From the Group $SL(2, C)$ to Gyrogroups and Gyrovector Spaces and Hyperbolic Geometry

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We show that the algebra of the group $SL(2, C)$ naturally leads to the notion of gyrogroups and gyrovector spaces for dealing with the Lorentz group and its underlying hyperbolic geometry. The superiority of the use of the gyrogroup formalism over the use of the $SL(2, C)$ formalism for dealing with the Lorentz group in some cases is indicated by (i) the validity of gyrogroups and gyrovector spaces in higher dimensions, by (ii) the analogies that they share with groups and vector spaces, and by (iii) the demonstration that gyrovector spaces form the setting for hyperbolic geometry in the same way that vector spaces form the setting for Euclidean geometry. As such, gyrogroups and gyrovector spaces provide powerful tools for the study of relativity physics.

1 INTRODUCTION

Relativistically admissible velocities are elements of the open ball $\mathbb{R}^3_c$ with radius c of the Euclidean three-space $\mathbb{R}^3$,

$$\mathbb{R}^3_c = \left\{\mathbf{v} \in \mathbb{R}^3 : \|\mathbf{v}\| < c\right\} \tag{1}$$

c being the vacuum speed of light. The Einstein velocity addition is a binary operation $\oplus$ in the ball $\mathbb{R}^3_c$ given by the equation

$$\mathbf{u}\oplus\mathbf{v} = \frac{1}{1+\frac{\mathbf{u}\cdot\mathbf{v}}{c^2}}\left\{\mathbf{u} + \frac{1}{\gamma_\mathbf{u}}\mathbf{v} + \frac{1}{c^2}\frac{\gamma_\mathbf{u}}{1+\gamma_\mathbf{u}}(\mathbf{u}\cdot\mathbf{v})\mathbf{u}\right\} \tag{2}$$

for $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3_c$. Here $\cdot$ represents the usual dot product in $\mathbb{R}^3$, and $\gamma_{\mathbf{u}}$ is the *Lorentz factor* given by

$$\gamma_{\mathbf{u}} = \frac{1}{\sqrt{1 - \frac{\|\mathbf{u}\|^2}{c^2}}} . \tag{3}$$

When $\mathbf{u}$ and $\mathbf{v}$ are non-parallel, Einstein's addition (2) is neither commutative nor associative. Einstein's addition in the ball $\mathbb{R}^3_c$ thus forms a nongroup *groupoid* $(\mathbb{R}^3_c, \oplus)$, a groupoid being a nonempty set with a binary operation. In contrast, Newtonian velocities do form a group, $(\mathbb{R}^3, +)$, under ordinary vector addition. Since groups measure symmetry and exhibit mathematical regularity, it seems that the transition from Newtonian velocities to Einsteinian velocities is associated with loss of both symmetry and mathematical regularity. This is, however, not the case since Einstein's addition admits a natural rotation that "repairs" the breakdown of its commutativity and associativity, giving rise to gyrocommutativity and gyroassociativity, as we explain below.

The rotation that repairs the breakdown of commutativity and associativity in Einstein's addition is the Thomas precession, $\mathrm{gyr}[\mathbf{u}, \mathbf{v}]$, $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3_c$, that rotates any vector $\mathbf{w} \in \mathbb{R}^3$ into the vector $\mathrm{gyr}[\mathbf{u}, \mathbf{v}]\mathbf{w} \in \mathbb{R}^3$ according to the equation

$$\mathrm{gyr}[\mathbf{u}, \mathbf{v}]\mathbf{w} = -(\mathbf{u}\oplus\mathbf{v})\oplus\{\mathbf{u}\oplus(\mathbf{v}\oplus\mathbf{w})\} \tag{4}$$

Extending the Thomas precession by abstraction, the abstract Thomas precession is called Thomas gyration, suggesting the prefix "gyro" that is used to emphasize analogies with classical notions. Thomas gyration gives rise to the gyrocommutative law,

$$\mathbf{u}\oplus\mathbf{v} = \mathrm{gyr}[\mathbf{u}, \mathbf{v}](\mathbf{v}\oplus\mathbf{u})$$

and to the gyroassociative law (left and right) as well,

$$\begin{aligned} \mathbf{u}\oplus(\mathbf{v}\oplus\mathbf{w}) &= (\mathbf{u}\oplus\mathbf{v})\oplus\mathrm{gyr}[\mathbf{u}, \mathbf{v}]\mathbf{w} \\ (\mathbf{u}\oplus\mathbf{v})\oplus\mathbf{w} &= \mathbf{u}\oplus(\mathbf{v}\oplus\mathrm{gyr}[\mathbf{v}, \mathbf{u}]\mathbf{w}) \end{aligned}$$

thus discovering a grouplike structure called a *gyrogroup*. Moreover, Thomas precession possesses the so called *loop property*,

$$\mathrm{gyr}[\mathbf{u}, \mathbf{v}] = \mathrm{gyr}[\mathbf{u}\oplus\mathbf{v}, \mathbf{v}]$$

which makes it algebraically effective in uncovering analogies shared with groups and gyrogroups [1].

Thus,

(1) Einstein's addition, $\oplus$, is a gyrocommutative gyrogroup operation in the Einstein relativity gyrogroup $(\mathbb{R}^3_c, \oplus)$ of relativistically admissible velocities, in full analogy with ordinary

(2) vector addition, +, which is a commutative group operation in the group $(\mathbb{R}^3, +)$ of prerelativity velocities.

Einstein's addition also admits scalar multiplication, $\otimes$, turning the Einstein gyrogroup into a gyrovector space $(\mathbb{R}^3_c, \oplus, \otimes)$. The latter, in turn, forms the setting for the Beltrami model of hyperbolic geometry in the same way that vector spaces form the setting for Euclidean geometry. Thus, for instance, the gyrovector space valued function

$$L_h(t) = \mathbf{u} \oplus (\ominus \mathbf{u} \oplus \mathbf{v}) \otimes t \tag{5}$$

$\mathbf{u}, \mathbf{v} \in \mathbb{R}^3_c$, $\ominus \mathbf{u} = -\mathbf{u}$, of a real variable $t \in \mathbb{R}$ represents a geodesic in the Beltrami model of hyperbolic geometry, Fig. 1, in the same way that the vector space valued function

$$L_e(t) = \mathbf{u} + (-\mathbf{u} + \mathbf{v})t \tag{6}$$

$\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$, of a real variable $t \in \mathbb{R}$ represents a straight line in Euclidean geometry.

The replacement of Einstein addition $\oplus = \oplus_{\rm E}$ in (5) by some other gyrocommutative gyrogroup operations results in geodesics of other models of hyperbolic geometry. These are shown in Fig. 1 for $\oplus = \oplus_{\rm E}$, in Fig. 2 for $\oplus = \oplus_{\rm M}$, in Fig. 3 for $\oplus = \oplus_{\rm U}$, and in Fig. 4 for $\oplus = \oplus_{\rm C}$.

The theory of gyrogroups and gyrovector spaces and its applications are presented in [1], [2], [3], and in a forthcoming book [4], where the advantages offered by the use of gyrogroup-theoretic techniques in the manipulation of the Lorentz group for solving important problems are demonstrated.

Taking the key features of Einstein's addition as axioms, and guided by analogies with group theory, we arrive at the following definition of a gyrogroup and a gyrocommutative gyrogroup. Here a groupoid $(G, \oplus)$ is a nonempty set G with a binary operation $\oplus$, and an automorphism of the groupoid $(G, \oplus)$ is a bijective self-map of the groupoid G which respects its binary operation $\oplus$. The set of all automorphisms of a groupoid $(G, \oplus)$ forms a group under bijection composition, denoted $Aut(G, \oplus)$.

Definition 1 (Gyrogroups). *The groupoid $(G, \oplus)$ is a gyrogroup if its binary operation satisfies the following axioms. In G there is at least one element, 0, called a left identity, satisfying*

(G1) $0 \oplus a = a$ *Left Identity*

for all $a \in G$. There is an element $0 \in G$ satisfying axiom $(G1)$ such that for each a in G there is an element $\ominus a$ in G, called a left inverse of a, satisfying

(G2) $\ominus a \oplus a = 0$ *Left Inverse*

Moreover, for any $a, b, z \in G$ *there exists a unique element* $\mathrm{gyr}[a, b]z \in G$ *such that*

(G3) $\quad a\oplus(b\oplus z) = (a\oplus b)\oplus\mathrm{gyr}[a, b]z \qquad$ *Left Gyroassociative Law*

If $gyr[a, b]$ *denotes the map* $gyr[a, b] : G \to G$ *given by* $z \mapsto gyr[a, b]z$ *then*

(G4) $\quad \mathrm{gyr}[a, b] \in \mathrm{Aut}(G, \oplus) \qquad$ *Gyroautomorphism*

and $gyr[a, b]$ *is called the Thomas gyration, or the gyroautomorphism of* G*, generated by* $a, b \in G$*. The operation* $\mathrm{gyr} : G \times G \to \mathrm{Aut}(G, \oplus)$ *is called the gyrooperation of* G*. Finally, the gyroautomorphism* $gyr[a, b]$ *generated by any* $a, b \in G$ *satisfies*

(G5) $\quad \mathrm{gyr}[a, b] = \mathrm{gyr}[a\oplus b, b] \qquad$ *Left Loop Property*

In full analogy with groups, gyrogroups are classified into gyrocommutative and non-gyrocommutative gyrogroups. The definition of gyrocommutativity in gyrogroups follows.

Definition 2 (Gyrocommutative Gyrogroups). *The gyrogroup* $(G, \oplus)$ *is gyrocommutative if for all* $a, b \in G$

(G6) $\quad a \oplus b = \mathrm{gyr}[a, b](b \oplus a) \qquad$ *Gyrocommutative Law*

The gyrogroup axioms, as well as resulting gyrogroup theorems, are studied in [1] and [4], and in references therein. Gyrogroup-theoretic techniques allow the discovery of the truly Hyperbolic Pythagorean Theorem in [5] [6], and the truly hyperbolic trigonometry in [7] [8].

Einstein's addition admits the scalar multiplication $\otimes$ given by the equation

$$\begin{aligned} r\otimes\mathbf{v} &= c\frac{(1 + \|\mathbf{v}\|/c)^r - (1 - \|\mathbf{v}\|/c)^r}{(1 + \|\mathbf{v}\|/c)^r + (1 - \|\mathbf{v}\|/c)^r}\frac{\mathbf{v}}{\|\mathbf{v}\|} \\ &= c\,\tanh\left(r\,\tanh^{-1}\frac{\|\mathbf{v}\|}{c}\right)\frac{\mathbf{v}}{\|\mathbf{v}\|} \end{aligned} \tag{7}$$

where $r \in \mathbb{R}$, $\mathbf{v} \in \mathbb{V}_c$, $\mathbf{v} \neq \mathbf{0}$, and $r\otimes_{\mathrm{E}}\mathbf{0} = \mathbf{0}$, and with which we use the notation $\mathbf{v}\otimes r = r\otimes\mathbf{v}$. This scalar multiplication plays harmoniously with Einstein's addition, satisfying the following properties [1] that give rise to the gyrovector space definition in [4].

For any positive integer n and for all $r, r_1, r_2 \in \mathbb{R}$ and $\mathbf{a}, \mathbf{b}, \mathbf{u}, \mathbf{v} \in \mathbb{V}_c$,

$$\begin{aligned}
n\otimes\mathbf{v} &= \mathbf{v}\oplus\ldots\oplus\mathbf{v} && n \text{ terms}\\
(r_1+r_2)\otimes\mathbf{v} &= r_1\otimes\mathbf{v}\oplus r_2\otimes\mathbf{v} && \text{Scalar Distributive Law}\\
(r_1 r_2)\otimes\mathbf{v} &= r_1\otimes(r_2\otimes\mathbf{v}) && \text{Scalar Associative Law}\\
r\otimes(r_1\otimes\mathbf{v}\oplus r_2\otimes\mathbf{v}) &= r\otimes(r_1\otimes\mathbf{v})\oplus r\otimes(r_2\otimes\mathbf{v}) && \text{Monodistributive Law}\\
\|r\otimes\mathbf{v}\| &= |r|\otimes\|\mathbf{v}\| && \text{Homogeneity Property}\\
\frac{|r|\otimes\mathbf{v}}{\|r\otimes\mathbf{v}\|} &= \frac{\mathbf{v}}{\|\mathbf{v}\|} && \text{Scaling Property}\\
\|\mathbf{u}\oplus\mathbf{v}\| &\le \|\mathbf{u}\|\oplus\|\mathbf{v}\| && \text{Triangle Inequality}\\
\mathrm{gyr}[\mathbf{a},\mathbf{b}](r\otimes\mathbf{v}) &= r\otimes\mathrm{gyr}[\mathbf{a},\mathbf{b}]\mathbf{v} && \text{Gyroautomorphism}\\
\mathrm{gyr}[r_1\otimes\mathbf{v}, r_2\otimes\mathbf{v}] &= I && \text{Identity Automorphism}
\end{aligned} \tag{8}$$

Demonstrations of the elegance and power of the tools that gyrogroup-theoretic techniques offer in the study of the Lorentz group and its applications are found in the book *Beyond the Einstein Addition Law and its Gyroscopic Thomas Precession: The Theory of Gyrogroups and Gyrovector Spaces* [4].

The aim of this paper is to present a natural way in which the algebra of the $SL(2, C)$ group leads to gyrogroups and gyrovector spaces. This natural way convincingly demonstrates that the theory of gyrogroups and gyrovector spaces provides a most powerful formalism for dealing with the Lorentz group and hyperbolic geometry, the geometry that governs the special theory of relativity as well as other areas of physics (see, for instance, [9] and [10]). It is therefore hoped that, following this article, gyrogroup and gyrovector space theoretic techniques will provide standard tools in the study of relativity physics and, as such, will become part of the lore learned by all explorers who are interested in relativity physics.

Links between gyrogroups and other mathematical objects are presented in [11] [12] [13] and [14]. Furthermore, our approach to gyrogroups and scalar multiplication in a gyrogroup of gyrovectors can be used as a preparation for the study of a related, but more abstract study of Sabinin's odules in [15]. A related study of quasigroups in differential geometry is presented by Sabinin and Miheev on pp. 357 – 430 of [16].

2 THE ALGEBRA OF THE $SL(2,C)$ GROUP

Let $\mathbb{R}^3_c$ be the set of all relativistically admissible velocities,

$$\mathbb{R}^3_c = \{\mathbf{v} \in \mathbb{R}^3 : \|\mathbf{v}\| < c\}$$

It is the ball of radius c, $c > 0$, of the Euclidean 3-space $\mathbb{R}^3$, c being the vacuum speed of light. A boost $L(\mathbf{v})$ is a pure Lorentz transformation, that is, a Lorentz transformation without rotation, parametrized by a velocity parameter $\mathbf{v} \in \mathbb{R}^3_c$. The boost $L(\mathbf{v})$ is a linear transformation of spacetime coordinates which has the matrix representation $L_m(\mathbf{v})$, $\mathbf{v} = (v_1, v_2, v_3)^t$,

$$L_m(\mathbf{v}) = \begin{pmatrix} \gamma_{\mathbf{v}} & c^{-2}\gamma_{\mathbf{v}} v_1 & c^{-2}\gamma_{\mathbf{v}} v_2 & c^{-2}\gamma_{\mathbf{v}} v_3 \\ \gamma_{\mathbf{v}} v_1 & 1 + c^{-2}\frac{\gamma_{\mathbf{v}}^2}{\gamma_{\mathbf{v}}+1} v_1^2 & c^{-2}\frac{\gamma_{\mathbf{v}}^2}{\gamma_{\mathbf{v}}+1} v_1 v_2 & c^{-2}\frac{\gamma_{\mathbf{v}}^2}{\gamma_{\mathbf{v}}+1} v_1 v_3 \\ \gamma_{\mathbf{v}} v_2 & c^{-2}\frac{\gamma_{\mathbf{v}}^2}{\gamma_{\mathbf{v}}+1} v_1 v_2 & 1 + c^{-2}\frac{\gamma_{\mathbf{v}}^2}{\gamma_{\mathbf{v}}+1} v_2^2 & c^{-2}\frac{\gamma_{\mathbf{v}}^2}{\gamma_{\mathbf{v}}+1} v_2 v_3 \\ \gamma_{\mathbf{v}} v_3 & c^{-2}\frac{\gamma_{\mathbf{v}}^2}{\gamma_{\mathbf{v}}+1} v_1 v_3 & c^{-2}\frac{\gamma_{\mathbf{v}}^2}{\gamma_{\mathbf{v}}+1} v_2 v_3 & 1 + c^{-2}\frac{\gamma_{\mathbf{v}}^2}{\gamma_{\mathbf{v}}+1} v_3^2 \end{pmatrix} \tag{10}$$

[17] so that

$$L(\mathbf{v}) \begin{pmatrix} t \\ \mathbf{x} \end{pmatrix} = L_m(\mathbf{v}) \begin{pmatrix} t \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} t' \\ x'_1 \\ x'_2 \\ x'_3 \end{pmatrix} \tag{11}$$

Here $\mathbf{v} = (v_1, v_2, v_3)^t \in \mathbb{R}^3_c$ and $\mathbf{x} = (x_1, x_2, x_3)^t \in \mathbb{R}^3$, where exponent t denotes transposition.

In the limit of large vacuum speed of light, $c \to \infty$, the Lorentz boost (11) reduces to the Galilei boost $G(\mathbf{v})$, $\mathbf{v} \in \mathbb{R}^3$,

$$G(\mathbf{v}) \begin{pmatrix} t \\ \mathbf{x} \end{pmatrix} = \lim_{c\to\infty} L(\mathbf{v}) \begin{pmatrix} t \\ \mathbf{x} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ v_1 & 1 & 0 & 0 \\ v_2 & 0 & 1 & 0 \\ v_3 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} t \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} t \\ x_1 + v_1 t \\ x_2 + v_2 t \\ x_3 + v_3 t \end{pmatrix} \tag{12}$$

The composition of two Galilei boosts is equivalent to a single Galilei boost according to the equation

$$G(\mathbf{u})G(\mathbf{v}) = G(\mathbf{u} + \mathbf{v}) \tag{13}$$

The composition of Lorentz boosts is more complicated than that of the Galilei boosts. In general, the composition of two Lorentz boosts $L(\mathbf{u})$ and $L(\mathbf{v})$ is not a boost but, rather, a boost preceded (or followed) by a space rotation $\mathrm{Gyr}[\mathbf{u}, \mathbf{v}]$,

$$L(\mathbf{u})L(\mathbf{v}) = L(\mathbf{w})\mathrm{Gyr}[\mathbf{u}, \mathbf{v}] \tag{14}$$

where $\mathbf{w} \in \mathbb{R}^3_c$ is the velocity composition of $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3_c$, yet to be determined, and $\mathrm{Gyr}[\mathbf{u}, \mathbf{v}]$ is the space rotation of spacetime coordinates that the two relativistically admissible velocities $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3_c$ generate. This space rotation is known as the *Thomas precession* [1], or the Thomas rotation, or the Thomas gyration.

The determination of the Thomas rotation in terms of its generating velocities $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3_c$ used to be a formidable task, as noted by Goldstein in his *Classical Mechanics* book [18]. Discouraging any attempt to simplify the Thomas rotation matrix to the point where its rotation-matrix behavior can be actually demonstrated and applied to related problems, Goldstein notes that

> The decomposition process [describing successive pure Lorentz transformations as a pure Lorentz transformation preceded, or followed, by a Thomas rotation] can be carried through on the product of two pure Lorentz transformations to obtain explicitly the rotation of the coordinate axes resulting from the two successive boosts [that is, the Thomas rotation]. In general, the algebra involved is *quite forbidding,* more than enough, usually, to *discourage any actual demonstration* of the rotation matrix [italics added].
>
> Herbert Goldstein, Classical Mechanics

Currently, the widely accepted way to simplify the algebra of the Lorentz transformation group is provided by the Pauli spin matrices σ_k, $k = 1, 2, 3$, which together with the identity 2×2 matrix $\mathbf{1}$, are the matrices [19]

$$\mathbf{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{15}$$

where $i = \sqrt{-1}$. Using the vector notation

$$\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3) \tag{16}$$

and

$$\boldsymbol{\sigma}\cdot\mathbf{v} = v_1\sigma_1 + v_2\sigma_2 + v_3\sigma_3 \tag{17}$$

for any $\mathbf{v} = (v_1, v_2, v_3)^t \in \mathbb{R}^3$, we have the multiplication rule in the following ([18], p. 186)

Theorem 3.

$$(\boldsymbol{\sigma}\cdot\mathbf{u})(\boldsymbol{\sigma}\cdot\mathbf{v}) = \mathbf{1}\mathbf{u}\cdot\mathbf{v} + i\boldsymbol{\sigma}\cdot(\mathbf{u}\times\mathbf{v}) \tag{18}$$

$\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$.

Proof.

$$
\begin{aligned}
(\boldsymbol{\sigma}\cdot\mathbf{u})(\boldsymbol{\sigma}\cdot\mathbf{v}) &= \begin{pmatrix} u_3 & u_1 - iu_2 \\ u_1 + iu_2 & -u_3 \end{pmatrix} \begin{pmatrix} v_3 & v_1 - iv_2 \\ v_1 + iv_2 & -v_3 \end{pmatrix} \\
&= \begin{pmatrix} \mathbf{u}\cdot\mathbf{v} & 0 \\ 0 & \mathbf{u}\cdot\mathbf{v} \end{pmatrix} + \begin{pmatrix} i(\mathbf{u}\times\mathbf{v})_3 & i(\mathbf{u}\times\mathbf{v})_1 + (\mathbf{u}\times\mathbf{v})_2 \\ i(\mathbf{u}\times\mathbf{v})_1 - (\mathbf{u}\times\mathbf{v})_2 & -i(\mathbf{u}\times\mathbf{v})_3 \end{pmatrix} \\
&= \mathbf{u}\cdot\mathbf{v} + i\boldsymbol{\sigma}\cdot(\mathbf{u}\times\mathbf{v})
\end{aligned}
\tag{19}
$$

□

Let $\hat{\mathbf{v}}$ be the unit vector in the direction of a nonzero vector $\mathbf{v} \in \mathbb{R}^3$,

$$
\hat{\mathbf{v}} = \frac{\mathbf{v}}{\|\mathbf{v}\|} \tag{20}
$$

We then have, as a special case of (18),

$$
(\boldsymbol{\sigma}\cdot\hat{\mathbf{v}})^2 = \mathbf{1} \tag{21}
$$

Let $\psi_{\mathbf{v}} \in \mathbb{R}^+$, $\mathbb{R}^+ = \{r \in \mathbb{R} : r > 0\}$, be a positive number given by the equation

$$
\cosh \psi_{\mathbf{v}} = \gamma_{\mathbf{v}} = \frac{1}{\sqrt{1 - \dfrac{\|\mathbf{v}\|^2}{c^2}}} \tag{22}
$$

where $\mathbf{v} \in \mathbb{V}_c$, so that

$$
\sinh \psi_{\mathbf{v}} = \frac{1}{\sqrt{1 - \dfrac{\|\mathbf{v}\|^2}{c^2}}} \frac{\|\mathbf{v}\|}{c} \,. \tag{23}
$$

The 2×2 complex matrix

$$
L(\hat{\mathbf{v}}, \psi_{\mathbf{v}}) = \mathbf{1} \cosh \frac{\psi_{\mathbf{v}}}{2} + \boldsymbol{\sigma}\cdot\hat{\mathbf{v}} \sinh \frac{\psi_{\mathbf{v}}}{2} \tag{24}
$$

is a parametric element of the group $SL(2, C)$ of all 2×2 complex matrices with determinant 1, parametrized by $\hat{\mathbf{v}}$, proved useful in the understanding of the Lorentz boost $L(\mathbf{v})$ in (11). Another parametric element of that group, proved useful in the understanding of rotations, is presented in (29).

For the application of the Lorentz boost $L(\mathbf{v})$ to spacetime coordinates in terms of the Pauli spin matrices let us consider the 2×2 spacetime matrix $S = S(t, \mathbf{x})$,

$$
S = \frac{1}{c} \begin{pmatrix} ct + x_3 & x_1 - ix_2 \\ x_1 + ix_2 & ct - x_3 \end{pmatrix} \tag{25}
$$

the determinant, $\det(S)$, of which is the squared relativistic norm,

$$\sqrt{\det(S)} = \left\| \begin{pmatrix} t \\ \mathbf{x} \end{pmatrix} \right\| = \sqrt{t^2 - \frac{\|\mathbf{x}\|^2}{c^2}}\,. \tag{26}$$

The adjoint matrix $M^\dagger$ of a complex matrix M is the transpose conjugate matrix of M. The adjoint of $L(\hat{\mathbf{v}}, \psi_{\mathbf{v}})$ is thus denoted $L^\dagger(\hat{\mathbf{v}}, \psi_{\mathbf{v}})$. A matrix that is identical with its adjoint is called self-adjoint or *hermitean*. The Pauli spin matrices and the matrix S are, thus, hermitean.

It turns out that the transformation

$$S' = L(\hat{\mathbf{v}}, \psi_{\mathbf{v}}) S L^\dagger(\hat{\mathbf{v}}, \psi_{\mathbf{v}}) \tag{27}$$

of the spacetime matrix S is, again, a spacetime matrix S', where

$$S' = S(t', \mathbf{x}') \tag{28}$$

and where $(t', \mathbf{x}')^t$ is identical with the extreme right hand side of (11). In that sense, the transformation from the spacetime matrix S to the spacetime matrix S' in (27) – (28) represents the pure Lorentz transformation $L(\mathbf{v})$ in (11) from spacetime events $(t, \mathbf{x})^t$ to spacetime events $(t', \mathbf{x}')^t$ ([18], p. 295). Accordingly,

(i) the Lorentz transformation $L(\mathbf{v})$ corresponds to the two $SL(2, C)$ matrices $\pm L(\hat{\mathbf{v}}, \psi_{\mathbf{v}})$; and

(ii) the composition $L(\mathbf{u})L(\mathbf{v})$ of two Lorentz boosts $L(\mathbf{u})$ and $L(\mathbf{v})$ corresponds to the matrix multiplication $\pm L(\hat{\mathbf{u}}, \psi_{\mathbf{u}}) L(\hat{\mathbf{v}}, \psi_{\mathbf{v}})$.

Clearly, the ambiguous sign in items (i) and (ii) stems from (27). The group $SL(2, C)$ is thus locally isomorphic to the (3+1)-dimensional Lorentz group $SO(3, 1)$ [20].

Goldstein [18] presents (24) with a negative sign, corresponding to the Lorentz boost $L(-\mathbf{v})$. A detailed discussion of the homomorphism between the restricted Lorentz group and $SL(2, C)$ is given by Halpern [21], Chapter 1. Appendix 3 of [21] contains an explicit $SL(2, C)$ calculation of the decomposition of the product of two boosts into the product of a rotation and a boost, as we do in (32) below.

A slight modification of ψ in (24), $\psi \to i\delta$ results in the 2×2 matrix

$$R(\hat{\mathbf{n}}, \delta) = \mathbf{1} \cos\frac{\delta}{2} + i\boldsymbol{\sigma}\cdot\hat{\mathbf{n}} \sin\frac{\delta}{2} \tag{29}$$

where $\hat{\mathbf{n}} \in \mathbb{R}^3$ is a unit vector, and $0 \le \delta < 2\pi$. It proves useful in the application of space rotations to spacetime coordinates. The application of

$R(\hat{\mathbf{n}}, \delta)$ to the spacetime matrix $S = S(t, \mathbf{x})$, (25), according to the equation similar to (27),

$$S'' = R(\hat{\mathbf{n}}, \delta) S R^{\dagger}(\hat{\mathbf{n}}, \delta) \tag{30}$$

results in a matrix which is, again, a spacetime matrix S'', where

$$S'' = S(t, \mathbf{x}'') \tag{31}$$

with $\|\mathbf{x}''\| = \|\mathbf{x}\|$. Indeed the application (30) of $R(\hat{\mathbf{n}}, \delta)$ represents a space rotation about the space origin of spacetime coordinates, with an axis of rotation parallel to $\hat{\mathbf{n}}$ through an angle δ ([18], p. 166), leaving time t intact.

The composition of two successive Lorentz boosts is not a boost but, rather, a boost preceded (or followed) by a space rotation ([18], p. 296). Accordingly, the composition of two successive Lorentz boosts is represented by the equation

$$L(\hat{\mathbf{u}}, \psi_{\mathbf{u}}) L(\hat{\mathbf{v}}, \psi_{\mathbf{v}}) = L(\hat{\mathbf{w}}, \psi_{\mathbf{w}}) R(\hat{\mathbf{n}}, \delta) \tag{32}$$

where $\mathbf{w}$ is the sum, in a sense that will become clear in the sequel, of $\mathbf{u}$ and $\mathbf{v}$ that we write as $\mathbf{w} = \mathbf{u} \oplus \mathbf{v}$, calling $\oplus$ the $SL(2, C)$ general vector addition.

Geometric phases in physics, known as *physical holonomy*, are linked to the Thomas precession [22], the abstraction of which results in the Thomas gyration that appears in the gyrogroup definition 1. The application of the $SL(2, C)$ formalism in the study of the Lorentz group, and its relevance to geometric phases of hyperbolic geometry in optical demonstrations are nicely explained by Monzón and Sánchez-Soto in [20].

3 EMPLOYING THE $SL(2, C)$ FORMALISM

Employing the $SL(2, C)$ formalism we present the following

Theorem 4. *The composition $L(\hat{\mathbf{u}}, \psi_{\mathbf{u}}) L(\hat{\mathbf{v}}, \psi_{\mathbf{v}})$ of the two successive Lorentz boosts $L(\hat{\mathbf{u}}, \psi_{\mathbf{u}})$ and $L(\hat{\mathbf{v}}, \psi_{\mathbf{v}})$ is equivalent to a single Lorentz boost $L(\hat{\mathbf{w}}, \psi_{\mathbf{w}})$ preceded by the space rotation $R(\hat{\mathbf{n}}, \delta)$,*

$$L(\hat{\mathbf{u}}, \psi_{\mathbf{u}}) L(\hat{\mathbf{v}}, \psi_{\mathbf{v}}) = L(\hat{\mathbf{w}}, \psi_{\mathbf{w}}) R(\hat{\mathbf{n}}, \delta) \tag{33}$$

where $\hat{\mathbf{w}}$, ψ_{w}, $\hat{\mathbf{n}}$ *and* δ *are related to the successive boosts parameters by the equation*

$$\cosh\frac{\psi_{\mathrm{w}}}{2}\cos\frac{\delta}{2} = \cosh\frac{\psi_{\mathrm{u}}}{2}\cosh\frac{\psi_{\mathrm{v}}}{2} + \hat{\mathbf{u}}\cdot\hat{\mathbf{v}}\sinh\frac{\psi_{\mathrm{u}}}{2}\sinh\frac{\psi_{\mathrm{v}}}{2} \tag{34a}$$

$$\hat{\mathbf{n}}\cosh\frac{\psi_{\mathrm{w}}}{2}\sin\frac{\delta}{2} = \hat{\mathbf{u}}\times\hat{\mathbf{v}}\sinh\frac{\psi_{\mathrm{u}}}{2}\sinh\frac{\psi_{\mathrm{v}}}{2} \tag{34b}$$

$$\begin{aligned}\hat{\mathbf{w}}\sinh\frac{\psi_{\mathrm{w}}}{2}\cos\frac{\delta}{2} &+ \hat{\mathbf{n}}\times\hat{\mathbf{w}}\sinh\frac{\psi_{\mathrm{w}}}{2}\sin\frac{\delta}{2} \\ &= \hat{\mathbf{u}}\sinh\frac{\psi_{\mathrm{u}}}{2}\cosh\frac{\psi_{\mathrm{v}}}{2} + \hat{\mathbf{v}}\cosh\frac{\psi_{\mathrm{u}}}{2}\sinh\frac{\psi_{\mathrm{v}}}{2}\end{aligned} \tag{34c}$$

$$\hat{\mathbf{n}}\cdot\hat{\mathbf{w}} = 0 \tag{34d}$$

Furthermore, $\hat{\mathbf{w}}$, ψ_{w}, $\hat{\mathbf{n}}$ *and* δ *satisfy the identities*

$$\cosh\psi_{\mathrm{w}} = \cosh\psi_{\mathrm{u}}\cosh\psi_{\mathrm{v}} + \sinh\psi_{\mathrm{u}}\sinh\psi_{\mathrm{v}}\,\hat{\mathbf{u}}\cdot\hat{\mathbf{v}} \tag{35a}$$

$$\tan\frac{\delta}{2} = \frac{|\hat{\mathbf{u}}\times\hat{\mathbf{v}}|\sinh\frac{\psi_{\mathrm{u}}}{2}\sinh\frac{\psi_{\mathrm{v}}}{2}}{\cosh\frac{\psi_{\mathrm{u}}}{2}\cosh\frac{\psi_{\mathrm{v}}}{2} + \sinh\frac{\psi_{\mathrm{u}}}{2}\sinh\frac{\psi_{\mathrm{v}}}{2}\hat{\mathbf{u}}\cdot\hat{\mathbf{v}}} \tag{35b}$$

$$\hat{\mathbf{n}} = \frac{\hat{\mathbf{u}}\times\hat{\mathbf{v}}}{\|\hat{\mathbf{u}}\times\hat{\mathbf{v}}\|} \tag{35c}$$

$$\begin{aligned}\hat{\mathbf{w}}\sinh\psi_{\mathrm{w}} &= \hat{\mathbf{u}}\{\sinh\psi_{\mathrm{u}}\cosh\psi_{\mathrm{v}} + \hat{\mathbf{u}}\cdot\hat{\mathbf{v}}\sinh\psi_{\mathrm{v}}(\cosh\psi_{\mathrm{u}} - 1)\} \\ &\quad +\hat{\mathbf{v}}\sinh\psi_{\mathrm{v}}\end{aligned} \tag{35d}$$

Proof. Employing the $SL(2,C)$ formalism we use (24) and (29) to expand each of the two sides of (32), obtaining

$$\begin{aligned}&L(\hat{\mathbf{u}},\psi_{\mathrm{u}})L(\hat{\mathbf{v}},\psi_{\mathrm{v}}) \\ &= \{\mathbf{1}\cosh\frac{\psi_{\mathrm{u}}}{2} + (\boldsymbol{\sigma}\cdot\hat{\mathbf{u}})\sinh\frac{\psi_{\mathrm{u}}}{2}\}\{\mathbf{1}\cosh\frac{\psi_{\mathrm{v}}}{2} + (\boldsymbol{\sigma}\cdot\hat{\mathbf{v}})\sinh\frac{\psi_{\mathrm{v}}}{2}\} \\ &= \mathbf{1}\cosh\frac{\psi_{\mathrm{u}}}{2}\cosh\frac{\psi_{\mathrm{v}}}{2} + (\boldsymbol{\sigma}\cdot\hat{\mathbf{v}})\cosh\frac{\psi_{\mathrm{u}}}{2}\sinh\frac{\psi_{\mathrm{v}}}{2} + (\boldsymbol{\sigma}\cdot\hat{\mathbf{u}})\sinh\frac{\psi_{\mathrm{u}}}{2}\cosh\frac{\psi_{\mathrm{v}}}{2} \\ &\quad + \mathbf{1}(\hat{\mathbf{u}}\cdot\hat{\mathbf{v}})\sinh\frac{\psi_{\mathrm{u}}}{2}\sinh\frac{\psi_{\mathrm{v}}}{2} + i(\boldsymbol{\sigma}\cdot(\hat{\mathbf{u}}\times\hat{\mathbf{v}}))\sinh\frac{\psi_{\mathrm{u}}}{2}\sinh\frac{\psi_{\mathrm{v}}}{2}\end{aligned} \tag{36}$$

and

$$
\begin{aligned}
&L(\hat{\mathbf{w}},\psi_{\mathrm{w}})R(\hat{\mathbf{n}},\delta)\\
&\quad=\{\mathbf{1}\cosh\frac{\psi_{\mathrm{w}}}{2}+(\boldsymbol{\sigma}\cdot\hat{\mathbf{w}})\sinh\frac{\psi_{\mathrm{w}}}{2}\}\{\mathbf{1}\cos\frac{\delta}{2}+i(\boldsymbol{\sigma}\cdot\hat{\mathbf{n}})\sin\frac{\delta}{2}\}\\
&\quad=\mathbf{1}\cosh\frac{\psi_{\mathrm{w}}}{2}\cos\frac{\delta}{2}+i(\boldsymbol{\sigma}\cdot\hat{\mathbf{n}})\cosh\frac{\psi_{\mathrm{w}}}{2}\sin\frac{\delta}{2}+(\boldsymbol{\sigma}\cdot\hat{\mathbf{w}})\sinh\frac{\psi_{\mathrm{w}}}{2}\cos\frac{\delta}{2}\\
&\qquad+\mathbf{1}i(\hat{\mathbf{n}}\cdot\hat{\mathbf{w}})\sinh\frac{\psi_{\mathrm{w}}}{2}\sin\frac{\delta}{2}+(\boldsymbol{\sigma}\cdot(\hat{\mathbf{n}}\times\hat{\mathbf{w}}))\sinh\frac{\psi_{\mathrm{w}}}{2}\sin\frac{\delta}{2}
\end{aligned}
\tag{37}
$$

It follows from (32) that the right hand sides of (36) and (37) are equal. Comparing coefficients of equal Pauli spin matrices in the right hand sides of (36) and (37) we obtain the desired identities in (34).

To complete the proof we note that, as required for the decomposition (33), (i) $0 \le \delta \le 2\pi$ since the right hand side of (35b) is real; (ii) the vector $\hat{\mathbf{n}}$ has the required magnitude 1, by (35c); and (iii) the vector $\hat{\mathbf{w}}$ has the required magnitude 1, as one can check from (35d) and (34a). □

Theorem 3 and the identities in (34) of Theorem 4 are known in the literature of quantum mechanics, for instance [23] and [24].

By the standard algebra of the circular function $\tan$ and the hyperbolic functions $\sinh$ and $\cosh$, the δ-identity (35b) for $\delta_{\mathbf{u},\mathbf{v}}/2$ can be manipulated into a δ-identity for the precession angle $\delta_{\mathbf{u},\mathbf{v}}$, obtaining

$$\tan\delta_{\mathbf{u},\mathbf{v}}=\frac{A}{B} \tag{38a}$$

where

$$A=\sinh\psi_{\mathbf{u}}\sinh\psi_{\mathbf{v}}\left\{1+\frac{\sinh\psi_{\mathbf{u}}}{1+\cosh\psi_{\mathbf{u}}}\frac{\sinh\psi_{\mathbf{v}}}{1+\cosh\psi_{\mathbf{v}}}\hat{\mathbf{u}}\cdot\hat{\mathbf{v}}\right\}\|\hat{\mathbf{u}}\times\hat{\mathbf{v}}\| \tag{38b}$$

and

$$
\begin{aligned}
B=&\cosh\psi_{\mathbf{u}}+\cosh\psi_{\mathbf{v}}+\sinh\psi_{\mathbf{u}}\sinh\psi_{\mathbf{v}}\hat{\mathbf{u}}\cdot\hat{\mathbf{v}}\\
&+\frac{\sinh^2\psi_{\mathbf{u}}}{1+\cosh\psi_{\mathbf{u}}}\frac{\sinh^2\psi_{\mathbf{v}}}{1+\cosh\psi_{\mathbf{v}}}(\hat{\mathbf{u}}\cdot\hat{\mathbf{v}})^2\,.
\end{aligned}
\tag{38c}
$$

4 THE $SL(2,C)$ GENERAL VECTOR ADDITION

The scalar identity (35a) follows from the unit vector identity (35d), as one can see by squaring both sides of each of these two identities, (35a) and (35d). We therefore consider (35d) as a peculiar unit vector "addition" that satisfies (35a).

To convert (35d) from an addition law of unit vectors to an addition law of vectors, called the "$SL(2,C)$ General Vector Addition", $\oplus$, we introduce the unspecified function $f(\psi)$ of a real variable and express the vectors $\mathbf{u},\mathbf{v},\mathbf{w}\in\mathbb{R}^3$ in terms of the unit vectors $\hat{\mathbf{u}},\hat{\mathbf{v}},\hat{\mathbf{w}}$ by the equations

$$\begin{aligned}\mathbf{u}&=c\hat{\mathbf{u}}f(\psi_{\mathbf{u}})\\ \mathbf{v}&=c\hat{\mathbf{v}}f(\psi_{\mathbf{v}})\\ \mathbf{w}&=c\hat{\mathbf{w}}f(\psi_{\mathbf{w}})\end{aligned}\tag{39}$$

where c is an arbitrarily fixed positive constant, so that identity (35d), that satisfies (35a), becomes the identity

$$\begin{aligned}&\mathbf{w}\frac{\sinh\psi_{\mathbf{w}}}{f(\psi_{\mathbf{w}})}\\ &=\mathbf{u}\left\{\frac{\sinh\psi_{\mathbf{u}}}{f(\psi_{\mathbf{u}})}\cosh\psi_{\mathbf{v}}+\frac{\sinh\psi_{\mathbf{v}}(\cosh\psi_{\mathbf{u}}-1)}{f^2(\psi_{\mathbf{u}})f(\psi_{\mathbf{v}})}\frac{\mathbf{u}\cdot\mathbf{v}}{c^2}\right\}+\mathbf{v}\frac{\sinh\psi_{\mathbf{v}}}{f(\mathbf{v})}\end{aligned}\tag{40}$$

that satisfies

$$\cosh\psi_{\mathbf{w}}=\cosh\psi_{\mathbf{u}}\cosh\psi_{\mathbf{v}}+\frac{\sinh\psi_{\mathbf{u}}}{f(\psi_{\mathbf{u}})}\frac{\sinh\psi_{\mathbf{v}}}{f(\psi_{\mathbf{v}})}\frac{\mathbf{u}\cdot\mathbf{v}}{c^2}\,.\tag{41}$$

Calling $\mathbf{w}$ the vector addition of $\mathbf{u}$ and $\mathbf{v}$, $\mathbf{w}=\mathbf{u}\oplus\mathbf{v}$ in $\mathbb{R}^3$, identity (40) with identity (41) gives the following $SL(2,C)$ General Vector Addition $\oplus$ in $\mathbb{R}^3$,

$$\mathbf{u}\oplus\mathbf{v}=\frac{1}{\frac{\sinh\psi_{\mathbf{u}\oplus\mathbf{v}}}{f(\psi_{\mathbf{u}\oplus\mathbf{v}})}}(C_{\mathbf{u}}\mathbf{u}+C_{\mathbf{v}}\mathbf{v})\tag{42}$$

where

$$\begin{aligned}C_{\mathbf{u}}&=\frac{\sinh\psi_{\mathbf{u}}}{f(\psi_{\mathbf{u}})}\cosh\psi_{\mathbf{v}}+\left(\frac{\sinh\psi_{\mathbf{u}}}{f(\psi_{\mathbf{u}})}\right)^2\frac{\sinh\psi_{\mathbf{v}}}{f(\psi_{\mathbf{v}})}\frac{1}{1+\cosh\psi_{\mathbf{u}}}\frac{\mathbf{u}\cdot\mathbf{v}}{c^2}\\ C_{\mathbf{v}}&=\frac{\sinh\psi_{\mathbf{v}}}{f(\psi_{\mathbf{v}})}\end{aligned}\tag{43}$$

and

$$\cosh\psi_{\mathbf{u}\oplus\mathbf{v}}=\cosh\psi_{\mathbf{u}}\cosh\psi_{\mathbf{v}}+\frac{\sinh\psi_{\mathbf{u}}}{f(\psi_{\mathbf{u}})}\frac{\sinh\psi_{\mathbf{v}}}{f(\psi_{\mathbf{v}})}\frac{\mathbf{u}\cdot\mathbf{v}}{c^2}\tag{44}$$

for any nonzero $\mathbf{u},\mathbf{v}\in\mathbb{V}_c$.

Allowing singularities, the binary operation $\oplus$, (42), in $\mathbb{R}^3$ is suggested by the $SL(2,C)$ algebra in $\mathbb{R}^3$. However, it can readily be extended to a binary operation in any real inner product space $\mathbb{V}$. Taking the necessary care to avoid singularities we are thus led to the definition of a binary operation $\oplus$ in a subset $\mathbb{V}_c$ of a real inner product space $\mathbb{V}$ that we present in Section 5.

5 THE $SL(2,C)$ GENERAL ADDITION

Definition 5. *Let* $\mathbb{V} = (\mathbb{V}, +, \cdot)$ *be a real inner product space, and let*

$$\phi : \mathbb{V} \to \mathbb{R}^+, \qquad \phi : \mathbf{v} \mapsto \phi_{\mathbf{v}}$$

be a continuous map of a subset $\mathbb{V}_c \subset \mathbb{V}$ *into the positive real ray* $\mathbb{R}^+$*, taking* $\mathbf{v} \in \mathbb{V}_c$ *into* $\phi(\mathbf{v}) = \phi_{\mathbf{v}} \in \mathbb{R}^+$*, normalized by the condition* $\phi(\mathbf{0}) = 0$*. We assume that* $\mathbf{0} \in \mathbb{V}_c$*, and that* $\mathbb{V}_c$ *is the identity-connected component of the preimage* $\{\mathbf{v} \in \mathbb{V} : \phi(\mathbf{v}) \in \mathbb{R}^+\}$ *of* $\mathbb{R}^+$ *under* ϕ*.*

Furthermore, we assume that there exists a continuous, bijective self-map f *of* $\mathbb{R}^+$*,* $f : \mathbb{R}^+ \to \mathbb{R}^+$*, satisfying the condition*

$$f(\phi_{\mathbf{v}}) = \frac{\|\mathbf{v}\|}{c} \tag{45}$$

for an arbitrarily fixed positive constant c, such that

$$\frac{\sinh \phi_{\mathbf{v}}}{f(\phi_{\mathbf{v}})} \neq 0 \tag{46}$$

for all $\mathbf{v} \in \mathbb{V}_c$*.*

Then, the $SL(2,C)$ *general binary operation* $\oplus$ *in* $\mathbb{V}_c$ *is given by the equation*

$$\mathbf{u} \oplus \mathbf{v} = \frac{1}{\frac{\sinh \phi_{\mathbf{u}\oplus\mathbf{v}}}{f(\phi_{\mathbf{u}\oplus\mathbf{v}})}} (C_{\mathbf{u}} \mathbf{u} + C_{\mathbf{v}} \mathbf{v}) \tag{47}$$

for all $\mathbf{u}, \mathbf{v} \in \mathbb{V}_c$*, where*

$$\begin{aligned} C_{\mathbf{u}} &= \frac{\sinh \phi_{\mathbf{u}}}{f(\phi_{\mathbf{u}})} \cosh \phi_{\mathbf{v}} + \left(\frac{\sinh \phi_{\mathbf{u}}}{f(\phi_{\mathbf{u}})} \right)^2 \frac{\sinh \phi_{\mathbf{v}}}{f(\phi_{\mathbf{v}})} \frac{1}{1 + \cosh \phi_{\mathbf{u}}} \frac{\mathbf{u} \cdot \mathbf{v}}{c^2} \\ C_{\mathbf{v}} &= \frac{\sinh \phi_{\mathbf{v}}}{f(\phi_{\mathbf{v}})} \end{aligned} \tag{48}$$

and where

$$\cosh \phi_{\mathbf{u}\oplus\mathbf{v}} = \cosh \phi_{\mathbf{u}} \cosh \phi_{\mathbf{v}} + \frac{\sinh \phi_{\mathbf{u}}}{f(\phi_{\mathbf{u}})} \frac{\sinh \phi_{\mathbf{v}}}{f(\phi_{\mathbf{v}})} \frac{\mathbf{u} \cdot \mathbf{v}}{c^2} . \tag{49}$$

Furthermore, the scalar multiplication in $\mathbb{V}_c$ *is given by the equation*

$$r \otimes \mathbf{v} = c f(r f^{-1} \frac{\|\mathbf{v}\|}{c}) \frac{\mathbf{v}}{\|\mathbf{v}\|}, \qquad \mathbf{v} \neq 0 \tag{50}$$

and $r \otimes \mathbf{0} = \mathbf{0}$*, for all* $\mathbf{v} \in \mathbb{V}_c$ *and* $r \in \mathbb{R}$*.*

The pair $(\oplus, \otimes)$, consisting of the $SL(2,C)$ general addition $\oplus$ and its corresponding scalar multiplication $\otimes$ in Definition 5, is general in the sense that it involves an unspecified function f. Specifying f in various ways results in various binary operations in various subsets $\mathbb{V}_c$ of $\mathbb{V}$, as we will see in the sequel.

The groupoid $(\mathbb{V}_c, \oplus)$ is a gyrocommutative gyrogroup. Moreover, it admits scalar multiplication, $\otimes$, turning itself into a gyrovector space $(\mathbb{V}_c, \oplus, \otimes)$. To see this, we examine several special cases as illustrative examples of Definition 5. In the first special case we will recover the well known Einstein addition of relativistically admissible velocities, and the scalar multiplication that it admits.

6 SPECIAL CASE I – THE EINSTEIN GYROVECTOR SPACES

To recover the Einstein addition of relativistically admissible velocities from the the $SL(2, C)$ addition (47), we specify the function $f : \mathbb{R}^+ \to \mathbb{R}^+$ in (45) to be

$$f(r) = \tanh r \tag{51}$$

so that, by condition (45),

$$f(\phi_{\mathbf{v}}) = \tanh \phi_{\mathbf{v}} = \frac{\|\mathbf{v}\|}{c} \tag{52}$$

and hence

$$\frac{\sinh \phi_{\mathbf{v}}}{f(\phi_{\mathbf{v}})} = \cosh \phi_{\mathbf{v}} \tag{53}$$

for all $\mathbf{v} \in \mathbb{V}_c$, where $\mathbb{V}_c \subset \mathbb{V}$ is to be determined. The condition (52) implies

$$\cosh \phi_{\mathbf{v}} = \frac{1}{\sqrt{1 - \tanh^2 \phi_{\mathbf{v}}}} = \gamma_{\mathbf{v}} \tag{54}$$

where $\gamma_{\mathbf{v}}$ is the Lorentz factor,

$$\gamma_{\mathbf{v}} = \frac{1}{\sqrt{1 - \frac{\|\mathbf{v}\|^2}{c^2}}} \,. \tag{55}$$

Hence,

$$\sinh \phi_{\mathbf{v}} = \gamma_{\mathbf{v}} \frac{\|\mathbf{v}\|}{c} \,. \tag{56}$$

The required continuity of $\cosh \phi_{\mathbf{v}} = \gamma_{\mathbf{v}}$ for $\mathbf{v} \in \mathbb{V}_c$ implies, by Definition 5, that

$$\mathbb{V}_c = \{\mathbf{v} \in \mathbb{V} : \|\mathbf{v}\| < c\}$$

so that $\mathbb{V}_c$ is the c-ball of the real inner product space $\mathbb{V}$.

Substituting (53) and (54) in (49) we obtain the gamma identity

$$\gamma_{\mathbf{u}\oplus\mathbf{v}} = \gamma_{\mathbf{u}} \gamma_{\mathbf{v}} \left(1 + \frac{\mathbf{u}\cdot\mathbf{v}}{c^2}\right) \,. \tag{57}$$

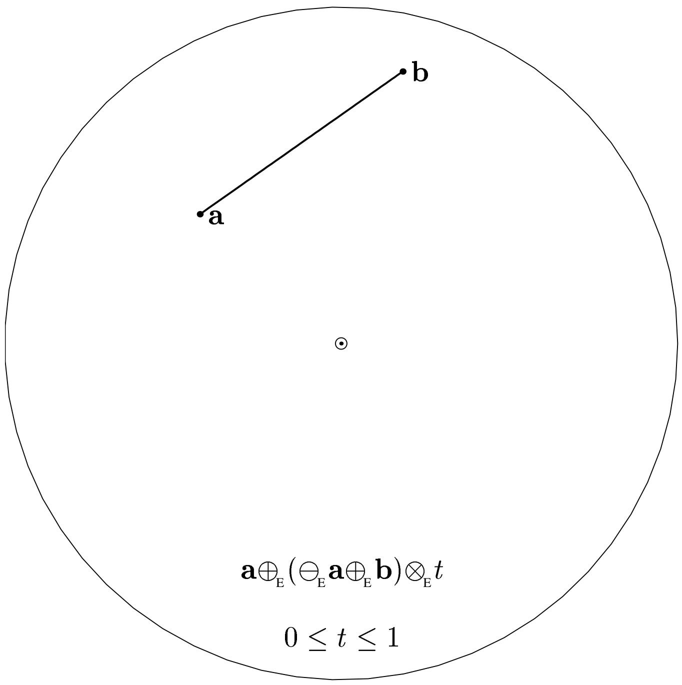

Figure 1. The geodesic segment joining two given points $\mathbf{a}$ and $\mathbf{b}$ in the Einstein gyrovector plane $(\mathbb{R}^2_c, \oplus_{E}, \otimes_{E})$, which is identical with the Beltrami disc model of hyperbolic geometry. The geodesic is generated by the graph of (63) with addition $\oplus_{E}$ and scalar multiplication $\otimes_{E}$ given by Einstein's addition (60) and Einstein's scalar multiplication (62).

Substituting (53) and (54) in (48) and (47) we have

$$\begin{aligned} C_{\mathbf{u}} &= \cosh\phi_{\mathbf{u}}\cosh\phi_{\mathbf{v}} + \frac{\cosh^2\phi_{\mathbf{u}}\cosh\phi_{\mathbf{v}}}{1+\cosh\phi_{\mathbf{u}}}\frac{\mathbf{u}\cdot\mathbf{v}}{c^2} \\ &= \gamma_{\mathbf{u}}\gamma_{\mathbf{v}} + \frac{\gamma^2_{\mathbf{u}}\gamma_{\mathbf{v}}}{1+\gamma_{\mathbf{u}}}\frac{\mathbf{u}\cdot\mathbf{v}}{c^2} \\ C_{\mathbf{v}} &= \cosh\phi_{\mathbf{v}} = \gamma_{\mathbf{v}} \end{aligned} \tag{58}$$

and

$$\mathbf{u}\oplus\mathbf{v} = \frac{1}{\gamma_{\mathbf{u}\oplus\mathbf{v}}}(C_{\mathbf{u}}\mathbf{u} + C_{\mathbf{v}}\mathbf{v})\,. \tag{59}$$

Substituting (57) and (58) in (59) we, finally recover the addition $\oplus = \oplus_{E}$,

$$\mathbf{u}\oplus_{E}\mathbf{v} = \frac{1}{1+\frac{\mathbf{u}\cdot\mathbf{v}}{c^2}}\left\{\mathbf{u} + \frac{1}{\gamma_{\mathbf{u}}}\mathbf{v} + \frac{1}{c^2}\frac{\gamma_{\mathbf{u}}}{1+\gamma_{\mathbf{u}}}(\mathbf{u}\cdot\mathbf{v})\mathbf{u}\right\} \tag{60}$$

which is recognized as the Einstein addition [1].

Thus, the $SL(2,C)$ general addition $\oplus$ in (47), that corresponds to the choice

$$f(r) = \tanh r \tag{61}$$

made in (51), specializes to the Einstein addition $\oplus = \oplus_{\text{E}}$ in (60).

According to (50) and (61), the scalar multiplication that the Einstein addition $\oplus_{\text{E}}$ in (60) admits is

$$r \otimes_{\text{E}} \mathbf{v} = c \tanh\left(r \tanh^{-1} \frac{\|\mathbf{v}\|}{c}\right) \frac{\mathbf{v}}{\|\mathbf{v}\|}, \qquad \mathbf{v} \neq \mathbf{0} \tag{62}$$

and $r \otimes_{\text{E}} \mathbf{0} = \mathbf{0}$. This scalar multiplication is indeed the Einstein scalar multiplication presented in [1].

Interestingly, the set of all points

$$\mathbf{a} \oplus_{\text{E}} (\ominus_{\text{E}} \mathbf{a} \oplus_{\text{E}} \mathbf{b}) \otimes_{\text{E}} t \tag{63}$$

for any two given points $\mathbf{a}, \mathbf{b} \in \mathbb{V}_c$ and all $t \in \mathbb{R}$, where $\oplus_{\text{E}}$ and $\otimes_{\text{E}}$ are the Einstein addition (60) and scalar multiplication (62), forms the unique geodesic passing through the points $\mathbf{a}$ (when $t = 0$) and $\mathbf{b}$ (when $t = 1$) in the Beltrami ball model of hyperbolic geometry [4]. The segment of this geodesic joining the points $\mathbf{a}$ and $\mathbf{b}$ in the Beltrami disc model of hyperbolic geometry is shown in Fig. 1. Placing the Einstein addition centrally in special relativity theory is thus an idea whose time has come back [25].

7 SPECIAL CASE II – THE MÖBIUS GYROVECTOR SPACES

To recover the Möbius addition in the ball of a real inner product space from the $SL(2,C)$ addition (47), we specify the function $f : \mathbb{R}^+ \to \mathbb{R}^+$ in (45) to be

$$f(r) = \tanh \frac{r}{2} \tag{64}$$

so that, by condition (45),

$$f(\phi_{\mathbf{v}}) = \tanh \frac{\phi_{\mathbf{v}}}{2} = \frac{\|\mathbf{v}\|}{c} \tag{65}$$

implying

$$\cosh \frac{\phi_{\mathbf{v}}}{2} = \gamma_{\mathbf{v}} \tag{66}$$

where $\gamma_{\mathbf{v}}$ is the Lorentz factor (55). Hence

$$\cosh \phi_{\mathbf{v}} = 2\cosh^2 \frac{\phi_{\mathbf{v}}}{2} - 1 = 2\gamma_{\mathbf{v}}^2 - 1 \tag{67}$$

$$\tanh\phi_{\mathbf{v}} = \left(1+\frac{1}{\cosh\phi_{\mathbf{v}}}\right)\tanh\frac{\phi_{\mathbf{v}}}{2} = \frac{2\gamma_{\mathbf{v}}^2}{2\gamma_{\mathbf{v}}^2-1}\frac{\|\mathbf{v}\|}{c} \tag{68}$$

and

$$\sinh\phi_{\mathbf{v}} = 2\gamma_{\mathbf{v}}^2\frac{\|\mathbf{v}\|}{c}\,. \tag{69}$$

It follows from (65) and (68) that

$$f(\phi_{\mathbf{v}}) = \tanh\phi_{\mathbf{v}}\left(1+\frac{1}{\cosh\phi_{\mathbf{v}}}\right)^{-1} \tag{70}$$

and hence

$$\frac{\sinh\phi_{\mathbf{v}}}{f(\phi_{\mathbf{v}})} = 1+\cosh\phi_{\mathbf{v}} \tag{71}$$

for all $\mathbf{v}\in\mathbb{V}_c$, where $\mathbb{V}_c\subset\mathbb{V}$ is to be determined.

The required continuity of $\cosh\phi_{\mathbf{v}}$ for $\mathbf{v}\in\mathbb{V}_c$ implies, by Definition 5, that

$$\mathbb{V}_c = \{\mathbf{v}\in\mathbb{V} : \|\mathbf{v}\| < c\} \tag{72}$$

so that $\mathbb{V}_c$ is the c-ball of the real inner product space $\mathbb{V}$.

Substituting (67) and (71) in (49) we obtain the identity

$$2\gamma_{\mathbf{u}\oplus\mathbf{v}}^2 - 1 = (2\gamma_{\mathbf{u}}^2-1)(2\gamma_{\mathbf{v}}^2-1)\left(1+\frac{2\gamma_{\mathbf{u}}^2}{2\gamma_{\mathbf{u}}^2-1}\frac{2\gamma_{\mathbf{v}}^2}{2\gamma_{\mathbf{v}}^2-1}\frac{\mathbf{u}\cdot\mathbf{v}}{c^2}\right) \tag{73}$$

that can be manipulated into the gamma identity for Möbius addition $\oplus = \oplus_{_M}$,

$$\gamma_{\mathbf{u}\oplus\mathbf{v}} = \gamma_{\mathbf{u}}\gamma_{\mathbf{v}}\sqrt{1+\frac{2}{c^2}\mathbf{u}\cdot\mathbf{v}+\frac{1}{c^4}\|\mathbf{u}\|^2\|\mathbf{v}\|^2}\,. \tag{74}$$

Substituting (71) and (67) in (48) and (47) we have,

$$\begin{aligned} C_{\mathbf{u}} &= (1+\cosh\phi_{\mathbf{u}})\cosh\phi_{\mathbf{v}} + (1+\cosh\phi_{\mathbf{u}})(1+\cosh\phi_{\mathbf{v}})\frac{\mathbf{u}\cdot\mathbf{v}}{c^2} \\ &= 2\gamma_{\mathbf{u}}^2(2\gamma_{\mathbf{v}}^2-1) + 4\gamma_{\mathbf{u}}^2\gamma_{\mathbf{v}}^2\frac{\mathbf{u}\cdot\mathbf{v}}{c^2} \\ C_{\mathbf{v}} &= 1+\cosh\phi_{\mathbf{v}} \\ &= 2\gamma_{\mathbf{v}}^2 \end{aligned} \tag{75}$$

and

$$\begin{aligned} \mathbf{u}\oplus\mathbf{v} &= \frac{1}{1+\cosh\phi_{\mathbf{u}\oplus\mathbf{v}}}(C_{\mathbf{u}}\mathbf{u}+C_{\mathbf{v}}\mathbf{v}) \\ &= \frac{1}{2\gamma_{\mathbf{u}\oplus\mathbf{v}}^2}(C_{\mathbf{u}}\mathbf{u}+C_{\mathbf{v}}\mathbf{v})\,. \end{aligned} \tag{76}$$

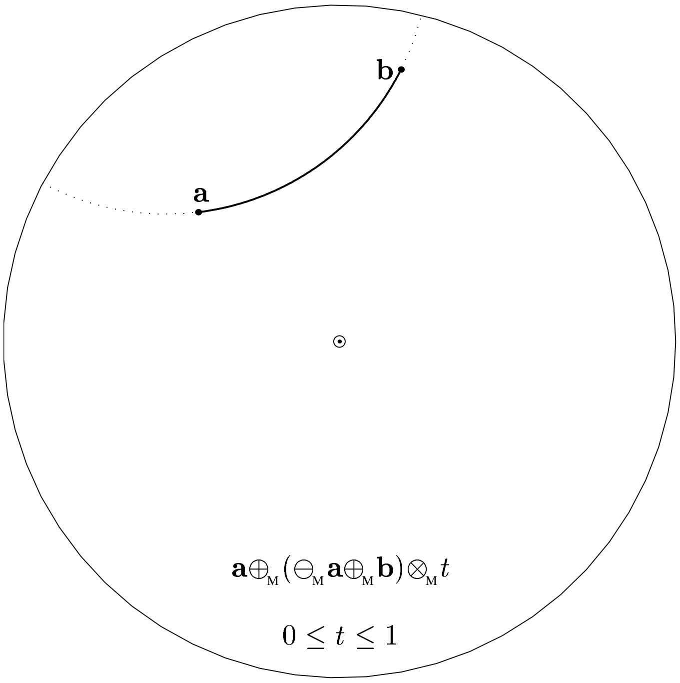

Figure 2. The geodesic segment joining two given points $\mathbf{a}$ and $\mathbf{b}$ in the Möbius gyrovector plane $(\mathbb{R}^2_c, \oplus_{\text{M}}, \otimes_{\text{M}})$, which is identical with the Poincaré disc model of hyperbolic geometry. The geodesic is generated by the graph of (80) with addition $\oplus_{\text{M}}$ and scalar multiplication $\otimes_{\text{M}}$ given by Möbius addition (77) and Möbiuss scalar multiplication (79).

Substituting (75) and (74) in (76) we, finally recover the Möbius addition $\oplus = \oplus_{\text{M}}$ [1] [26],

$$\mathbf{u}\oplus_{\text{M}}\mathbf{v} = \frac{\left(2 - \dfrac{1}{\gamma_{\mathbf{v}}^2} + 2\dfrac{\mathbf{u}\cdot\mathbf{v}}{c^2}\right)\mathbf{u} + \dfrac{1}{\gamma_{\mathbf{u}}^2}\mathbf{v}}{1 + \dfrac{2}{c^2}\mathbf{u}\cdot\mathbf{v} + \dfrac{1}{c^4}\|\mathbf{u}\|^2\|\mathbf{v}\|^2} \tag{77}$$

as defined in Definition 8 of [2].

Thus, the $SL(2,C)$ addition (47) that corresponds to the choice

$$f(r) = \tanh\frac{r}{2} \tag{78}$$

made in (64), specializes to the Möbius addition $\oplus = \oplus_{\text{M}}$ in (77).

According to (50) and (78), the scalar multiplication that the Möbius addition $\oplus_{_M}$ in (77) admits is

$$\begin{aligned} r\otimes_{_M}\mathbf{v} &= c\tanh\left(\frac{r}{2}2\tanh^{-1}\frac{\|\mathbf{v}\|}{c}\right)\frac{\mathbf{v}}{\|\mathbf{v}\|} \\ &= c\tanh\left(r\tanh^{-1}\frac{\|\mathbf{v}\|}{c}\right)\frac{\mathbf{v}}{\|\mathbf{v}\|} \end{aligned} \tag{79}$$

for $\mathbf{v} \neq \mathbf{0}$, and $r\otimes_{_M}\mathbf{0} = \mathbf{0}$. This scalar multiplication is indeed the Möbius scalar multiplication presented in [1], which is, in turn, identical to the Einstein scalar multiplication (62).

Interestingly, the set of all points

$$\mathbf{a}\oplus_{_M}(\ominus_{_M}\mathbf{a}\oplus_{_M}\mathbf{b})\otimes_{_M}t \tag{80}$$

for any two given points $\mathbf{a}, \mathbf{b} \in \mathbb{V}_c$ and all $t \in \mathbb{R}$, where $\oplus_{_M}$ and $\otimes_{_M}$ are the Möbius addition (77) and scalar multiplication (79), forms the unique geodesic passing through the points $\mathbf{a}$ (when $t = 0$) and $\mathbf{b}$ (when $t = 1$) in the Poincaré ball model of hyperbolic geometry. The segment of this geodesic joining the points $\mathbf{a}$ and $\mathbf{b}$ in the Poincaré disc model of hyperbolic geometry is shown in Fig. 2.

8 SPECIAL CASE III

Proper velocities in special relativity theory are velocities measured by proper time, that is, by traveler's time rather than by observer's time [4]. The addition of proper velocities is called Ungar's addition, giving rise to the Ungar model of hyperbolic geometry [4]. To recover the Ungar addition in the whole of a real inner product space $\mathbb{V}$ from the $SL(2, C)$ addition (47), we specify the function $f : \mathbb{R}^+ \to \mathbb{R}^+$ in (45) to be

$$f(r) = \sinh r \tag{81}$$

and, furthermore, we specify $\mathbb{V}_c = \mathbb{V}$ to comply with Definition 5, so that, by condition (45),

$$f(\phi_{\mathbf{v}}) = \sinh\phi_{\mathbf{v}} = \frac{\|\mathbf{v}\|}{c} \tag{82}$$

and hence

$$\frac{\sinh\phi_{\mathbf{v}}}{f(\phi_{\mathbf{v}})} = 1 \tag{83}$$

for all $\mathbf{v} \in \mathbb{V}$. The condition (82) implies

$$\cosh\phi_{\mathbf{v}} = \frac{1}{\beta_{\mathbf{v}}} = \sqrt{1 + \frac{\|\mathbf{v}\|^2}{c^2}} \tag{84}$$

where $\beta_{\mathbf{v}}$ is the beta factor,

$$\beta_{\mathbf{v}} = \frac{1}{\sqrt{1 + \frac{\|\mathbf{v}\|^2}{c^2}}} \,. \tag{85}$$

Hence

$$\tanh \phi_{\mathbf{v}} = \beta_{\mathbf{v}} \frac{\|\mathbf{v}\|}{c} \,. \tag{86}$$

Substituting (83) and (84) in (49) we obtain the beta identity

$$\frac{1}{\beta_{\mathbf{u}\oplus\mathbf{v}}} = \frac{1}{\beta_{\mathbf{u}}} \frac{1}{\beta_{\mathbf{v}}} + \frac{\mathbf{u}\cdot\mathbf{v}}{c^2} \,. \tag{87}$$

Substituting (83) and (84) in (47) and (48) we have,

$$\begin{aligned} C_{\mathbf{u}} &= \cosh \phi_{\mathbf{v}} + \frac{1}{1 + \cosh \phi_{\mathbf{u}}} \frac{\mathbf{u}\cdot\mathbf{v}}{c^2} \\ &= \frac{1}{\beta_{\mathbf{v}}} + \frac{\beta_{\mathbf{u}}}{1 + \beta_{\mathbf{u}}} \frac{\mathbf{u}\cdot\mathbf{v}}{c^2} \\ C_{\mathbf{v}} &= 1 \end{aligned} \tag{88}$$

and with $\oplus_{_U} = \oplus$ we have

$$\begin{aligned} \mathbf{u} \oplus_{_U} \mathbf{v} &= C_{\mathbf{u}} \mathbf{u} + C_{\mathbf{v}} \mathbf{v} \\ &= \left(\frac{1}{\beta_{\mathbf{v}}} + \frac{\beta_{\mathbf{u}}}{1 + \beta_{\mathbf{u}}} \frac{\mathbf{u}\cdot\mathbf{v}}{c^2} \right) \mathbf{u} + \mathbf{v} \end{aligned} \tag{89}$$

thus recovering the Ungar addition $\oplus = \oplus_{_U}$, (89), in $\mathbb{V}$ [1][4].

Hence, the $SL(2, C)$ addition (47) that corresponds to the choice

$$f(r) = \sinh r \tag{90}$$

made in (81) specializes to the Ungar addition (89).

According to (50) and (90), the scalar multiplication that the Ungar addition $\oplus_{_U}$ in (89) admits is

$$r \otimes_{_U} \mathbf{v} = c \sinh \left(r \sinh^{-1} \frac{\|\mathbf{v}\|}{c} \right) \frac{\mathbf{v}}{\|\mathbf{v}\|} \,, \qquad \mathbf{v} \neq \mathbf{0} \tag{91}$$

and $r \otimes_{_U} \mathbf{0} = \mathbf{0}$. This scalar multiplication is indeed the Ungar scalar multiplication presented in [1] and [4].

The set of all points

$$\mathbf{a} \oplus_{_U} (\ominus_{_U} \mathbf{a} \oplus_{_U} \mathbf{b}) \otimes_{_U} t \tag{92}$$

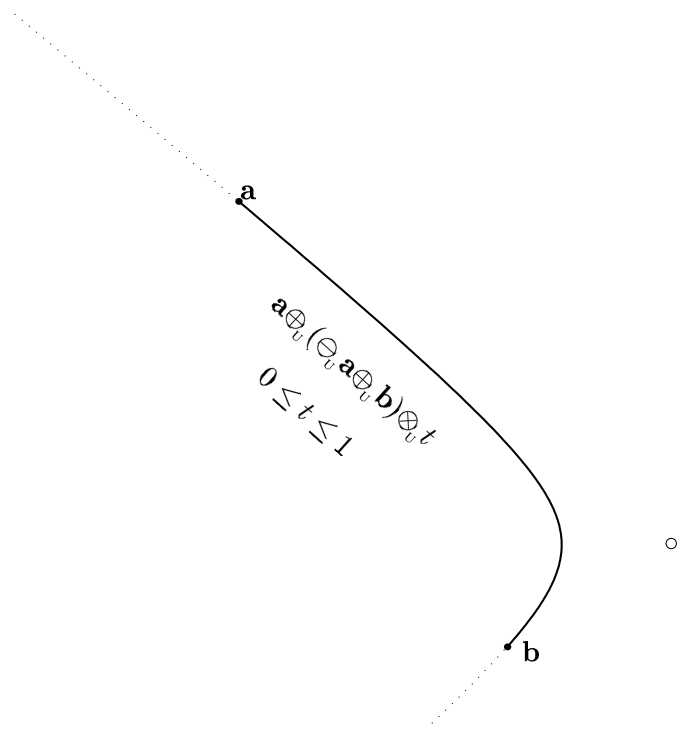

Figure 3. The geodesic segment joining two given points $\mathbf{a}$ and $\mathbf{b}$ in the Ungar plane model of hyperbolic geometry. The geodesic is generated by the graph of (92) with addition $\oplus_U$ and scalar multiplication $\otimes_U$ given by Ungar addition (89) and Ungar scalar multiplication (91).

for any two given points $\mathbf{a}, \mathbf{b} \in \mathbb{V}$ and all $t \in \mathbb{R}$, where $\oplus_U$ and $\otimes_U$ are the Ungar addition (89) and scalar multiplication (91), forms the unique geodesic passing through the points $\mathbf{a}$ (when $t = 0$) and $\mathbf{b}$ (when $t = 1$) in the Ungar model of hyperbolic geometry [4]. The segment of this geodesic joining the points $\mathbf{a}$ and $\mathbf{b}$ in the Ungar plane model of hyperbolic geometry is shown in Fig. 3.

9 SPECIAL CASE IV

To discover a new addition that gives rise to a new model of hyperbolic geometry, that we call Chen's addition, we specify the function $f : \mathbb{R}^+ \to \mathbb{R}^+$ in (45) to be

$$f(r) = \sinh \frac{r}{2} \tag{93}$$

so that, by condition (45),

$$f(\phi_{\mathbf{v}}) = \sinh \frac{\phi_{\mathbf{v}}}{2} = \frac{\|\mathbf{v}\|}{c} \tag{94}$$

implying

$$\cosh\frac{\phi_{\mathbf{v}}}{2} = \sqrt{1+\frac{\|\mathbf{v}\|^2}{c^2}} = \frac{1}{\beta_{\mathbf{v}}} \tag{95}$$

and

$$\tanh\frac{\phi_{\mathbf{v}}}{2} = \beta_{\mathbf{v}}\frac{\|\mathbf{v}\|}{c} \tag{96}$$

where $\beta_{\mathbf{v}}$ is the beta factor (85). Hence

$$\cosh\phi_{\mathbf{v}} = 2\cosh^2\frac{\phi_{\mathbf{v}}}{2} - 1 = \frac{2}{\beta_{\mathbf{v}}^2} - 1 \tag{97}$$

and

$$\tanh\phi_{\mathbf{v}} = \left(1+\frac{1}{\cosh\phi_{\mathbf{v}}}\right)\tanh\frac{\phi_{\mathbf{v}}}{2} = \frac{2\beta_{\mathbf{v}}}{2-\beta_{\mathbf{v}}^2}\frac{\|\mathbf{v}\|}{c}\,. \tag{98}$$

It follows from (94) and (97) that

$$f(\phi_{\mathbf{v}}) = \sinh\frac{\phi_{\mathbf{v}}}{2} = \frac{\sinh\phi_{\mathbf{v}}}{\sqrt{2(1+\cosh\phi_{\mathbf{v}})}} = \frac{1}{2}\beta_{\mathbf{v}}\sinh\phi_{\mathbf{v}} \tag{99}$$

and hence

$$\frac{\sinh\phi_{\mathbf{v}}}{f(\phi_{\mathbf{v}})} = \frac{2}{\beta_{\mathbf{v}}} \tag{100}$$

for all $\mathbf{v}\in\mathbb{V}_c=\mathbb{V}$.

Substituting (97) and (100) in (49) we obtain the identity

$$\frac{2}{\beta_{\mathbf{u}\oplus\mathbf{v}}^2} - 1 = \left(\frac{2}{\beta_{\mathbf{u}}^2}-1\right)\left(\frac{2}{\beta_{\mathbf{v}}^2}-1\right) + \frac{4}{\beta_{\mathbf{u}}\beta_{\mathbf{v}}}\frac{\mathbf{u}\cdot\mathbf{v}}{c^2} \tag{101}$$

that can be manipulated into the identity

$$\frac{1}{\beta_{\mathbf{u}\oplus\mathbf{v}}} = \frac{1}{\beta_{\mathbf{u}}}\frac{1}{\beta_{\mathbf{v}}}\sqrt{\beta_{\mathbf{u}}^2\beta_{\mathbf{v}}^2 - (\beta_{\mathbf{u}}^2+\beta_{\mathbf{v}}^2) + 2\left(1+\beta_{\mathbf{u}}\beta_{\mathbf{v}}\frac{\mathbf{u}\cdot\mathbf{v}}{c^2}\right)} \tag{102}$$

called the beta identity for the Chen addition $\oplus = \oplus_{\mathrm{c}}$ in (105) below.

Substituting (97) and (100) in (48) and (47) we have,

$$\begin{aligned} C_{\mathbf{u}} &= \frac{2}{\beta_{\mathbf{u}}}\left(\frac{2}{\beta_{\mathbf{v}}^2}-1\right) + \frac{4}{\beta_{\mathbf{v}}}\frac{\mathbf{u}\cdot\mathbf{v}}{c^2} \\ C_{\mathbf{v}} &= \frac{2}{\beta_{\mathbf{v}}} \end{aligned} \tag{103}$$

and

$$\mathbf{u}\oplus\mathbf{v} = \frac{\beta_{\mathbf{u}\oplus\mathbf{v}}}{2}(C_{\mathbf{u}}\mathbf{u} + C_{\mathbf{v}}\mathbf{v}) \tag{104}$$

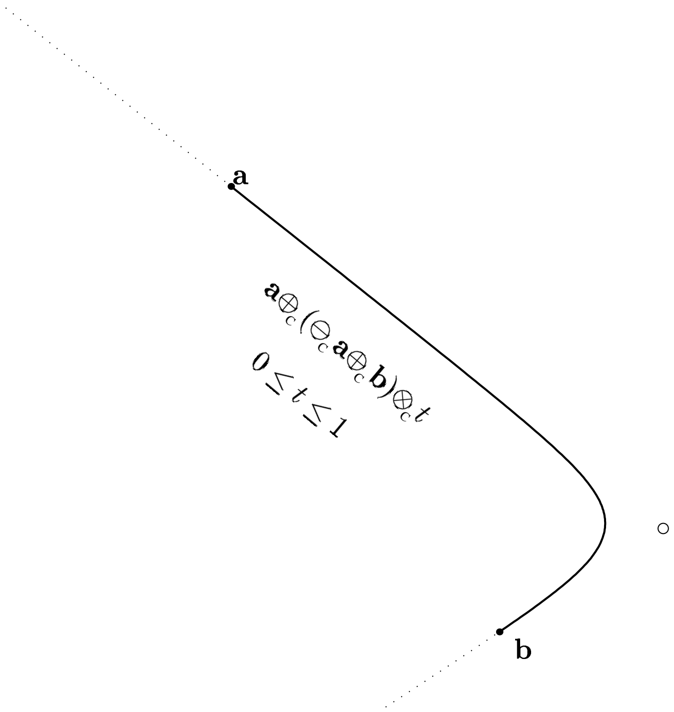

Figure 4. The geodesic segment joining two given points $\mathbf{a}$ and $\mathbf{b}$ in the Chen plane model of hyperbolic geometry. The geodesic is generated by the graph of (108) with addition $\oplus_c$ and scalar multiplication $\otimes_c$ given by Chen addition (105) and Chen scalar multiplication (107).

Substituting (103) and (102) in (104) we finally obtain the Chen addition $\oplus = \oplus_c$,

$$\begin{aligned}
\mathbf{u}\oplus_c\mathbf{v} &= \frac{\beta_{\mathbf{u}\oplus\mathbf{v}}}{2}\left\{\left[\frac{2}{\beta_\mathbf{u}}\left(\frac{2}{\beta_\mathbf{v}^2}-1\right)+\frac{4}{\beta_\mathbf{v}}\frac{\mathbf{u}\cdot\mathbf{v}}{c^2}\right]\mathbf{u}+\frac{2}{\beta_\mathbf{v}}\mathbf{v}\right\} \\
&= \frac{\frac{1}{2}\beta_\mathbf{u}\beta_\mathbf{v}\left\{\left[\frac{2}{\beta_\mathbf{u}}\left(\frac{2}{\beta_\mathbf{v}^2}-1\right)+\frac{4}{\beta_\mathbf{v}}\frac{\mathbf{u}\cdot\mathbf{v}}{c^2}\right]\mathbf{u}+\frac{2}{\beta_\mathbf{v}}\mathbf{v}\right\}}{\sqrt{\beta_\mathbf{u}^2\beta_\mathbf{v}^2-(\beta_\mathbf{u}^2+\beta_\mathbf{v}^2)+2\left(1+\beta_\mathbf{u}\beta_\mathbf{v}\frac{\mathbf{u}\cdot\mathbf{v}}{c^2}\right)}} \qquad (105) \\
&= \frac{\left[\frac{2}{\beta_\mathbf{v}}-\beta_\mathbf{v}+2\beta_\mathbf{u}\frac{\mathbf{u}\cdot\mathbf{v}}{c^2}\right]\mathbf{u}+\beta_\mathbf{u}\mathbf{v}}{\sqrt{\beta_\mathbf{u}^2\beta_\mathbf{v}^2-(\beta_\mathbf{u}^2+\beta_\mathbf{v}^2)+2\left(1+\beta_\mathbf{u}\beta_\mathbf{v}\frac{\mathbf{u}\cdot\mathbf{v}}{c^2}\right)}}\,.
\end{aligned}$$

Thus, the $SL(2,C)$ addition $\oplus$ in (47), that corresponds to the choice

$$f(r) = \sinh\frac{r}{2} \tag{106}$$

made in (93), specializes to the Chen addition $\oplus_{\rm c}$ in (105).

According to (50) and (106), the scalar multiplication that the Chen addition $\oplus_{\rm c}$ in (105) admits is

$$\begin{aligned} r\otimes_{\rm c}\mathbf{v} &= c\sinh\left(\frac{r}{2}2\sinh^{-1}\frac{\|\mathbf{v}\|}{c}\right)\frac{\mathbf{v}}{\|\mathbf{v}\|} \\ &= c\sinh\left(r\sinh^{-1}\frac{\|\mathbf{v}\|}{c}\right)\frac{\mathbf{v}}{\|\mathbf{v}\|} \end{aligned} \tag{107}$$

for $\mathbf{v} \neq \mathbf{0}$, and $r\otimes_{\rm c}\mathbf{0} = \mathbf{0}$. This scalar multiplication is identical with the Ungar scalar multiplication in (91) $\otimes_{\rm c} = \otimes_{\rm U}$.

The set of all points

$$\mathbf{a}\oplus_{\rm c}(\ominus_{\rm c}\mathbf{a}\oplus_{\rm c}\mathbf{b})\otimes_{\rm c}t \tag{108}$$

for any two given points $\mathbf{a}, \mathbf{b} \in \mathbb{V}$ and all $t \in \mathbb{R}$, where $\oplus_{\rm c}$ and $\otimes_{\rm c}$ are the Chen addition (105) and scalar multiplication (107), forms the unique geodesic passing through the points $\mathbf{a}$ (when $t = 0$) and $\mathbf{b}$ (when $t = 1$) in the Chen model of hyperbolic geometry [4]. The segment of this geodesic joining the points $\mathbf{a}$ and $\mathbf{b}$ in the resulting Chen plane model of hyperbolic geometry is shown in Fig. 4.

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