf{v}$ is any vector in $\mathbb{R}^3$, then

$$proj_W(\mathbf{v}) = \frac{\mathbf{v} \cdot \mathbf{w_1}}{\mathbf{w_1} \cdot \mathbf{w_1}}\mathbf{w_1} + \frac{\mathbf{v} \cdot \mathbf{w_2}}{\mathbf{w_2} \cdot \mathbf{w_2}}\mathbf{w_2}.$$

(e) If A is a 2×2 diagonalizable matrix, then A is invertible.

(f) The polynomials $1 - x, 1 + x^2, x + x^2$ form a linearly dependent set in $\mathcal{P}_2$.

(g) Let V and W be vector spaces. Let $\mathbf{v_1}, \ldots, \mathbf{v_n}$ be in V and assume $T : V \rightarrow W$ is a linear transformation. If $\{T(\mathbf{v_1}), \ldots, T(\mathbf{v_n})\}$ is linearly independent then $\{\mathbf{v_1}, \ldots, \mathbf{v_n}\}$ is linearly independent.

(h) Let V and W be vector spaces. Let $\mathbf{v_1}, \ldots, \mathbf{v_n}$ be in V and assume $T : V \rightarrow W$ is a linear transformation. If $\{\mathbf{v_1}, \ldots, \mathbf{v_n}\}$ is linearly independent then $\{T(\mathbf{v_1}), \ldots, T(\mathbf{v_n})\}$ is linearly independent.

(i) Suppose $\mathcal{B}$ and $\mathcal{C}$ are two bases for a vector space V of dimension n. If $P_{\mathcal{C}\leftarrow\mathcal{B}} = I_n$ then $\mathcal{B} = \mathcal{C}$.

(j) For all square matrices A, $\det(-A) = -\det(A)$

(k) If A is invertible, then $\det(A^{-1}) = \det(A^T)$.

(l) If A is an $n \times n$ invertible matrix, then 0 is an eigenvalue of A.

(23) Be prepared to state any definition or repeat a proof which was covered in class.

Answers

(1) -22

(2) -14

(3) $A^{-1} = \begin{bmatrix} 9 & -3/2 & -5 \\ -5 & 1 & 3 \\ -2 & 1/2 & 1 \end{bmatrix}$

(4) $A\mathbf{u} = -4\mathbf{u}$ so the eigenvalue is -4.

(5) (a) $\lambda_1 = 1$ and $\lambda_2 = -2$

(b) Basis for $\lambda_1 = 1$: $\left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\}$; Basis for $\lambda_2 = -2$: $\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$

(c) Yes! $P = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$

(6) No

(7) $\begin{bmatrix} 342 & 341 \\ 682 & 683 \end{bmatrix}$

(8) (a) Yes

(b) $L = \begin{bmatrix} 0.304 & 0.304 & 0.304 \\ 0.354 & 0.354 & 0.354 \\ 0.342 & 0.342 & 0.342 \end{bmatrix}$

(9)

$$\begin{aligned} x_1(t) &= (2+\sqrt{2})e^{\sqrt{2}t}/4 + (2-\sqrt{2})e^{-\sqrt{2}t}/4 \\ x_2(t) &= \sqrt{2}e^{\sqrt{2}t}/4 - \sqrt{2}e^{-\sqrt{2}t}/4 \end{aligned}$$

(10) Check that $\mathbf{v_i} \cdot \mathbf{v_j} = 0$ for each i and j; $[\mathbf{w}]_{\mathcal{B}} = \begin{bmatrix} 1 \\ -2 \\ -2 \end{bmatrix}$

(11) Yes; $Q^{-1} = Q^T$

(12) $W^\perp = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : x - y + 3z = 0 \right\}$ and has basis $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix} \right\}$

(13) $\mathbf{v} = \begin{bmatrix} -2/5 \\ 2 \\ 1/5 \end{bmatrix} + \begin{bmatrix} 7/5 \\ 014/5 \end{bmatrix}$

(14) $\left\{ \begin{bmatrix} 3 \\ 1 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 3 \\ -1 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \\ 1 \\ 3 \end{bmatrix} \right\}$

(15) $\overline{\mathbf{x}} = \begin{bmatrix} 3-t \\ -5+t \\ -2+t \\ t \end{bmatrix}$

(16) Yes

(17) (a) No

(b) Yes; $\dim(W) = 1$

(c) Yes; $\dim(W) = 2$

(18) The polynomials are linearly independent and $\dim(\mathcal{P}_2) = 3$

(19) $P_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{bmatrix} 1/2 & 1/2 & -1/2 \\ -1/2 & 1/2 & 1/2 \\ 1/2 & -1/2 & 1/2 \end{bmatrix}$ and $[1+2x-x^2]_{\mathcal{C}} = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$

(20) (a) Yes

(b) No

(21) $3x + x^2$

(22) (a) True

(b) False

(c) False

(d) False

(e) False

(f) True

(g) True

(h) False

(i) True

(j) False

(k) False

(l) False