Take-Home Test 1

Due: Thursday, September 10

Collaboration Allowed

Collaboration on this test is both allowed and encouraged. By collaboration, I mean that you are encouraged to discuss the problems, test out your ideas, check your reasoning and arguments, etc., with other people. *However, there is a big difference between collaborating and copying. Each student must write up his or her own solutions to the problems in his or her own words.*

You will be graded both on mathematical content and on clarity of expression. Each problem is worth a maximum of 12 points. Some of the questions are a bit open-ended, so be creative, make educated guesses if you have to, but back up your assertions by providing proofs, counter-examples, or (in the case of guesses) numerical evidence. In writing your answers, use complete sentences (with punctuation!) and be sure to say exactly what you mean. Papers will be graded on the basis of what you have written, so be sure to take the time to express yourself clearly. If you are stuck on a problem and have no idea where to begin, a good way to get started is to look at lots of specific examples and try to find a pattern.

Exercises:

(1) The set of all even integers can be divided into three classes, depending on their remainders when divided by 6:
- All those which are of the form $6n$, namely 6, 12, 18, 24, etc. Let's call these the *six-zero* integers.
- All those which are of the form $6n + 2$, namely 2, 8, 14, 20, etc. Let's call these the *six-two* integers.
- All those which are of the form $6n + 4$, namely 4, 10, 16, 22, etc. Let's call these the *six-four* integers.

In class we proved that the product of an odd integer and an even integer is even. What kinds of analogous statements can we make about two six-zero integers, two six-two integers, two six-four integers, a six-zero integer and a six-two integer, a six-zero integer and a six-four integer, and a six-two integer and a six-four integer? (We know all these products will be even, but will they be six-zero integers, six-two integers, or six-four integers?) Don't forget to include proofs of your claims.

(2) We say that an ordered triple (a, b, c) of positive integers (written in ascending order) is a *Pythagorean Triple* if $a^2 + b^2 = c^2$. There are many examples of Pythagorean triples. For example, $(3, 4, 5), (5, 12, 13)$ and $(6, 8, 10)$ are all Pythagorean triples.

(i) Show that if (a, b, c) is a Pythagorean triple and d is any positive integer, then (da, db, dc) is also a Pythagorean triple.

(ii) Let $a > 1$ be an odd integer. Show that a begins a Pythagorean triple; i.e., there exist positive integers b and c such that (a, b, c) is a Pythagorean triple. (Hint: Let $b = \frac{a^2-1}{2}$.)

(iii) Let a be any positive even integer which is not a power of 2. Show that a begins a Pythagorean triple. (Hint: Find a way to write a in general which involves an odd factor and then apply parts (i) and (ii).)

(3) Observe that for the first few positive integers n, the number $n^2 - n$ is a multiple of 2. For example, $1^2 - 1 = 0, 2^2 - 2 = 2, 3^2 - 3 = 6, 4^2 - 4 = 12$, etc. Is this always true? In other words, is $n^2 - n$ always a multiple of 2?

Similarly, the numbers $1^3 - 1 = 0, 2^3 - 2 = 6, 3^3 - 3 = 24, 4^3 - 4 = 60$, etc., all seem to be multiples of 3. Is $n^3 - n$ always a multiple of 3 for a positive integer n?

More generally, if n and k are positive integers, will $n^k - n$ always be a multiple of k?

(4) For each positive integer n between 1 and 30 (inclusive), list all of the positive divisors of n. For example, 12 has the six positive divisors 1, 2, 3, 4, 6, and 12. You will notice that "most" positive integers have an even number of divisors, but a few have an odd number of divisors. Classify those positive integers which have an even number of divisors and those which have an odd number of divisors. Try to explain why your classification works in general.

(5) Fun with GCDs. Let a be an integer.
  (a) Prove that $\gcd(a, a+1) = 1$.
  (b) Prove that $\gcd(a, a+2) = 1$ or 2. For which integers a does $\gcd(a, a+2) = 2$?
  (c) Prove that $\gcd(a, a+3) = 1$ or 3.
  (d) Let n be a positive integer. Give an explicit counter-example to show that the statement "$\gcd(a, a+n) = 1$ or n" is not true in general. Add an additional assumption that makes this statement true and prove the corrected statement.

Statement of Resources: Give a list of all people with whom you discussed the exercises on this test. Also, if you used any references besides the class notes, list them as well.