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Abstract. A result of Foxby states that if there exists a complex with finite depth, finite flat dimension, and finite injective dimension over a local ring R, then R is Gorenstein. In this paper we investigate some homological dimensions involving a semidualizing complex and improve on Foxby's result by answering a question of Takahashi and White. In particular, we prove for a semidualizing complex C, if there exists a complex with finite depth, finite $\mathcal{F}_C$-projective dimension, and finite $\mathcal{I}_C$-injective dimension over a local ring R, then R is Gorenstein.

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Keywords. Depth, Flat dimension, Gorenstein rings, Injective dimension, Semidualizing complex, Semidualizing module, Small support.

1. Introduction. Throughout this paper let R be a commutative noetherian ring with identity. A result of Foxby states that, if there exists an R-complex X that has finite flat dimension and finite injective dimension, then $R_\mathfrak{p}$ is a Gorenstein ring for all $\mathfrak{p} \in \operatorname{supp}_R(X)$; see Section 2 for definitions. In this paper we generalize this theorem using a semidualizing R-module. A finitely generated R-module C is *semidualizing* if $R \cong \operatorname{Hom}_R(C, C)$ and $\operatorname{Ext}_R^{\geqslant 1}(C, C) = 0$. Semidualizing modules are useful, e.g., for proving results about Bass numbers [4, 14] and compositions of local ring homomorphisms [4, 13].

Takahashi and White [17] define the C-projective dimension for an R-module M (denoted $\mathcal{P}_C\text{-}\operatorname{pd}_R(M)$) to be the length of the shortest resolution by modules of the form $C \otimes_R P$ where P is a projective R-module. They define C-injective dimension ($\mathcal{I}_C\text{-}\operatorname{id}$) dually. In their investigation Takahashi and White posed the following question: When R is a local Cohen-Macaulay ring

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admitting a dualizing module and C is a semidualizing R-module, if there exists an R-module M such that $\mathcal{P}_C\text{-}\operatorname{pd}_R(M) < \infty$ and $\mathcal{I}_C\text{-}\operatorname{id}_R(M) < \infty$, must R be Gorenstein? If M has infinite depth, then the answer is false. However, if we additionally assume that M has finite depth, then an affirmative answer to this question would yield a generalization of Foxby's theorem.

Partial answers to Takahashi and White's question is given by Araya and Takahashi [1] and Sather-Wagstaff and Yassemi [16]. We give a complete answer to this question in the following result; see Section 2 for background on complexes and the derived category, and Theorem 3.2 for the proof.

Theorem 1.1. *Let C be a semidualizing R-complex. If there exists an R-complex $X \in \mathcal{D}_{\mathrm{b}}(R)$ such that $\mathcal{F}_C\text{-}\operatorname{pd}_R(X) < \infty$ and $\mathcal{I}_C\text{-}\operatorname{id}_R(X) < \infty$, then $R_\mathfrak{p}$ is Gorenstein for all $\mathfrak{p} \in \operatorname{supp}_R(X)$.*

2. Background. Let $\mathcal{D}(R)$ denote the derived category of complexes of R-modules, indexed homologically (see e.g. [11,12]). A complex $X \in \mathcal{D}(R)$ is *homologically bounded below*, denoted $X \in \mathcal{D}_+(R)$, if $\mathrm{H}_i(X) = 0$ for all $i \ll 0$. It is *homologically bounded above*, denoted $X \in \mathcal{D}_-(R)$, if $\mathrm{H}_i(X) = 0$ for all $i \gg 0$. It is *homologically degreewise finite*, denoted $X \in \mathcal{D}^{\mathrm{f}}(R)$, if $\mathrm{H}_i(X)$ is finitely generated for all i. Set $\mathcal{D}_{\mathrm{b}}(R) = \mathcal{D}_+(R) \cap \mathcal{D}_-(R)$ and $\mathcal{D}^{\mathrm{f}}_*(R) = \mathcal{D}^{\mathrm{f}}(R) \cap \mathcal{D}_*(R)$ for each $* \in \{+, -, \mathrm{b}\}$. Complexes in $\mathcal{D}^{\mathrm{f}}_{\mathrm{b}}(R)$ are called *homologically finite*. Isomorphisms in $\mathcal{D}(R)$ are identified by the symbol $\simeq$.

For R-complexes X and Y, let $\inf(X)$ and $\sup(X)$ denote the infimum and supremum, respectively, of the set $\{i \in \mathbb{Z} \mid \mathrm{H}_i(X) = 0\}$ with the convention $\sup(\emptyset) = -\infty$ and $\inf(\emptyset) = \infty$. Let $X \otimes^{\mathbf{L}}_R Y$ and $\mathbf{R}\mathrm{Hom}_R(X, Y)$ denote the left-derived tensor product and right-derived homomorphism complexes, respectively.

If $(R, \mathfrak{m}, k)$ is local, the *depth* and *width* of an R-complex $X \in \mathcal{D}(R)$ are defined by Foxby [7] and Yassemi [19] as

$$\operatorname{depth}_R(X) := -\sup(\mathbf{R}\mathrm{Hom}_R(k, X))$$
$$\operatorname{width}_R(X) := \inf(k \otimes^{\mathbf{L}}_R X).$$

One relation between these quantities is given in the following.

The *small support* of an R-complex $X \in \mathcal{D}(R)$ is defined by Foxby [7] as follows:

$$\operatorname{supp}_R(X) := \left\{\mathfrak{p} \in \operatorname{Spec}(R) \mid \kappa(\mathfrak{p}) \otimes^{\mathbf{L}}_R X \not\simeq 0\right\}.$$

An important property of the small support is given in the following.

Fact 2.1 ([7, Proposition 2.7]). If $X, Y \in \mathcal{D}(R)$, then

$$\operatorname{supp}_R\left(X \otimes^{\mathbf{L}}_R Y\right) = \operatorname{supp}_R(X) \cap \operatorname{supp}_R(Y).$$

The *flat dimension* of an R-complex $X \in \mathcal{D}_+(R)$ is

$$\operatorname{fd}_R(X) := \inf\left\{ n \in \mathbb{Z} \,\middle|\, \begin{array}{l} F \xrightarrow{\simeq} X \text{ where } F \text{ is a bounded below complex of} \\ \text{flat } R\text{-modules such that } F_i = 0 \text{ for all } i > n \end{array} \right\}.$$

The *injective dimension* of an R-complex $Y \in \mathcal{D}_-(R)$ is

$$\mathrm{id}_R(X) := \inf \left\{ n \in \mathbb{Z} \,\middle|\, \begin{array}{l} Y \xrightarrow{\simeq} I \text{ where } I \text{ is a bounded above complex of} \\ \text{injective } R\text{-modules such that } I_j = 0 \text{ for all } j > -n \end{array} \right\}.$$

A homologically finite R-complex C is *semidualizing* if the homothety morphism $\chi_C^R : R \to \mathbf{R}\mathrm{Hom}_R(C, C)$ is an isomorphism in $\mathcal{D}(R)$. An R-complex D is *dualizing* if it is semidualizing and has finite injective dimension. Dualizing complexes were introduced by Grothendieck and Hartshorne [12], and semidualizing complexes originate in work of Foxby [6], Avramov and Foxby [4], and Christensen [5]. For the non-commutative case, see, e.g., Araya, Takahashi, and Yoshino [2].

Assumption 2.2. For the rest of this paper, let C be a semidualizing R-complex.

The following classes were defined in [4,5]. The *Auslander Class* with respect to C is the full subcategory $\mathcal{A}_C(R) \subseteq \mathcal{D}_\mathrm{b}(R)$ such that a complex X is in $\mathcal{A}_C(R)$ if and only if $C \otimes_R^{\mathbf{L}} X \in \mathcal{D}_\mathrm{b}(R)$ and the natural morphism $\gamma_X^C : X \to \mathbf{R}\mathrm{Hom}_R(C, C \otimes_R^{\mathbf{L}} X)$ is an isomorphism in $\mathcal{D}(R)$. Dually, the *Bass Class* with respect to C is the full subcategory $\mathcal{B}_C(R) \subseteq \mathcal{D}_\mathrm{b}(R)$ such that a complex Y is in $\mathcal{B}_C(R)$ if and only if $\mathbf{R}\mathrm{Hom}_R(C, Y) \in \mathcal{D}_\mathrm{b}(R)$ and the natural morphism $\xi_Y^C : C \otimes_R^{\mathbf{L}} \mathbf{R}\mathrm{Hom}_R(C, Y) \to Y$ is an isomorphism in $\mathcal{D}(R)$.

The $\mathcal{F}_C$-projective dimension and $\mathcal{I}_C$-injective dimension of an R-complex $X \in \mathcal{D}_\mathrm{b}(R)$ are defined in [18] as follows:

$$\mathcal{F}_C\text{-}\,\mathrm{pd}_R(X) := \sup(C) + \mathrm{fd}_R(\mathbf{R}\mathrm{Hom}_R(C, X))$$
$$\mathcal{I}_C\text{-}\,\mathrm{id}_R(X) := \sup(C) + \mathrm{id}_R(C \otimes_R^{\mathbf{L}} X).$$

The following fact shows that the above definitions are consistent with the ones given by Takahashi and White [17] when C is a semidualizing module.

Fact 2.3 ([18, Theorem 3.9]). Let $X \in \mathcal{D}_\mathrm{b}(R)$.

(a) We have $\mathcal{F}_C\text{-}\,\mathrm{pd}_R(X) < \infty$ if and only if there exists an R-complex $F \in \mathcal{D}_\mathrm{b}(R)$ such that $\mathrm{fd}_R(F) < \infty$ and $X \simeq C \otimes_R^{\mathbf{L}} F$. When these conditions are satisfied, one has $F \simeq \mathbf{R}\mathrm{Hom}_R(C, X)$ and $X \in \mathcal{B}_C(R)$.

(b) We have $\mathcal{I}_C\text{-}\,\mathrm{id}_R(X) < \infty$ if and only if there exists an R-complex $J \in \mathcal{D}_\mathrm{b}(R)$ such that $\mathrm{id}_R(J) < \infty$ and $X \simeq \mathbf{R}\mathrm{Hom}_R(C, J)$. When these conditions are satisfied, one has $J \simeq C \otimes_R^{\mathbf{L}} X$ and $X \in \mathcal{A}_C(R)$.

3. Results. The next result fully answers the question of Takahashi and White discussed in the introduction.

Theorem 3.1. *Let $(R, \mathfrak{m}, k)$ be a local ring. If there is an R-complex $X \in \mathcal{D}_\mathrm{b}(R)$ with finite depth, $\mathcal{F}_C\text{-}\,\mathrm{pd}_R(X) < \infty$ and $\mathcal{I}_C\text{-}\,\mathrm{id}_R(X) < \infty$, then R is Gorenstein.*

Proof. Case 1:$\mathrm{depth}_R(X) < \infty$ *and* R has a dualizing complex D.

We first observe that by [8, Theorem 4.6] we have that the following:

$$\mathrm{depth}_R(\mathbf{R}\mathrm{Hom}_R(C, X)) = \mathrm{width}_R(C) + \mathrm{depth}_R(X) < \infty. \qquad (3.1.1)$$

Note that $\mathrm{depth}_R(X)$ is finite by assumption, and $\mathrm{width}_R(C)$ is finite by Nakayama's Lemma as C is homologically finite: see [7, Lemma 2.1].

Set $C^\dagger := \mathbf{R}\mathrm{Hom}_R(C,D)$. The assumption $\mathcal{I}_C\text{-}\mathrm{id}_R(X) < \infty$ with [18, Theorem 1.2] implies $\mathcal{F}_{C^\dagger}\text{-}\mathrm{pd}_R(X) < \infty$. Hence by Fact 2.3(a) there exist R-complexes F, G of finite flat dimension such that $C \otimes^{\mathbf{L}}_R F \simeq X \simeq C^\dagger \otimes^{\mathbf{L}}_R G$. Since G has finite flat dimension, [5, Proposotion 4.4] implies $G \in \mathcal{A}_{C^\dagger}(R)$, which explains the first isomorphism in the following display:

$$G \simeq \mathbf{R}\mathrm{Hom}_R(C^\dagger, C^\dagger \otimes^{\mathbf{L}}_R G) \simeq \mathbf{R}\mathrm{Hom}_R(C^\dagger, C \otimes^{\mathbf{L}}_R F) \simeq \mathbf{R}\mathrm{Hom}_R(C^\dagger, C) \otimes^{\mathbf{L}}_R F.$$

The last isomorphism is by tensor evaluation [3, Lemma 4.4(F)].

Fact 2.3(a) implies $F \simeq \mathbf{R}\mathrm{Hom}_R(C,X)$. By (3.1.1) we have $\mathrm{depth}_R(F) < \infty$. It follows from [7, Proposition 2.8] that $k \otimes^{\mathbf{L}}_R F \not\simeq 0$. For the rest of the proof, set $U := \mathbf{R}\mathrm{Hom}_R(C^\dagger, C)$. Since C and $C^\dagger$ are in $\mathcal{D}^{\mathrm{f}}_{\mathrm{b}}(R)$, we have $U \in \mathcal{D}^{\mathrm{f}}_{-}(R)$.

Claim A: $U \in \mathcal{D}^{\mathrm{f}}_{\mathrm{b}}(R)$.

To prove this claim it suffices to show that $U \in \mathcal{D}_{+}(R)$. Assume by way of contradiction that $\inf(U) = -\infty$. Then by [8, 4.5] we know that $\inf(k \otimes^{\mathbf{L}}_R U) = -\infty$. By tensor cancellation and the Künneth formula, we have isomorphisms

$$\begin{aligned} \mathrm{H}_n\left(k \otimes^{\mathbf{L}}_R \left(F \otimes^{\mathbf{L}}_R U\right)\right) &\cong \mathrm{H}_n\left(\left(k \otimes^{\mathbf{L}}_R F\right) \otimes^{\mathbf{L}}_k \left(k \otimes^{\mathbf{L}}_R U\right)\right) \\ &\cong \bigoplus_{p+q=n} \mathrm{H}_p\left(k \otimes^{\mathbf{L}}_R F\right) \otimes_k \mathrm{H}_q\left(k \otimes^{\mathbf{L}}_R U\right). \end{aligned}$$

Since $k \otimes^{\mathbf{L}}_R F \not\simeq 0$ and $\inf(k \otimes^{\mathbf{L}}_R U) = -\infty$, it follows that $\inf(k \otimes^{\mathbf{L}}_R (F \otimes^{\mathbf{L}}_R U)) = -\infty$. On the other hand, since $F \otimes^{\mathbf{L}}_R U \simeq G \in \mathcal{D}_{\mathrm{b}}(R)$ we have $k \otimes^{\mathbf{L}}_R (F \otimes^{\mathbf{L}}_R U) \simeq k \otimes^{\mathbf{L}}_R G \in \mathcal{D}_{+}(R)$, so $\inf(k \otimes^{\mathbf{L}}_R (F \otimes^{\mathbf{L}}_R U)) > -\infty$, a contradiction. This establishes Claim A.

Claim B: The complex U has finite projective dimension.

To show this claim, assume by way of contradiction that $\mathrm{pd}_R(U) = \infty$. Then because $U \in \mathcal{D}^{\mathrm{f}}_{\mathrm{b}}(R)$ we have $\sup(k \otimes^{\mathbf{L}}_R U) = \infty$ by [3, Proposition 5.5]. As in the proof of Claim A, we conclude that $\sup(k \otimes^{\mathbf{L}}_R (F \otimes^{\mathbf{L}}_R U)) = \infty$. On the other hand, we have $k \otimes^{\mathbf{L}}_R (F \otimes^{\mathbf{L}}_R U) \simeq k \otimes^{\mathbf{L}}_R G$. Since G has finite flat dimension, this implies that $\sup(k \otimes^{\mathbf{L}}_R (F \otimes^{\mathbf{L}}_R U)) < \infty$, a contradiction. This concludes the proof of Claim B.

Now [10, Theorem 1.4] implies that $\Sigma^n C \simeq C^\dagger = \mathbf{R}\mathrm{Hom}_R(C,D)$ for some $n \in \mathbb{Z}$. Hence by [9, Corollary 3.4] we deduce that R is Gorenstein. This concludes the proof of Case 1.

Case 2: $\mathrm{supp}_R(X) = \{\mathfrak{m}\}$.

For the proof of Case 2, first observe that R is Gorenstein if and only if $\widehat{R}$ is Gorenstein. Since $\widehat{R}$ has a dualizing complex, by Case 1 it suffices to show that

(1) $\widehat{R} \otimes^{\mathbf{L}}_R X \in \mathcal{D}_{\mathrm{b}}(\widehat{R})$,
(2) $\widehat{R} \otimes^{\mathbf{L}}_R C$ is a semidualizing $\widehat{R}$-complex,
(3) $\mathcal{F}_{\widehat{R} \otimes^{\mathbf{L}}_R C}\text{-}\mathrm{pd}_{\widehat{R}}(\widehat{R} \otimes^{\mathbf{L}}_R X) < \infty$,
(4) $\mathcal{I}_{\widehat{R} \otimes^{\mathbf{L}}_R C}\text{-}\mathrm{id}_{\widehat{R}}(\widehat{R} \otimes^{\mathbf{L}}_R X) < \infty$, and
(5) $\mathrm{depth}_{\widehat{R}}(\widehat{R} \otimes^{\mathbf{L}}_R X) < \infty$.

Item (1) follows from the fact that $\widehat{R}$ is flat over R. Items (2) and (3) follow from [5, Lemma 2.6] and [18, Proposition 3.11], respectively.

To prove (4) note that the first equality in the next sequence is by definition:

$$\begin{aligned}\mathcal{I}_{\widehat{R}\otimes^{\mathbf{L}}_R C}\text{-}\operatorname{id}_{\widehat{R}}\left(\widehat{R}\otimes^{\mathbf{L}}_R X\right) &= \operatorname{id}_{\widehat{R}}\left(\left(\widehat{R}\otimes^{\mathbf{L}}_R C\right)\otimes^{\mathbf{L}}_{\widehat{R}}\left(\widehat{R}\otimes^{\mathbf{L}}_R X\right)\right)+\sup\left(\widehat{R}\otimes^{\mathbf{L}}_R C\right)\\ &= \operatorname{id}_{\widehat{R}}\left(\widehat{R}\otimes^{\mathbf{L}}_R\left(C\otimes^{\mathbf{L}}_R X\right)\right)+\sup\left(\widehat{R}\otimes^{\mathbf{L}}_R C\right).\end{aligned}$$

The second equality is by tensor cancellation. From the condition $\mathcal{I}_C\text{-}\operatorname{id}_R(X) < \infty$, we have $\operatorname{id}_R(C \otimes^{\mathbf{L}}_R X) < \infty$ by definition. Note that $\mathfrak{m} \in \operatorname{Spec}(R) = \operatorname{supp}_R(C)$ by [15, Proposition 6.6]. Therefore Fact 2.1 implies

$$\operatorname{supp}_R\left(C\otimes^{\mathbf{L}}_R X\right) = \operatorname{supp}_R(C)\cap\operatorname{supp}_R(X) = \{\mathfrak{m}\}.$$

Hence by [13, Lemma 3.4] the complex $\widehat{R}\otimes^{\mathbf{L}}_R (C\otimes^{\mathbf{L}}_R X)$ has finite injective dimension over $\widehat{R}$, so (4) holds.

For the proof of (5) consider the following sequence:

$$\begin{aligned}\operatorname{depth}_{\widehat{R}}\left(\widehat{R}\otimes^{\mathbf{L}}_R X\right) &= -\sup\left(\mathbf{R}\operatorname{Hom}_{\widehat{R}}\left(k,\widehat{R}\otimes^{\mathbf{L}}_R X\right)\right)\\ &= -\sup\left(\mathbf{R}\operatorname{Hom}_{\widehat{R}}\left(\widehat{R}\otimes^{\mathbf{L}}_R k,\widehat{R}\otimes^{\mathbf{L}}_R X\right)\right)\\ &= -\sup\left(\widehat{R}\otimes^{\mathbf{L}}_R \mathbf{R}\operatorname{Hom}_R(k,X)\right)\\ &= -\sup\left(\mathbf{R}\operatorname{Hom}_R(k,X)\right)\\ &= \operatorname{depth}_R(X)\\ &< \infty.\end{aligned}$$

The second equality is because $k \cong \widehat{R}\otimes^{\mathbf{L}}_R k$, and the fourth equality is because $\widehat{R}$ is faithfully flat over R. This establishes (5) and concludes Case 2.

Case 3: general case.

Let $\mathbf{x}$ be a generating sequence for $\mathfrak{m}$, and let $K = K^R(\mathbf{x})$ be the Koszul complex. Then $\operatorname{supp}_R(K) = \{\mathfrak{m}\}$. Since $\operatorname{depth}_R(X) < \infty$, we have that $\mathfrak{m} \in \operatorname{supp}_R(X)$ by [7, Proposition 2.8]. Hence, we conclude from Fact 2.1 that

$$\operatorname{supp}_R\left(K\otimes^{\mathbf{L}}_R X\right) = \operatorname{supp}_R(K)\cap\operatorname{supp}_R(X) = \{\mathfrak{m}\}.$$

By Case 2 it suffices to show that

(a) $\operatorname{depth}_R(K\otimes^{\mathbf{L}}_R X) < \infty$,
(b) $K\otimes^{\mathbf{L}}_R X \in \mathcal{D}_{\mathrm{b}}(R)$,
(c) $\mathcal{F}_C\text{-}\operatorname{pd}_R(K\otimes^{\mathbf{L}}_R X) < \infty$, and
(d) $\mathcal{I}_C\text{-}\operatorname{id}_R(K\otimes^{\mathbf{L}}_R X) < \infty$.

Item (a) follows from [7, Proposition 2.8]. For (b), use the conditions $\operatorname{pd}_R(K) < \infty$ and $X \in \mathcal{D}_{\mathrm{b}}(R)$. Items (c) and (d) follow from [18, Proposition 4.5 and 4.7]. This concludes the proof of Case 3. □

The following result is Theorem 1.1 from the introduction.

Theorem 3.2. *If there is an R-complex $X \in \mathcal{D}_{\mathrm{b}}(R)$ such that $\mathcal{F}_C\text{-}\operatorname{pd}_R(X) < \infty$ and $\mathcal{I}_C\text{-}\operatorname{id}_R(X) < \infty$, then $R_{\mathfrak{p}}$ is Gorenstein for all $\mathfrak{p} \in \operatorname{supp}_R(X)$.*

Proof. By Theorem 3.1 it suffices to show the following:

(i) $X_{\mathfrak{p}} \in \mathcal{D}_{\mathrm{b}}(R_{\mathfrak{p}})$,

(ii) $C_{\mathfrak{p}}$ is a semidualizing $R_{\mathfrak{p}}$-complex,
(iii) $\mathcal{F}_{C_{\mathfrak{p}}}\text{-}\operatorname{pd}_{R_{\mathfrak{p}}}(X_{\mathfrak{p}}) < \infty$,
(iv) $\mathcal{I}_{C_{\mathfrak{p}}}\text{-}\operatorname{id}_{R_{\mathfrak{p}}}(X_{\mathfrak{p}}) < \infty$, and
(v) $\operatorname{depth}_{R_{\mathfrak{p}}}(X_{\mathfrak{p}}) < \infty$.

Item (i) follows from the fact that $R_{\mathfrak{p}}$ is a flat over R, and item (ii) follows from [5, Lemma 2.5]. Items (iii) and (iv) are by [18, Corollary 3.12].

(v) As $\mathfrak{p} \in \operatorname{supp}_R(X)$, we have $\mathfrak{p}R_{\mathfrak{p}} \in \operatorname{supp}_{R_{\mathfrak{p}}}(X_{\mathfrak{p}})$ by [15, Proposition 3.6]. Since $\mathfrak{p}R_{\mathfrak{p}}$ is the maximal ideal of the local ring $R_{\mathfrak{p}}$, we deduce from [7, Proposition 2.8] that $\operatorname{depth}_{R_{\mathfrak{p}}}(X_{\mathfrak{p}}) < \infty$. □

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