Take-Home Test 3

Due: Thursday, October 8

Collaboration Allowed

Collaboration on this test is both allowed and encouraged. By collaboration, I mean that you are encouraged to discuss the problems, test out your ideas, check your reasoning and arguments, etc., with other people. *However, there is a big difference between collaborating and copying. Each student must write up his or her own solutions to the problems in his or her own words.*

As before, you will be graded both on mathematical content and on clarity of expression. Exercises 1-3 are worth a maximum of 12 points each and Exercise 4 is worth 24 points. Some of the questions are a bit open-ended, so be creative, make educated guesses if you have to, but back up your assertions by providing proofs, counter-examples, or (in the case of guesses) numerical evidence. In writing your answers, use complete sentences (with punctuation!) and be sure to say exactly what you mean. Papers will be graded on the basis of what you have written, so be sure to take the time to express yourself clearly. If you are stuck on a problem and have no idea where to begin, a good way to get started is to look at lots of specific examples and try to find a pattern.

Exercises:

(1) *The Fibonacci Sequence:* The *Fibonacci sequence* is a sequence of integers $a_0, a_1, a_2, \ldots$ which is defined as follows: $a_0 = 1, a_1 = 1$ and for all $n \geq 2$, $a_n = a_{n-1} + a_{n-2}$.
   (a) Find a_i for $i = 2, 3, 4, 5, 6, 7, 8$ and 9.
   (b) Use the Principle of Mathematical Induction to prove that $a_n \leq 2^n$ for all $n \geq 0$.

(2) *Russian Peasant Arithmetic:* Here is an algorithm for multiplying positive integers. The method has been attributed to Russian peasants who could: add; multiply by 2; and divide by 2. This method was also used by ancient Egyptians and is of interest to computer programmers.

To multiply two positive integers a and b, put a and b at the top of two columns and fill in the columns below by multiplying the left number by 2 and dividing the right number by 2. Whenever division of the right number by 2 yields a new right number which is odd, subtract one from that new number before dividing by 2 again, and put the corresponding left number in the sum column. When you reach the number 1 in the right column, add the sum column to get the answer. Here's an example, in which we show $311 \times 116 = 36076$.

Left	Right	Sum
311	116	
622	58	
1244	29	1244
2488	14	
4976	7	4976
9952	3	9952
19904	1	19904
		36706

(a) Verify that Russian peasant arithmetic works to compute 195×218.

(b) Let $r(a,b)$ denote the result of doing Russian peasant arithmetic to compute $a \times b$. After doing some examples, you will see that $r(a,b) = r(2a, \frac{b}{2})$ if b is even and $r(a,b) = r\left(2a, \frac{b-1}{2}\right) + a$ if b is odd (you do not have to prove these two observations). Use the Principle of Mathematical Induction to prove that $r(a,b) = a \times b$ for all positive integers a and b. (*Hint:* To do this, induct on b so that your base case is $b = 1$ in which you need to show that $r(a,1) = a$ for all positive integers a. Break the inductive step into 2 cases: (1) b is even; and (2) b is odd.)

(3) *Mersenne Primes:* A prime integer which is of the form $2^n - 1$ for some integer n is called a *Mersenne prime*. Mersenne primes are named after the French monk Marin Mersenne (1588 - 1648) who studied them. It is still unknown whether there are infinitely may Mersenne primes. In Fall 2008, a research team at UCLA announced the discovery of a 13 million digit prime number - it is a Mersenne prime with $n = 43,112,609$.

(a) Find four Mersenne primes.

(b) Let p be a prime integer. Must $2^p - 1$ be a prime integer also? If yes, prove your claim. If no, give a specific counter-example.

(c) Preparation for part (d): Let x and y be integers. Recall the well-known formulas: $x^2 - y^2 = (x-y)(x+y)$ and $x^3 - y^3 = (x-y)(x^2 + xy + y^2)$. Prove that for any integer $n \geq 1$

$$x^n - y^n = (x-y)(x^{n-1} + x^{n-2}y + x^{n-3}y^2 + \cdots + xy^{n-2} + y^{n-1}).$$

(*Note:* You do not need to use induction to solve this exercise. Rather, try to simplify the right-hand side of the equation.)

(d) Prove that if $2^n - 1$ is a Mersenne prime, then n must also be a prime integer. (*Hint:* Note that $x^{cd} - 1 = (x^c)^d - 1^d$ for any integer x and positive integers c, d.)

(e) *Bonus Exercise:* The Electronic Frontier Foundation has a number of what it calls Cooperative Computing Awards. What are the four awards worth and what must one do to win? What is the largest known prime? Is it a Mersenne prime? State whatever reference(s) you used for this question.

(4) *The E-zone:* The set consisting of the positive even integers along with 1, i.e., $E = \{1, 2, 4, 6, 8, 10, \ldots\}$, is called the *E-zone*. Any integer in the E-zone is called an *E-number*. We call an E-number an *E-prime* if it is greater than 1 and its only E-zone factors are 1 and itself. For example, $2, 6, 10, 14$ and 18 are the first few E-primes.

(a) Prove that an E-number is an E-prime if and only if it is twice an odd integer. (*Note:* There are two things to be shown here.)

(b) Prove that every E-number can be factored into a product of E-primes.

(c) Find all possible factorizations of 36, 60, 72, and 360 into E-primes.

(d) Although part (4b) shows the *existence* of E-prime factorization of E-numbers, you should have noticed that part (4c) shows that this factorization is not always *unique*. However, you should have noticed that the *number* of E-primes occurring in every factorization of a given E-number was always the same. For example, every factorization you found of 36 should have involved 2 E-primes, and every factorization you found of 72 should have involved 3 E-primes. Is this always the case? In other words, either prove or disprove (by giving a counter-example) the following conjecture:

Conjecture: Suppose the E-number n can be factored as a product of k E-primes. Then *every* factorization of n into E-primes involves exactly k E-prime factors.

(e) Suppose $p_1, \ldots, p_k$ are distinct odd prime integers, and consider the E-number $n = 2^k p_1^{a_1} \cdots p_k^{a_k}$ where each a_i is an integer which is at least 1. Determine, with explanation, how many distinct E-prime factorizations n has.

(f) We say that an E-number x divides an E-number y in the E-zone if there exists an E-number u such that $y = xu$. Let p be an E-prime and let a and b be E-numbers. Suppose that p divides ab in the E-zone. Must p divide a or b in the E-zone? If yes, provide a proof. If no, give a specific counter-example.

Statement of Sources: Give a list of all people with whom you discussed the exercises on this test. Also, if you used any references besides the class notes, list them as well.