The Bifurcation Approach To Hyperbolic Geometry

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ABSTRACT The Thomas precession of relativity physics gives rise to important isometries in hyperbolic geometry that expose analogies with Euclidean geometry. These, in turn, suggest our bifurcation approach to hyperbolic geometry, according to which Euclidean geometry bifurcates into two mutually dual branches of hyperbolic geometry in its transition to non-Euclidean geometry. One of the two resulting branches turns out to be the standard hyperbolic geometry of Bolyai and Lobachevsky. The corresponding bifurcation of Newtonian mechanics in the transition to Einsteinian mechanics indicates that there are two, mutually dual, kinds of uniform accelerations. Furthermore, while current hyperbolic geometry "does not use the notion of vector at all" (I.M. Yaglom, *Geometric Transformations III*, p. 135, trans. by Abe Shenitzer, Random House, New York, 1973), our bifurcation approach exposes the elusive hyperbolic vectors, that we call gyrovectors.

1 INTRODUCTION

Since it was created in the first half of the nineteenth century in the midst of attempts to understand Euclid's axiomatic basis for geometry, hyperbolic geometry has been studied in terms of several models [3]. Two of its models are of special interest in this article since they arise from the Möbius transformations of the complex unit disc and from Einstein's addition. These models of hyperbolic geometry are

(1) the Poincaré ball model, which governs the Möbius addition in the ball, that will be introduced in this article; and

(2) the Beltrami (also known as the Klein) ball model, which governs the Einstein velocity addition of relativistically admissible velocities.

Hyperbolic geometry is a fundamental aspect of modern physics. The Thomas precession of relativity physics, in turn, is becoming increasingly important in hyperbolic geometry, where it has been extended by abstraction to the so called *Thomas gyration* [25]. Thomas gyration is an isometry of hyperbolic geometry that any two points of the geometry generate, that allows Euclidean and hyperbolic geometry to be unified in terms of analogies that it exposes. The novel analogies, in turn, uncover duality symmetries that demonstrate that in the transition to non-Euclidean geometry, Euclidean geometry bifurcates into two mutually dual branches of hyperbolic geometry one of which is the standard hyperbolic geometry, Fig. 0.9.

Thomas precession is a relativistic effect that has no classical counterpart. Its sensitivity to the non-Euclidean nature of the geometry of spacetime attracts NASA's interest to measure the Thomas precession of gyroscopes of unprecedented accuracy in Earth orbit; see the NASA/Stanford GP-B Program in URL http://einstein.stanford.edu.

Unexpected mathematical regularity that the Thomas precession stores was exposed in 1988 [21] and, subsequently, its key features were abstracted. The resulting abstract Thomas precession is called *Thomas gyration*, suggesting the prefix "gyro" that we use in gyroterminology to coin terms and notions that emphasize analogies, like the gyrocommutative law, the gyroassociative law, gyrogroups, gyroautomorphisms, gyrovectors and gyrovector spaces [25]. In fact, the essence of the present article is to demonstrate that hyperbolic geometry is the gyro-Euclidean geometry.

Gyrogroups are generalized groups that share remarkable analogies with groups [25]. In full analogy with groups

(1) gyrogroups are classified into gyrocommutative gyrogroups and non gyrocommutative gyrogroups; and

(2) some gyrocommutative gyrogroups admit scalar multiplication, turning them to gyrovector spaces.

(3) Gyrovector spaces, in turn, provide the setting for hyperbolic geometry in the same way that vector spaces provide the setting for Euclidean geometry, thus enabling the two geometries to be unified. Specifically:

Vector addition, $+$, in abstract spaces is a commutative and associative binary operation in a group of vectors, which supports scalar multiplication giving rise to vector spaces. These, in turn, provide the setting for Euclidean geometry. In full analogy, Einstein's addition, $\oplus$, of relativistically admissible velocities is abstracted to a *gyrocommutative* and *gyroassociative* binary operation in a *gyrogroup* of *gyrovectors*, which supports scalar multiplication giving rise to gyrovector spaces. These, in turn, provide the setting for hyperbolic geometry.

The isometry group of the resulting hyperbolic geometry is the *gyrosemidirect product group* of a gyrogroup of left gyrotranslations and a group of rotations [25]. This gyrosemidirect product structure of the isometry group of hyperbolic geometry is fully analogous to the semidirect product structure of the isometry group of Euclidean geometry, which is the semidirect product group of a group of translations and a group of rotations.

The power and elegance of the capability of gyrovector spaces to unify Euclidean and hyperbolic geometry is demonstrated in this article, leading naturally to our bifurcation approach to hyperbolic geometry.

2 GYROGROUPS AND GYROVECTOR SPACES

Historically, the first gyrogroup structure was discovered in the study of Einstein's velocity addition [21] [22] [25] [26]. However, the best way to introduce the gyrogroup notion by example is provided by the Möbius transformation group of the complex open unit disc.

The most general Möbius transformation of the complex open unit disc $\mathbb{D} = \{z : |z| < 1\}$ in the complex z-plane [1][9],

$$z \mapsto e^{i\theta} \frac{a+z}{1+\overline{a}z} = e^{i\theta}(a \oplus z) \tag{0.1}$$

induces the Möbius addition $\oplus$ in the disc, allowing the Möbius transformation of the disc to be viewed as a Möbius left *gyrotranslation*

$$z \mapsto a \oplus z = \frac{a+z}{1+\overline{a}z} \tag{0.2}$$

followed by a rotation. Here $\theta \in \mathbb{R}$ is a real number, $a, z \in \mathbb{D}$, and $\overline{a}$ is the complex conjugate of a. Möbius addition $\oplus$ is neither commutative nor associative. The breakdown of commutativity in Möbius addition is "repaired" by the introduction of gyration, $\mathrm{gyr} : \mathbb{D} \times \mathbb{D} \to \mathrm{Aut}(\mathbb{D}, \oplus)$, given by the equation

$$\mathrm{gyr}[a, b] = \frac{a \oplus b}{b \oplus a} = \frac{1 + a\overline{b}}{1 + \overline{a}b} \tag{0.3}$$

where $\mathrm{Aut}(\mathbb{D}, \oplus)$ is the automorphism group of the groupoid $(\mathbb{D}, \oplus)$. We recall that a groupoid $(G, +)$ is a nonempty set G with a binary operation $+$, and an automorphism of a groupoid $(G, +)$ is a bijective self-map of the groupoid G which respects its binary operation $+$. The set of all automorphisms of a groupoid $(G, +)$ forms a group under bijection composition, denoted $Aut(G, +)$.

The *gyrocommutative law* of Möbius addition $\oplus$ that follows from the definition of gyr in (0.3),

$$a \oplus b = \mathrm{gyr}[a, b](b \oplus a) \tag{0.4}$$

is not terribly surprising, but we are not finished.

Coincidentally, the gyration $\mathrm{gyr}[a, b]$ that repairs the breakdown of the commutative law of $\oplus$ in (0.4), repairs the breakdown of the associative law of $\oplus$

as well, giving rise to the respective *left and right gyroassociative laws*

$$\begin{aligned} a\oplus(b\oplus z) &= (a\oplus b)\oplus\mathrm{gyr}[a,b]z \\ (a\oplus b)\oplus z &= a\oplus(b\oplus\mathrm{gyr}[b,a]z) \end{aligned} \tag{0.5}$$

for all $a, b, z \in \mathbb{D}$.

Coincidences in mathematics do not emerge accidentally. Hence, the discovery that the gyrocommutative law of Möbius' (and Einstein's) addition accompanies a gyroassociative law (left and right) in [21] was a significant development, signaling the birth of gyrogroup theory and gyrovector space theory.

Guided by analogies with groups we take the (gyrocommutative and the) gyroassociative laws and other key features of the Möbius addition as a model of a (gyrocommutative) gyrogroup, obtaining the following

Definition 1 (Gyrogroups). *The groupoid $(G,\oplus)$ is a gyrogroup if its binary operation satisfies the following axioms and properties. In G there exists a unique element, 0, called the identity, satisfying*

(g1)	$0\oplus a = a\oplus 0 = a$	*Identity*

for all $a \in G$. For each a in G there exists a unique inverse $\ominus a$ in G, satisfying

(g2)	$\ominus a\oplus a = a\ominus a = 0$	*Inverse*

where we use the notation $a\ominus b = a\oplus(\ominus b)$, $a, b \in G$. Moreover, if for any $a, b \in G$ the self-map $\mathrm{gyr}[a,b]$ of G is given by the equation

$$\mathrm{gyr}[a,b]z = \ominus(a\oplus b)\oplus(a\oplus(b\oplus z)) \tag{0.6}$$

for all $z \in G$, then the following hold for all $a, b, c \in G$:

(g3)	$\mathrm{gyr}[a,b] \in \mathrm{Aut}(G,\oplus)$	*Gyroautomorphism Property*
(g4a)	$a\oplus(b\oplus c) = (a\oplus b)\oplus\mathrm{gyr}[a,b]c$	*Left Gyroassociative Law*
(g4b)	$(a\oplus b)\oplus c = a\oplus(b\oplus\mathrm{gyr}[b,a]c)$	*Right Gyroassociative Law*
(g5a)	$\mathrm{gyr}[a,b] = \mathrm{gyr}[a\oplus b, b]$	*Left Loop Property*
(g5b)	$\mathrm{gyr}[a,b] = \mathrm{gyr}[a, b\oplus a]$	*Right Loop Property*
(g6)	$\ominus(a\oplus b) = \mathrm{gyr}[a,b](\ominus b\ominus a)$	*Gyrosum Inversion Law*
(g7)	$\mathrm{gyr}^{-1}[a,b] = \mathrm{gyr}[b,a]$	*Gyroautomorphism Inversion*

A gyrogroup $(G,\oplus)$ is gyrocommutative if it satisfies

(g8)	$a\oplus b = \mathrm{gyr}[a,b](b\oplus a)$	*Gyrocommutative Law*

Definition 2 (The Gyrogroup Dual Operations). *Let $(G, \oplus)$ be a gyrogroup. A secondary binary operation $\boxplus$ in G is defined by the equation*

(g9) $a \boxplus b = a \oplus \mathrm{gyr}[a, \ominus b]b$ *Secondary Operation*

The primary operation $\oplus$ and the secondary operation $\boxplus$ of a gyrogroup $(G, \oplus)$ are collectively called the mutually dual operations of the gyrogroup. The secondary operation is also called the dual operation.

Theorem 3. *Let $(G, \oplus)$ be a gyrogroup. Its primary binary operation $\oplus$ is given in terms of its secondary binary operation $\boxplus$ by the identity*

(g10) $a \oplus b = a \boxplus \mathrm{gyr}[a, b]b$ *Primary Operation*

Theorem 4. *A gyrogroup $(G, \oplus)$ is gyrocommutative if and only if*

(g11) $\ominus(a \oplus b) = \ominus a \ominus b$ *Automorphic Inverse*

Theorem 5. *A gyrogroup $(G, \oplus)$ is gyrocommutative if and only if its associated dual groupoid $(G, \boxplus)$ is commutative,*

(g12) $a \boxplus b = b \boxplus a$ *Commutative Dual Operation*

Remark 6. *Considering the binary operations $\oplus$ and $\boxplus$ of a gyrogroup $(G, \oplus)$ mutually dual is justified by $(g9)$ and $(g10)$ provided that the gyrations $\mathrm{gyr}[a, b]$ and $\mathrm{gyr}[a, -b]$ are considered mutually dual in some sense as well. The sense in which $\mathrm{gyr}[a, b]$ and $\mathrm{gyr}[a, -b]$ are considered mutually dual will emerge naturally in Theorem 13.*

Gyrogroup theory is presented in [25], where Theorems 3, 4 and 5 are verified. The two dual binary operations in a gyrogroup expose useful duality symmetries in gyrogroups and in gyrovector spaces, as well as in hyperbolic geometry. In particular, they give rise to the *left cancellation law* [25]

$$a \oplus (\ominus a \oplus b) = b \tag{0.7}$$

and to the two *dual right cancellation laws* [25]

$$\begin{aligned} (b \ominus a) \boxplus a &= a \\ (b \boxminus a) \oplus a &= a \end{aligned} \tag{0.8}$$

where $b \boxminus a = b \boxplus (\ominus a)$. In addition, the secondary binary operation provides dual loop properties,

$(g5c)$ $\mathrm{gyr}[a, b] = \mathrm{gyr}[a \boxminus b, b]$ *Dual Left Loop Property*
$(g5d)$ $\mathrm{gyr}[a, b] = \mathrm{gyr}[a, b \boxminus a]$ *Dual Right Loop Property*

The left and the right cancellation laws indicate that in order to capture analogies that gyrogroups share with groups, the two mutually dual binary operations in a gyrogroup are needed. Their presence, in turn, gives rise to duality symmetries, and generate the *Euclidean bifurcation* shown in Fig. 0.9, that suggests our new approach to hyperbolic geometry.

A gyrogroup is a loop [13] [17], since it possesses an identity element, and each of the two equations

$$\begin{aligned} a\oplus x &= b \\ y\oplus a &= b \end{aligned} \tag{0.9}$$

for x and y in any gyrogroup $(G, \oplus)$ possesses a unique solution [25],

$$\begin{aligned} x &= \ominus a\oplus b \\ y &= b \boxminus a \end{aligned} \tag{0.10}$$

In the same way that we say that (a commutative and) an associative loop is a (commutative) group, one may say that (a gyrocommutative and) a gyroassociative loop is a (gyrocommutative) gyrogroup.

For the sake of clarity, Definition 1 of a gyrogroup involves both axioms and properties. However, for an axiomatic approach, a definition that involves only axioms is preferable. An equivalent definition, based on axioms alone, is therefore presented below.

Definition 7 (Gyrogroups: A Second, Equivalent Definition). *The groupoid* $(G, \oplus)$ *is a gyrogroup if its binary operation satisfies the following axioms. In* G *there is at least one element,* 0*, called a left identity, satisfying*

(G1) $\quad 0\oplus a = a$ $\qquad$ *Left Identity*

for all $a \in G$*. There is an element* $0 \in G$ *satisfying axiom* $(G1)$ *such that for each* a *in* G *there is an element* $\ominus a$ *in* G*, called a left inverse of* a*, satisfying*

(G2) $\quad \ominus a\oplus a = 0$ $\qquad$ *Left Inverse*

Moreover, for any $a, b, z \in G$ *there exists a unique element* $\mathrm{gyr}[a, b]z \in G$ *such that*

(G3) $\quad a\oplus(b\oplus z) = (a\oplus b)\oplus\mathrm{gyr}[a, b]z$ $\qquad$ *Left Gyroassociative Law*

If $gyr[a, b]$ *denotes the map* $gyr[a, b] : G \to G$ *given by* $z \mapsto gyr[a, b]z$ *then*

(G4) $\quad \mathrm{gyr}[a, b] \in \mathrm{Aut}(G, \oplus)$ $\qquad$ *Gyroautomorphism*

and $gyr[a, b]$ *is called the Thomas gyration, or the gyroautomorphism of* G*, generated by* $a, b \in G$*. Finally, the gyroautomorphism* $gyr[a, b]$ *generated by any* $a, b \in G$ *satisfies*

(G5) $\quad \mathrm{gyr}[a,b] = \mathrm{gyr}[a\oplus b, b]$ *Left Loop Property*

The equivalence of Definitions 1 and 7 is verified in [25].

3 THE MÖBIUS GYROGROUPS AND GYROVECTOR SPACES

Definition 8 (The Möbius Addition). *Let $\mathbb{V}$ be a real inner product space, and let $\mathbb{V}_c = \{\mathbf{v} \in \mathbb{V} : \|\mathbf{v}\| < c\}$ be the open ball of $\mathbb{V}$ of radius $c > 0$. Möbius addition $\oplus_{_M}$ in the ball $\mathbb{V}_c$, is a binary operation in $\mathbb{V}_c$ given by the equation*

$$\mathbf{u}\oplus_{_M}\mathbf{v} = \frac{(1 + \frac{2}{c^2}\mathbf{u}\cdot\mathbf{v} + \frac{1}{c^2}\|\mathbf{v}\|^2)\mathbf{u} + (1 - \frac{1}{c^2}\|\mathbf{u}\|^2)\mathbf{v}}{1 + \frac{2}{c^2}\mathbf{u}\cdot\mathbf{v} + \frac{1}{c^4}\|\mathbf{u}\|^2\|\mathbf{v}\|^2} \tag{0.11}$$

where $\cdot$ and $\|\cdot\|$ are the inner product and norm that the ball $\mathbb{V}_c$ inherits from its space $\mathbb{V}$.

Möbius addition satisfies the useful identity

$$\gamma_{\mathbf{u}\oplus_{_M}\mathbf{v}} = \gamma_{\mathbf{u}}\gamma_{\mathbf{v}}\sqrt{1 + \frac{2}{c^2}\mathbf{u}\cdot\mathbf{v} + \frac{1}{c^4}\|\mathbf{u}\|^2\|\mathbf{v}\|^2} \tag{0.12}$$

where $\gamma_{\mathbf{u}}$ is the Lorentz factor $\gamma_{\mathbf{u}} = (1 - \|\mathbf{u}\|^2/c^2)^{-1/2}$.

The groupoid $(\mathbb{V}_c, \oplus_{_M})$ is a gyrocommutative gyrogroup [25], called the Möbius gyrogroup. Its dual binary operation, $\boxplus_{_M}$, is related to $\oplus_{_M}$ by Definition 2. It turns out to be

$$\begin{aligned}\mathbf{u} \boxplus_{_M} \mathbf{v} &= \frac{(1 - \frac{1}{c^2}\|\mathbf{v}\|^2)\mathbf{u} + (1 - \frac{1}{c^2}\|\mathbf{u}\|^2)\mathbf{v}}{1 - \frac{1}{c^4}\|\mathbf{u}\|^2\|\mathbf{v}\|^2} \\ &= \frac{\gamma_{\mathbf{u}}^2\mathbf{u} + \gamma_{\mathbf{v}}^2\mathbf{v}}{\gamma_{\mathbf{u}}^2 + \gamma_{\mathbf{v}}^2 - 1}\end{aligned} \tag{0.13}$$

for all $\mathbf{u}, \mathbf{v} \in \mathbb{V}_c$. The dual binary operation $\boxplus_{_M}$ is commutative, as expected from Theorem 5. The inverse $\boxminus_{_M}\mathbf{v}$ of $\mathbf{v}$ relative to the secondary binary operation $\boxplus_{_M}$ satisfies $\boxminus_{_M}\mathbf{v} = \ominus_{_M}\mathbf{v} = -\mathbf{v}$ for all $\mathbf{v} \in \mathbb{V}_c$.

Remark 9. *In the special case when $\mathbb{V}$ is realized by the real line $\mathbb{R}$, the two mutually dual binary operations coincide, resulting in a commutative group operation in the open interval $\mathbb{R}_c = (-c, c)$.*

To justify calling $\oplus_{_M}$ in Definition 8 a Möbius addition we will now show that $\oplus_{_M}$ of (0.11) in two dimensions, when $\mathbb{V} = \mathbb{R}^2$ and $\mathbb{V}_c = \mathbb{R}^2_c$, is equivalent to $\oplus_{_M}$ of (0.2). We, therefore, identify vectors in $\mathbb{R}^2$ with complex numbers in the usual way,

$$\mathbf{u} = (u_1, u_2) = u_1 + iu_2 = u \tag{0.14}$$

The inner product and the norm in $\mathbb{R}^2$ then become the real numbers

$$
\begin{aligned}
\mathbf{u}{\cdot}\mathbf{v} &= Re(\bar{u}v) = \frac{\bar{u}v + u\bar{v}}{2} \\
\|\mathbf{u}\| &= |u|
\end{aligned}
\tag{0.15}
$$

where $\bar{u}$ is the complex conjugate of u.

Under the translation (0.15) of elements of the open unit disc $\mathbb{R}^2_{c=1}$ of $\mathbb{R}^2$ to elements of the complex unit disc $\mathbb{D}$, the Möbius addition (0.11) in the open ball $\mathbb{V}_c = \mathbb{R}^2_{c=1}$ of $\mathbb{V} = \mathbb{R}^2$, with $c = 1$ for simplicity, takes the form

$$
\begin{aligned}
\mathbf{u}\oplus_{\mathrm{M}}\mathbf{v} &= \frac{(1 + 2\mathbf{u}{\cdot}\mathbf{v} + \|\mathbf{v}\|^2)\mathbf{u} + (1 - \|\mathbf{u}\|^2)\mathbf{v}}{1 + 2\mathbf{u}{\cdot}\mathbf{v} + \|\mathbf{u}\|^2\|\mathbf{v}\|^2} \\
&= \frac{(1 + \bar{u}v + u\bar{v} + |v|^2)u + (1 - |u|^2)v}{1 + \bar{u}v + u\bar{v} + |u|^2|v|^2} \\
&= \frac{(1 + u\bar{v})(u + v)}{(1 + \bar{u}v)(1 + u\bar{v})} \\
&= \frac{u + v}{1 + \bar{u}v} \\
&= u\oplus_{\mathrm{M}}v
\end{aligned}
\tag{0.16}
$$

for all $\mathbf{u}, \mathbf{v} \in \mathbb{R}^2_{c=1}$, and all $u, v \in \mathbb{D}$. In (0.16) we recover the Möbius addition $\oplus_{\mathrm{M}}$, (0.1), in the open unit disc $\mathbb{D}$ of $\mathbf{C}$. Hence, the Möbius addition in Definition 8 is a natural generalization of the Möbius addition (0.2) in the complex disc. For more about the generalization see [23].

Definition 10 (Möbius Scalar Multiplication). *Let* $(\mathbb{V}_c, \oplus)$ *be a Möbius gyrogroup. The Möbius scalar multiplication* $r\otimes\mathbf{v} = \mathbf{v}\otimes r$ *in* $\mathbb{V}_c$ *is given by the equation*

$$
\begin{aligned}
r\otimes\mathbf{v} &= c\frac{(1 + \|\mathbf{v}\|/c)^r - (1 - \|\mathbf{v}\|/c)^r}{(1 + \|\mathbf{v}\|/c)^r + (1 - \|\mathbf{v}\|/c)^r}\frac{\mathbf{v}}{\|\mathbf{v}\|} \\
&= c\,tanh\left(r\,tanh^{-1}\frac{\|\mathbf{v}\|}{c}\right)\frac{\mathbf{v}}{\|\mathbf{v}\|}
\end{aligned}
\tag{0.17}
$$

where $r \in \mathbb{R}$*,* $\mathbf{v} \in \mathbb{V}_c$*,* $\mathbf{v} \neq \mathbf{0}$*; and* $r\otimes\mathbf{0} = \mathbf{0}$*.*

The Möbius addition $\oplus_{\mathrm{M}} = \oplus$ and its associated scalar multiplication $\otimes$ possess the following properties. For any positive integer n and for all $r, r_1, r_2 \in \mathbb{R}$ and $\mathbf{a}, \mathbf{b}, \mathbf{u}, \mathbf{v} \in \mathbb{V}_c$,

(V1) $n\otimes\mathbf{v} = \mathbf{v}\oplus \ldots \oplus\mathbf{v}$ (n terms)

(V2)	$(r_1 + r_2)\otimes\mathbf{v} = r_1\otimes\mathbf{v}\oplus r_2\otimes\mathbf{v}$	Scalar Distributive Law
(V3)	$(r_1 r_2)\otimes\mathbf{v} = r_1\otimes(r_2\otimes\mathbf{v})$	Scalar Associative Law
(V4)	$r\otimes(r_1\otimes\mathbf{v}\oplus r_2\otimes\mathbf{v}) = r\otimes(r_1\otimes\mathbf{v})\oplus r\otimes(r_2\otimes\mathbf{v})$	Monodistributive Law
(V5)	$\dfrac{\lvert r\rvert\otimes\mathbf{v}}{\lVert r\otimes\mathbf{v}\rVert} = \dfrac{\mathbf{v}}{\lVert\mathbf{v}\rVert}$	Scaling Property
(V6)	$\lVert r\otimes\mathbf{v}\rVert = \lvert r\rvert\otimes\lVert\mathbf{v}\rVert$	Homogeneity Property
(V7)	$\lVert\mathbf{u}\oplus\mathbf{v}\rVert \le \lVert\mathbf{u}\rVert\oplus\lVert\mathbf{v}\rVert$	Gyrotriangle inequality
(V8)	$\mathrm{gyr}[\mathbf{a},\mathbf{b}](r\otimes\mathbf{v} = r\otimes\mathrm{gyr}[\mathbf{a},\mathbf{b}]\mathbf{v}\rVert$	Gyroautomorphism Property

We may note that, ambiguously, $\oplus$ in $(V7)$ is a binary operation in $\mathbb{V}_c$ and in $\mathbb{R}_c$, Remark 9. Obviously, the condition in $(V1)$ can be relaxed to $1\otimes\mathbf{v} = \mathbf{v}$. A Möbius gyrogroup $(\mathbb{V}_c, \oplus_{\text{M}})$ with scalar multiplication $\otimes$ is called a Möbius gyrovector space, denoted $(\mathbb{V}_c, \oplus_{\text{M}}, \otimes)$.

4 THE EINSTEIN GYROGROUPS AND GYROVECTOR SPACES

Einstein's velocity addition $\oplus_{\text{E}}$ of relativistically admissible velocities is a gyrocommutative gyrogroup operation [22] [25] in the c-ball $\mathbb{R}^3_c$,

$$\mathbb{R}^3_c = \{\mathbf{v} \in \mathbb{R}^3 : \lVert\mathbf{v}\rVert < c\} \tag{0.18}$$

of the Euclidean 3-space $\mathbb{R}^3$, giving rise to the relativity gyrocommutative gyrogroup $(\mathbb{R}^3_c, \oplus_{\text{E}})$. It is given by the equation

$$\mathbf{u}\oplus_{\text{E}}\mathbf{v} = \frac{1}{1 + \frac{\mathbf{u}\cdot\mathbf{v}}{c^2}}\left\{\mathbf{u} + \frac{1}{\gamma_\mathbf{u}}\mathbf{v} + \frac{1}{c^2}\frac{\gamma_\mathbf{u}}{1+\gamma_\mathbf{u}}(\mathbf{u}\cdot\mathbf{v})\mathbf{u}\right\} \tag{0.19}$$

for all $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3_c$, where $\gamma_\mathbf{v} = (1 - \lVert\mathbf{v}\rVert^2/c^2)^{-1/2}$. An extension of the relativity gyrocommutative gyrogroup $(\mathbb{R}^3_c, \oplus_{\text{E}})$ to a gyrogroup of boosts (a boost is a Lorentz transformation without rotation in the jargon) is presented in [18].

Einstein's addition $\oplus_{\text{E}}$ is thus a binary operation in the open c-ball $\mathbb{R}^3_c$ of the Euclidean 3-space $\mathbb{R}^3$. Most texts on relativity present Einstein's addition only for parallel velocities, for the sake of simplicity. Several outstanding exceptions to this are the books by Fock [6], by Bacry [2], and by Sexl and Urbantke [16], where, as in (0.19), the Einstein velocity addition law is presented for the general case in which velocities need not be collinear. Unlike its simple special case, when all velocities are parallel, in its full generality the Einstein relativistic velocity addition $\oplus_{\text{E}}$ is neither commutative nor associative.

The groupoid of classical velocities $(\mathbb{R}^3, +)$ under ordinary vector addition forms a **commutative group** as opposed to the Einstein groupoid of relativistically admissible velocities $(\mathbb{R}^3_c, \oplus_{\text{E}})$ which, under Einstein's velocity addition, forms a non-group **gyrocommutative gyrogroup**. Furthermore, like vector addition, Einstein's addition remains valid in higher dimensions. We therefore extend the Einstein addition (0.19) by abstraction.

Definition 11 (The Abstract Einstein Addition). *Let $\mathbb{V}$ be a real inner product space, and let $\mathbb{V}_c = \{\mathbf{v} \in \mathbb{V} : \|\mathbf{v}\| < c\}$ be the open ball of $\mathbb{V}$ of radius $c > 0$. Einstein's addition $\oplus_E$ in the ball $\mathbb{V}_c$, is a binary operation in $\mathbb{V}_c$ given by the equation*

$$\mathbf{u}\oplus_E\mathbf{v} = \frac{1}{1+\frac{\mathbf{u}\cdot\mathbf{v}}{c^2}}\left\{\mathbf{u} + \frac{1}{\gamma_\mathbf{u}}\mathbf{v} + \frac{1}{c^2}\frac{\gamma_\mathbf{u}}{1+\gamma_\mathbf{u}}(\mathbf{u}\cdot\mathbf{v})\mathbf{u}\right\} \tag{0.20}$$

where $\gamma_\mathbf{v} = (1 - \|\mathbf{v}\|^2/c^2)^{-1/2}$ and where $\cdot$ and $\|\cdot\|$ are the inner product and norm that the ball $\mathbb{V}_c$ inherits from its space $\mathbb{V}$.

Einstein's addition satisfies the useful identity

$$\gamma_{\mathbf{u}\oplus_E\mathbf{v}} = \gamma_\mathbf{u}\gamma_\mathbf{v}(1 + \frac{\mathbf{u}\cdot\mathbf{v}}{c^2}) \tag{0.21}$$

The groupoid $(\mathbb{V}_c, \oplus_E)$ is a gyrocommutative gyrogroup [25], called the Einstein gyrogroup.

The formal definition of Einstein's scalar multiplication is identical with that of Möbius scalar multiplication,

Definition 12 (Einstein's Scalar Multiplication). *Let $(\mathbb{V}_c, \oplus)$ be an Einstein's gyrogroup. The Einstein scalar multiplication $r\otimes\mathbf{v} = \mathbf{v}\otimes r$ in $\mathbb{V}_c$ is given by the equation*

$$\begin{aligned} r\otimes\mathbf{v} &= c\frac{(1+\|\mathbf{v}\|/c)^r - (1-\|\mathbf{v}\|/c)^r}{(1+\|\mathbf{v}\|/c)^r + (1-\|\mathbf{v}\|/c)^r}\frac{\mathbf{v}}{\|\mathbf{v}\|} \\ &= c\,tanh\left(r\,tanh^{-1}\frac{\|\mathbf{v}\|}{c}\right)\frac{\mathbf{v}}{\|\mathbf{v}\|} \end{aligned} \tag{0.22}$$

where $r \in \mathbb{R}$, $\mathbf{v} \in \mathbb{V}_c$, $\mathbf{v} \neq \mathbf{0}$; and $r\otimes\mathbf{0} = \mathbf{0}$.

Like Möbius addition and scalar multiplication, Einstein's addition and its associated scalar multiplication satisfy properties $(V1) - (V7)$.

The Einstein dual addition $\boxplus_E$, Definition 2, is given by the equation

$$\mathbf{u} \boxplus_E \mathbf{v} = 2\otimes\frac{\gamma_\mathbf{u}\mathbf{u} + \gamma_\mathbf{v}\mathbf{v}}{\gamma_\mathbf{u} + \gamma_\mathbf{v}} \tag{0.23}$$

or equivalently, by the equation

$$\mathbf{u} \boxplus_E \mathbf{v} = C_{\mathbf{u},\mathbf{v}}\gamma_\mathbf{u}\mathbf{u} + C_{\mathbf{u},\mathbf{v}}\gamma_\mathbf{v}\mathbf{v} \tag{0.24}$$

where $C_{\mathbf{u},\mathbf{v}}$ is the coefficient

$$C_{\mathbf{u},\mathbf{v}} = \frac{\gamma_{\mathbf{u}} + \gamma_{\mathbf{v}}}{\gamma_{\mathbf{u}}^2 + \gamma_{\mathbf{v}}^2 + \gamma_{\mathbf{u}}\gamma_{\mathbf{v}}(1 + \frac{\mathbf{u}\cdot\mathbf{v}}{c^2}) - 1} \tag{0.25}$$

which is symmetric in $\mathbf{u}$ and $\mathbf{v}$.

The inverse $\boxminus_{\scriptscriptstyle E}\mathbf{v}$ of $\mathbf{v}$ relative to the secondary binary operation $\boxplus_{\scriptscriptstyle E}$ satisfies $\boxminus_{\scriptscriptstyle E}\mathbf{v} = \ominus_{\scriptscriptstyle E}\mathbf{v} = -\mathbf{v}$ for all $\mathbf{v} \in \mathbb{V}_c$.

5 RELATIVISTIC UNIFORM ACCELERATION

Einstein's and Möbius' additions are related by the equations

$$\begin{aligned} \mathbf{u}\oplus_{\scriptscriptstyle E}\mathbf{v} &= 2\otimes(\frac{1}{2}\otimes\mathbf{u}\oplus_{\scriptscriptstyle M}\frac{1}{2}\otimes\mathbf{v}) \\ \mathbf{u}\oplus_{\scriptscriptstyle M}\mathbf{v} &= \frac{1}{2}\otimes(2\otimes\mathbf{u}\oplus_{\scriptscriptstyle E}2\otimes\mathbf{v}) \end{aligned} \tag{0.26}$$

Their corresponding gyrogroups are, accordingly, isomorphic in an obvious sense [27].

Motivated by duality symmetries and by analogies in relativity physics we introduce in this section two kinds of hyperbolic geodesics, called the mutually dual geodesics, and their corresponding mutually dual metrics. We will use the notation $\oplus$ for both $\oplus_{\scriptscriptstyle E}$ and $\oplus_{\scriptscriptstyle M}$ when no ambiguity arises or when no distinction is needed.

The hyperbolic geodesic

$$\mathbf{v}(t) = \mathbf{v}_0\oplus\mathbf{a}\otimes t \tag{0.27}$$

$\mathbf{v}_0, \mathbf{a} \in \mathbb{R}_c^3$, $t \in \mathbb{R}$, in the Einstein or Moöbius gyrovector space $(\mathbb{R}_c^3, \oplus, \otimes)$ represents the time dependent velocity of uniform acceleration, where $\mathbf{v}_0$ is an initial velocity and $\mathbf{a}$ is a constant acceleration. The analogies that the relativistic uniform acceleration (0.27) shares with its Newtonian counterpart,

$$\mathbf{v}(t) = \mathbf{v}_0 + \mathbf{a}t \tag{0.28}$$

are clear, but incomplete. The non commutativity of Einstein's and Moöbius' addition $\oplus$ raises the question as to whether the right relativistic counterpart of (0.28) is the gyroline (0.27), shown in Figs. 0.1 (Einstein) and 0.3 (Möbius), or, perhaps, the dual gyroline (0.29),

$$\mathbf{v}(t) = \mathbf{a}\otimes t\oplus\mathbf{v}_0 \tag{0.29}$$

shown in Figs. 0.2 (Einstein) and 0.4 (Möbius).

The geodesic in Fig. 0.1 is recognized as a geodesic in the Beltrami disc model of hyperbolic geometry. It is a Euclidean straight line [11]. Similarly,

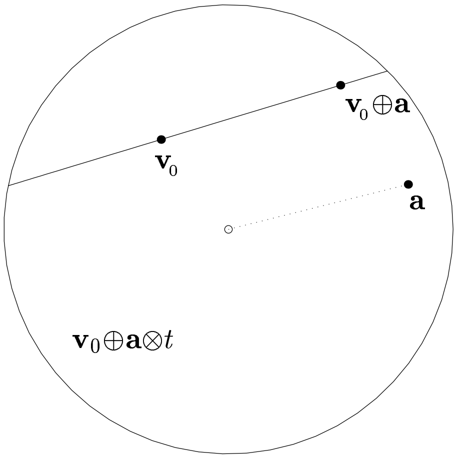

Figure 0.1. The gyroline $\mathbf{v}_0 \oplus \mathbf{a} \otimes t$, $t \in \mathbb{R}$, in an Einstein gyrovector space $(\mathbb{V}_c, \oplus, \otimes)$ is a Euclidean straight line which is a geodesic relative to the Einstein metric (0.33). The Einstein gyrovector space governs the Beltrami model of hyperbolic geometry.

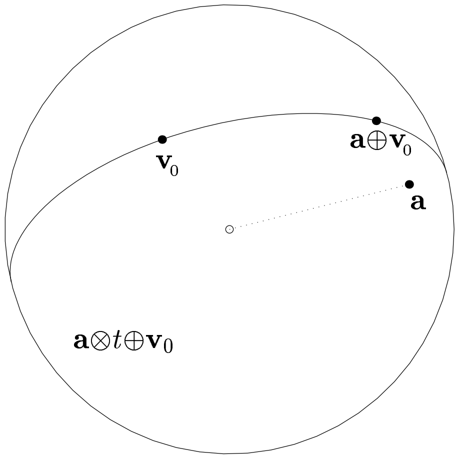

Figure 0.2. The dual gyroline $\mathbf{a} \otimes t \oplus \mathbf{v}_0$, $t \in \mathbb{R}$ in an Einstein gyrovector space $(\mathbb{V}_c, \oplus, \otimes)$ is a Euclidean semi elliptic arc which is also a dual geodesic relative to the Einstein dual metric (0.34).

the geodesic in Fig. 0.3 is recognized as a geodesic in the Poincaré disc model of hyperbolic geometry. It is a circular arc that intersects the boundary of the disc orthogonally [11]. The dual geodesic in Fig. 0.2 is a semiellipse whose major axis coincides with a diameter of the disc, called the *supporting diameter*. Similarly, the dual geodesic in Fig. 0.4 is a circular arc which intersects the boundary of the disc diametrically, that is, at two diametrically opposite points, called the antipodal points of the supporting diameter.

In the Beltrami model, the gyrovector $\mathbf{a}$ in Fig. 0.1 shows no clear geometrical relationship with the geodesics $\mathbf{v}_0 \oplus \mathbf{a} \otimes t$ that it generates. But, in the Poincaré model, the gyrovector $\mathbf{a}$ in Fig. 0.3 shows clear geometrical relationship with the geodesics $\mathbf{v}_0 \oplus \mathbf{a} \otimes t$ that it generates: it is Euclidean parallel to the tangent line of the geodesic $\mathbf{v}_0 \oplus \mathbf{a} \otimes t$ at the point corresponding to $t = 0$. Clear relationship between the gyrovector $\mathbf{a}$ and the dual geodesic $\mathbf{a} \otimes t \oplus \mathbf{v}_0$ that it generates in the two models is observed in Figs. 0.2 and 0.4: it is Euclidean parallel to the supporting diameter of the dual geodesic that it generates.

Various definitions of uniform acceleration are possible in relativity theory [10]. However, a widely accepted one states that uniformly accelerated velocities are those which traverse along geodesics in velocity spaces [29]. One therefore may argue that (0.27) is superior over (0.29) as a candidate for the hyperbolic counterpart of the Newtonian uniform acceleration (0.28) since unlike

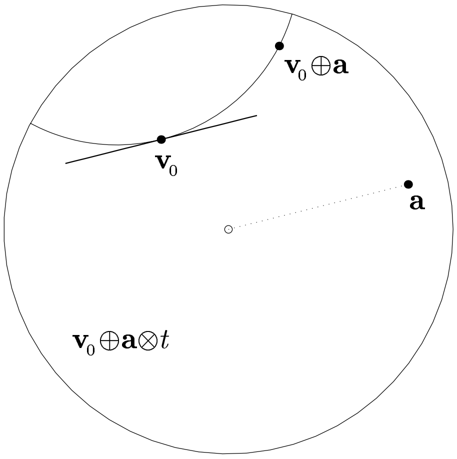

Figure 0.3. The gyroline $\mathbf{v}_0 \oplus \mathbf{a} \otimes t$, $t \in \mathbb{R}$, in a Möbius disc gyrovector space $(\mathbb{R}^2_{c=1}, \oplus, \otimes)$ is a Euclidean circular arc that intersect the boundary of the disc orthogonally. The Möbius gyrovector space governs the Poincaré model of hyperbolic geometry.

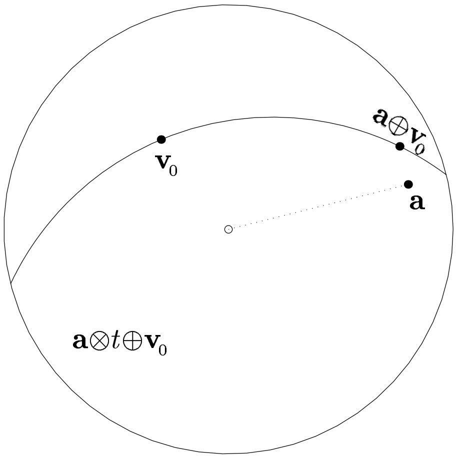

Figure 0.4. The dual gyroline $\mathbf{a} \otimes t \oplus \mathbf{v}_0$, $t \in \mathbb{R}$ in a Möbius gyrovector space $(\mathbb{R}^2_{c=1}, \oplus, \otimes)$ is a Euclidean circular arc which intersects the boundary of the disc at diametrically opposite points.

(0.29), the relativistic uniform acceleration representation (0.27) is a geodesic in the ball $\mathbb{R}^3_c$ relative to the metric (0.33) of the ball. But, what if (0.29) is also a geodesic relative to some as yet unknown metric of the ball? To answer that question we explore the geometric significance of the gyrolines (0.27), shown in Figs. 0.1 and 0.3, and the dual gyrolines (0.29), shown in Figs. 0.2 and 0.4, in terms of analogies that they share with their Euclidean counterpart, (0.28).

It is convenient to write the Euclidean geodesic (0.28) in the form

$$\mathbf{a} + (-\mathbf{a} + \mathbf{b})t \tag{0.30}$$

to display the property of being the unique geodesic passing through two given points $\mathbf{a}$ (when $t = 0$) and $\mathbf{b}$ (when $t = 1$).

The two hyperbolic counterparts, (0.27) and (0.29), of (0.28) can be written in a form analogous to (0.30) as

$$\mathbf{a} \oplus (\ominus \mathbf{a} \oplus \mathbf{b}) \otimes t \tag{0.31}$$

and

$$(\mathbf{b} \boxminus \mathbf{a}) \otimes t \oplus \mathbf{a} \tag{0.32}$$

The hyperbolic gyroline (0.31) is a geodesic in the ball $\mathbb{V}_c$ of a real inner product space $\mathbb{V}$ relative to the metric,

$$d_\ominus(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} \ominus \mathbf{v}\| \tag{0.33}$$

of the gyrovector space $(\mathbb{V}_c, \oplus, \otimes)$. It is the unique geodesic passing through the points $\mathbf{a}$ (when $t = 0$) and $\mathbf{b}$ (when $t = 1$). We should notice that it is due to the left cancellation law in (0.7), that when $t = 1$ the geodesic (0.31) passes through the point $\mathbf{b}$.

Similarly, the *dual* gyroline (0.32) passes through the point $\mathbf{a}$ when $t = 0$ and, due to the right cancellation law (0.8), it passes through the point $\mathbf{b}$ when $t = 1$. It is this need to employ the right cancellation law that dictates that the term $\ominus\mathbf{a}\oplus\mathbf{b}$ in (0.31) be replaced by the term $\boxminus\mathbf{a} \boxplus \mathbf{b} = \mathbf{b} \boxminus \mathbf{a}$ in (0.32). The dual gyrolines (0.32) are geodesics, called dual geodesics, relative to the *dual metric*,

$$d_{\boxminus}(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} \boxminus \mathbf{v}\| \tag{0.34}$$

Gyrovector spaces are thus bimetric, possessing the two mutually dual metrics (0.33) and (0.34) [28]. The primary metric satisfies the *primary gyrotriangle inequality*

$$\|\mathbf{u}\ominus\mathbf{v}\|\oplus\|\mathbf{v}\ominus\mathbf{w}\| \geq \|\mathbf{u}\ominus\mathbf{w}\| \tag{0.35}$$

and the secondary metric satisfies the *secondary gyrotriangle inequality*

$$\|\mathbf{u} \boxminus \mathbf{v}\| \boxplus \|\mathbf{v} \boxminus \mathbf{w}\| \geq \|\mathbf{u} \boxminus \mathrm{gyr}[\mathbf{u} \boxminus \mathbf{v}, \mathbf{v} \boxminus \mathbf{w}]\mathbf{w}\| \tag{0.36}$$

One may note that $\oplus$ and $\boxplus$ in (0.35) (0.36) are, ambigiously, binary operations in $\mathbb{V}_c$ and in $\mathbb{R}_c$, discussed in Remark 9.

The two mutually dual geodesics admit two mutually dual gyration gyrotransitive laws, (0.37) and (0.38), as we state in the following

Theorem 13 (The Gyration Gyrotransitive Laws). *Let L_p and L_s be a primary and a secondary geodesics, respectively, in a gyrovector space $(\mathbb{V}_c, \oplus, \otimes)$. Let $\{\mathbf{a}_1, \cdots, \mathbf{a}_n\}$ be a set of n points lying on L_p (in any order), and let $\{\mathbf{b}_1, \cdots, \mathbf{b}_n\}$ be a set of n points lying on L_s (in any order). Then we have the gyration gyrotransitive law for geodesics,*

$$\mathrm{gyr}[\mathbf{a}_1, -\mathbf{a}_2]\mathrm{gyr}[\mathbf{a}_2, -\mathbf{a}_3] \cdots \mathrm{gyr}[\mathbf{a}_{n-1}, -\mathbf{a}_n] = \mathrm{gyr}[\mathbf{a}_1, -\mathbf{a}_n] \tag{0.37}$$

and the dual gyration gyrotransitive law for dual geodesics,

$$\mathrm{gyr}[\mathbf{b}_1, \mathbf{b}_2]\mathrm{gyr}[\mathbf{b}_2, \mathbf{b}_3] \cdots \mathrm{gyr}[\mathbf{b}_{n-1}, \mathbf{b}_n] = \mathrm{gyr}[\mathbf{b}_1, \mathbf{b}_n] \tag{0.38}$$

The proof of the first identity in the theorem, (0.37), is found in Eq. (7.3b) of [24] and the proof of the second identity in the theorem, (0.38), is left to the interested reader.

The identities (0.37) and (0.38) in Theorem 13 are called the *gyration gyrotransitive laws* of, respectively, primary and secondary geodesics. They prove useful in the study of hyperbolic triangle angular defects and parallel transport [28], and emphasize the duality symmetries in the mutually dual geodesics and in the mutually dual gyrations, as remarked in Remark 6.

6 HYPERBOLIC ANGLES AND DUAL HYPERBOLIC ANGLES

Definition 14 (The Hyperbolic Angle). *The measure of the hyperbolic angle α, $\alpha \leq \pi$, between two geometric gyrovectors $\ominus\mathbf{a}\oplus\mathbf{b}$ and $\ominus\mathbf{a}\oplus\mathbf{c}$ that emanate from a point $\mathbf{a}$ in a gyrovector space $(\mathbb{V}_c, \oplus, \otimes)$ is given by the equation*

$$\cos\alpha = \frac{\ominus\mathbf{a}\oplus\mathbf{b}}{\|\ominus\mathbf{a}\oplus\mathbf{b}\|} \cdot \frac{\ominus\mathbf{a}\oplus\mathbf{c}}{\|\ominus\mathbf{a}\oplus\mathbf{c}\|} \tag{0.39}$$

Definition 14 of the hyperbolic angle is illustrated in Fig. 4 of [26], where it is shown that the hyperbolic angle is invariant under left gyrotranslations and rotations. Furthermore, Definition 14 is equivalent to the standard definition of the hyperbolic angle in the Beltrami model of hyperbolic geometry (when $\oplus = \oplus_{\text{E}}$) and in the Poincaré model of hyperbolic geometry (when $\oplus = \oplus_{\text{M}}$) [26]. But, Definition 14 appears in a novel form that exhibits remarkable analogies. As opposed to Euclidean geometry, the triangle angle sum in hyperbolic geometry is less than π [7].

Equations (0.31) and (0.39) capture the essence of the hyperbolic vectors which, being analogous to Euclidean vectors, are called gyrovectors. Indeed, Walter [31] points out that Varičak [30] had to admit in 1924 that the adaption of ordinary vector algebra for use in hyperbolic space was just not possible. Here, however, we find that the lost vectors reappear as gyrovectors.

Being guided by duality symmetries, we are naturally led to explore the hyperbolic dual angle. We will find in Theorem 16 below that, as in Euclidean geometry, the triangle dual angle sum in hyperbolic geometry is π.

Definition 15 (The Hyperbolic Dual Angle). *The measure of the hyperbolic dual angle α between two geometric dual gyrovectors $\boxminus\mathbf{a} \boxplus \mathbf{b}$ and $\boxminus\mathbf{a} \boxplus \mathbf{c}$ that emanate from a point a in a gyrovector space $(\mathbb{V}_c, \oplus, \otimes)$ is given by the equation*

$$\cos\alpha = \frac{\boxminus\mathbf{a} \boxplus \mathbf{b}}{\| \boxminus \mathbf{a} \boxplus \mathbf{b}\|} \cdot \frac{\boxminus\mathbf{a} \boxplus \mathbf{c}}{\| \boxminus \mathbf{a} \boxplus \mathbf{c}\|} \tag{0.40}$$

Hyperbolic dual angles are invariant under rotations, but are not invariant under left gyrotranslations. However, they carry an important Euclidean property that we state in the following

Theorem 16 (The Hyperbolic π Theorem). *Let $\Delta\mathbf{abc}$ be the hyperbolic dual triangle in a gyrovector space $(\mathbb{V}_c, \oplus, \otimes)$ whose three vertices are the points $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{V}_c$. Then, its three hyperbolic dual angles α, β and γ, given* *by*

$$\cos\alpha = \frac{\mathbf{b}\boxminus\mathbf{a}}{\|\mathbf{b}\boxminus\mathbf{a}\|}\cdot\frac{\mathbf{c}\boxminus\mathbf{a}}{\|\mathbf{c}\boxminus\mathbf{a}\|}$$
$$\cos\beta = \frac{\mathbf{a}\boxminus\mathbf{b}}{\|\mathbf{a}\boxminus\mathbf{b}\|}\cdot\frac{\mathbf{c}\boxminus\mathbf{b}}{\|\mathbf{c}\boxminus\mathbf{b}\|} \tag{0.41}$$
$$\cos\gamma = \frac{\mathbf{a}\boxminus\mathbf{c}}{\|\mathbf{a}\boxminus\mathbf{c}\|}\cdot\frac{\mathbf{b}\boxminus\mathbf{c}}{\|\mathbf{b}\boxminus\mathbf{c}\|}$$

satisfy the identity

$$\alpha+\beta+\gamma=\pi \tag{0.42}$$

The hyperbolic orientation of a dual geodesic is defined to be the Euclidean orientation of its supporting diameter. Accordingly, the two hyperbolic dual geodesics L_1 and L_2 in Figs. 0.5 and 0.6 are parallel since they share their supporting diameter. Parallelism thus reappears in the dual branch of hyperbolic geometry, indicating the possibility that no Euclidean property has been lost in the transition to non-Euclidean geometry. The seemingly lost properties, in fact, reappear in the dual branch of hyperbolic geometry. The bifurcation of Euclidean geometry into two hyperbolic mutually dual geometries has initially been encountered in Section 5 in our attempt to understand relativistic accelerations in terms of analogies with classical accelerations. We now see it at work with the hyperbolic dual angle as well.

As in Euclidean geometry, the proof of Theorem 16 follows immediately from the *hyperbolic alternate interior dual angles theorem* that we will present and verify in Section 7.

7 PARALLELISM IN HYPERBOLIC DUAL GEOMETRY

Two dual (or, secondary) geodesics $\mathbf{a}\otimes t\oplus\mathbf{v}_0$ and $\mathbf{b}\otimes t\oplus\mathbf{w}_0$ in a Möbius or in an Einstein gyrovector space $(\mathbb{V}_c,\oplus,\otimes)$, where $\mathbf{a}$ and $\mathbf{b}$ are nonzero elements of $\mathbb{V}_c$, share a supporting diameter if and only if $\mathbf{a}$ and $\mathbf{b}$ are Euclidean parallel, as shown in Figs. 0.2 and 0.4. Coincidentally, two dual (or, secondary) geodesics do not intersect if and only if their supporting diameters coincide. This coincidence suggests the following

Definition 17. *Two dual geodesics* $\mathbf{a}\otimes t\oplus\mathbf{v}_0$ *and* $\mathbf{b}\otimes t\oplus\mathbf{w}_0$ *in any Einstein or Möbius gyrovector space* $(\mathbb{V}_c,\oplus,\otimes)$ *are parallel if* $\mathbf{a}$ *and* $\mathbf{b}$ *are Euclidean parallel.*

We may note that in a formal abstract gyrovector space one may take the resulting Theorem 20 below as a definition (rather than as a result) of parallel dual geodesics. The following Lemma has a Corollary that will prove useful in the proof of Theorem 20.

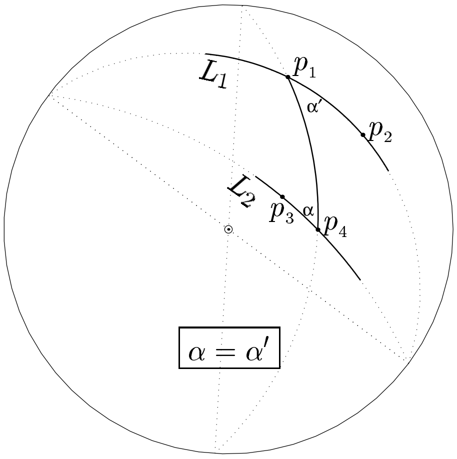

Figure 0.5. Two hyperbolic alternate interior dual angles, α and α', generated in any Möbius gyrovector space $(\mathbb{V}_c, \oplus, \otimes)$ by a hyperbolic dual geodesic intersecting two parallel hyperbolic dual geodesics are equal.

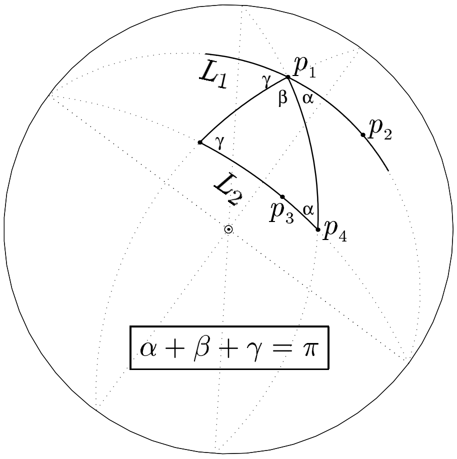

Figure 0.6. The sum of the dual angles of a hyperbolic dual triangle in any Möbius gyrovector space $(\mathbb{V}_c, \oplus, \otimes)$ is π. The proof follows from the equality of hyperbolic alternate interior dual angles, as in the Euclidean case.

Lemma 18. *Let* $(G, \oplus)$ *be a gyrogroup. Then*

$$(\mathbf{a}\oplus\mathrm{gyr}[\mathbf{a},\mathbf{b}]\mathbf{x}) \boxminus (\mathbf{b}\oplus\mathbf{x}) = \mathbf{a} \boxminus \mathbf{b} \tag{0.43}$$

Proof. By the right cancellation law, (0.8), by the dual left loop property $(g5c)$ and by the left gyroassociative law we have

$$\begin{aligned} \mathbf{a}\oplus\mathrm{gyr}[\mathbf{a},\mathbf{b}]\mathbf{x} &= ((\mathbf{a} \boxminus \mathbf{b})\oplus\mathbf{b})\oplus\mathrm{gyr}[\mathbf{a} \boxminus \mathbf{b}, \mathbf{b}]\mathbf{x} \\ &= (\mathbf{a} \boxminus \mathbf{b})\oplus(\mathbf{b}\oplus\mathbf{x}) \end{aligned} \tag{0.44}$$

Hence, by the right cancellation law or, equivalently, by the second equation in each of (0.9) and (0.10) we have

$$(\mathbf{a}\oplus\mathrm{gyr}[\mathbf{a},\mathbf{b}]\mathbf{x}) \boxminus (\mathbf{b}\oplus\mathbf{x}) = \mathbf{a} \boxminus \mathbf{b}$$

as desired. □

Corollary 19. *Let* $(G, \oplus)$ *be a gyrogroup and let* $\mathbf{a}$ *and* $\mathbf{b}$ *be any two elements of* G *satisfying* $\mathrm{gyr}[\mathbf{a},\mathbf{b}] = I$*,* I *being the identity automorphism of* G*. Then*

$$(\mathbf{a}\oplus\mathbf{x}) \boxminus (\mathbf{b}\oplus\mathbf{x}) = \mathbf{a} \boxminus \mathbf{b} = \mathbf{a}\ominus\mathbf{b} \tag{0.46}$$

Proof. The first and the second equalities in (0.46) follow from the substitution of $\mathrm{gyr}[\mathbf{a},\mathbf{b}] = I$ (i) in the identity of Lemma 18; and (ii) in the equation $\mathbf{a} \boxminus \mathbf{b} = \mathbf{a}\ominus\mathrm{gyr}[\mathbf{a},\mathbf{b}]\mathbf{b}$ that follows from Definition 2. □

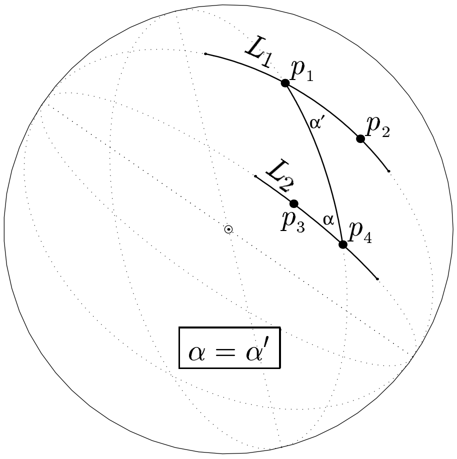

Figure 0.7. Two hyperbolic alternate interior dual angles, α and α', generated in any Einstein's gyrovector space $(\mathbb{V}_c, \oplus, \otimes)$ by a hyperbolic dual geodesic intersecting two parallel hyperbolic dual geodesics are equal.

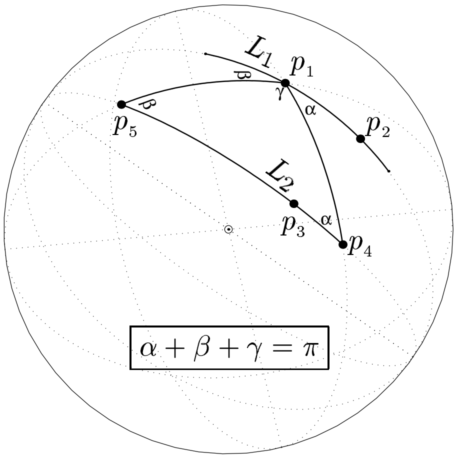

Figure 0.8. The sum of the dual angles of a hyperbolic dual triangle in any Einstein's gyrovector space $(\mathbb{V}_c, \oplus, \otimes)$ is π. The proof follows from the equality of hyperbolic alternate interior dual angles, as in the Euclidean case.

Theorem 20 (Hyperbolic Alternate Interior Dual Angles Theorem). *Let L_1 and L_2 be two parallel dual geodesics in an Einstein or a Möbius gyrovector space $(\mathbb{V}_c, \oplus, \otimes)$ that are intersected by a dual geodesic at the points P_1 and P_4 of L_1 and L_2 respectively, Figs. 0.5 and 0.7. Furthermore, let P_2 and P_3 be points on L_1 and L_2 respectively, which are located on opposite sides of the intersecting dual geodesic. Then, the two alternate interior dual angles are equal,*

$$\angle P_1 P_4 P_3 = \angle P_4 P_1 P_2 \tag{0.47}$$

Proof. The proof is instructive, involving elegant applications of gyrogroup formalism. Let

$$\begin{aligned} L_1 &: \mathbf{a}\otimes t \oplus \mathbf{v}_1, & P_1 &= \mathbf{a}\otimes t_1 \oplus \mathbf{v}_1, & P_2 &= \mathbf{a}\otimes t_2 \oplus \mathbf{v}_1 \\ L_2 &: \mathbf{a}\otimes t \oplus \mathbf{v}_2, & P_3 &= \mathbf{a}\otimes t_3 \oplus \mathbf{v}_2, & P_4 &= \mathbf{a}\otimes t_4 \oplus \mathbf{v}_2 \end{aligned} \tag{0.48}$$

be two hyperbolic parallel dual geodesics in a Möbius or in an Einstein gyrovector space ball $(\mathbb{V}_c, \oplus, \otimes)$, with four of their points, P_1 and P_2 on L_1 and P_3 and P_4 on L_2, where $\mathbf{a}, \mathbf{v}_1, \mathbf{v}_2 \in \mathbb{V}_c$, and where $t \in \mathbb{R}$ is a parameter. The special case when the ball $\mathbb{V}_c$ of a real inner product space $\mathbb{V}$ reduces to the disc $\mathbb{R}^2_c$ of $\mathbb{R}^2$ is shown in Figs. 0.5 and 0.7. The points P_1 and P_2 on L_1 and the points P_3 and P_4 on L_2 correspond to arbitrarily selected values t_1, t_2, t_3 and t_4 of the parameter t in (0.48). These values are arbitrarily selected such that $t_1 - t_2$ and $t_4 - t_3$ have opposite signs so that the points P_2 and P_3 are located on opposite sides of the dual geodesic passing through P_1 and P_4, as shown in Figs. 0.5 and 0.7.

It follows from Definition 15 of the hyperbolic dual angle, that in order to verify (0.47) it is enough to establish the equation

$$(\boxminus P_1 \boxplus P_4){\cdot}(\boxminus P_1 \boxplus P_2) = k^2(\boxminus P_4 \boxplus P_1){\cdot}(\boxminus P_4 \boxplus P_3) \tag{0.49}$$

in any Möbius or Einstein gyrovector space $(\mathbb{V}_c, \oplus, \otimes)$ where k^2 is a positive constant of proportionality that need not be specified. In both Möbius and Einstein gyrovector spaces $(\mathbb{V}_c, \oplus, \otimes)$ we clearly have $-\mathbf{v} = \ominus\mathbf{v} = \boxminus\mathbf{v}$ for all $\mathbf{v} \in \mathbb{V}_c$.

The gyrogroup structure of a gyrovector space is, by definition, gyrocommutative (in the same way that the group structure of a vector space is commutative by definition). The dual binary operation $\boxplus$ in a gyrocommutative gyrogroup, in turn, is commutative by Theorem 5 and, moreover, it satisfies the equation $-(\mathbf{x} \boxminus \mathbf{y}) = -\mathbf{x} \boxplus \mathbf{y}$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{V}_c$, as one can readily check. Hence, the left factor in the inner product on the left hand side of (0.49) equals the negative of the left factor in the inner product on the right hand side of (0.49). Therefore, to establish the validity of (0.49) it is enough to verify the equation

$$P_1 \boxminus P_2 = -k^2(P_4 \boxminus P_3) \tag{0.50}$$

Equation (0.50) can be written by means of (0.48) as

$$(\mathbf{a}{\otimes}t_1{\oplus}\mathbf{v}_1) \boxminus (\mathbf{a}{\otimes}t_2{\oplus}\mathbf{v}_1) = -k^2(\mathbf{a}{\otimes}t_4{\oplus}\mathbf{v}_2) \boxminus (\mathbf{a}{\otimes}t_3{\oplus}\mathbf{v}_2) \tag{0.51}$$

Since $\mathrm{gyr}[\mathbf{a}{\otimes}s, \mathbf{a}{\otimes}t] = I$ for any $\mathbf{a} \in (G, \oplus)$ and any $s, t \in \mathbb{R}$ (follows from Theorem 5.3 in [24]), (0.51) can be simplified by means of Corollary 19 to the equation

$$\mathbf{a}{\otimes}t_1{\ominus}\mathbf{a}{\otimes}t_2 = -k^2(\mathbf{a}{\otimes}t_4{\ominus}\mathbf{a}{\otimes}t_3)$$

which, by the scalar distributive law $(V2)$ can be further simplified to the equation

$$\mathbf{a}{\otimes}(t_1 - t_2) = -k^2(\mathbf{a}{\otimes}(t_4 - t_3)) \tag{0.53}$$

Finally, the validity of (0.53) is obvious since $t_1 - t_2$ and $t_4 - t_3$ have opposite signs. Hence, Equation (0.51) is valid, implying the validity of (0.49) and hence of (0.47) as desired. □

8 THE RIEMANNIAN GEOMETRY OF GYROVECTOR SPACES

The line element ds^2 of the n-dimensional Riemannian manifold that corresponds to the n-dimensional ball gyrovector space $\mathbb{R}^n_c = (\mathbb{R}^n_c, \oplus, \otimes)$ of the Euclidean n-space $\mathbb{R}^n$ is obtained from the *gyrodifferential*

$$\Delta\mathbf{s} = (\mathbf{v} + \Delta\mathbf{v}){\ominus}\mathbf{v} \tag{0.54}$$

whose squared norm in $\mathbb{R}^n$ gives the distance

$$\|\Delta\mathbf{s}\|^2 = \|(\mathbf{v}+\Delta\mathbf{v})\ominus\mathbf{v}\|^2$$

between the two neighboring points $\mathbf{v}$ and $\mathbf{v}+\Delta\mathbf{v}$ of $\mathbb{R}^n_c$, $\Delta\mathbf{v}$ being a gyrovector element of sufficiently small length, $\|\Delta\mathbf{v}\| < \varepsilon$ for some $\varepsilon > 0$, insuring the condition $\mathbf{v}+\Delta\mathbf{v} \in \mathbb{R}^n_c$.

Let $\mathbf{v}, \Delta\mathbf{v} \in \mathbb{R}^n_c$ be represented by their components relative to rectangular Cartesian coordinates as $\mathbf{v} = (x_1, \ldots x_n)$ and $\Delta\mathbf{v} = (\Delta x_1, \ldots \Delta x_n)$ in $\mathbb{R}^n$. Then, the gyrodifferential $\Delta\mathbf{s}$ can be written as

$$\begin{aligned}\Delta\mathbf{s} = \left[\frac{\partial\Delta\mathbf{s}}{\partial\Delta x_1}\right]_{\Delta\mathbf{v}=0}\Delta x_1 + \quad \ldots \quad + \left[\frac{\partial\Delta\mathbf{s}}{\partial\Delta x_n}\right]_{\Delta\mathbf{v}=0}\Delta x_n \\ + \varepsilon_1\Delta x_1 + \quad \ldots \quad + \varepsilon_n\Delta x_n\end{aligned} \tag{0.56}$$

where $\varepsilon_1, \ldots, \varepsilon_n \to 0$ as $\varepsilon \to 0$. We write (0.56) as

$$d\mathbf{s} = \left[\frac{\partial\Delta\mathbf{s}}{\partial\Delta x_1}\right]_{\Delta\mathbf{v}=0}dx_1 + \quad \ldots \quad + \left[\frac{\partial\Delta\mathbf{s}}{\partial\Delta x_n}\right]_{\Delta\mathbf{v}=0}dx_n \tag{0.57}$$

and, accordingly, we write (0.54) as

$$d\mathbf{s} = (\mathbf{v}+d\mathbf{v})\ominus\mathbf{v} \tag{0.58}$$

so that

$$ds^2 = \|d\mathbf{s}\|^2 = \|(\mathbf{v}+d\mathbf{v})\ominus\mathbf{v}\|^2 \tag{0.59}$$

ds^2 in (0.59) is the Riemannian line element of the n-dimensional ball gyrogroup $(\mathbb{R}^n_c, \oplus, \otimes)$, considered as an n-dimensional Riemannian manifold. Straightforward calculations give for Möbius' ball gyrovector spaces $(\mathbb{R}^n_c, \oplus_{\text{M}}, \otimes)$

$$\begin{aligned} ds^2 &= \|d\mathbf{s}\|^2 \\ &= \|(\mathbf{v}+d\mathbf{v})\ominus_{\text{M}}\mathbf{v}\|^2 \\ &= \frac{\sum_{i=1}^n dx_i^2}{\left(1+\frac{1}{4}K\sum_{i=1}^n x_i^2\right)^2}\end{aligned} \tag{0.60}$$

with curvature $K = -4/c^2$. The study of this line element by Riemann is described on p. 12 of [4], and in two dimensions it is known as the Poincaré metric of the Poincaré disc [11].

Similarly, straightforward calculations give for Einstein's ball gyrovector spaces $(\mathbb{R}^n_c, \oplus_{\text{E}}, \otimes)$

$$
\begin{aligned}
ds^2 &= \|d\mathbf{s}\|^2 \\
&= \|(\mathbf{v} + d\mathbf{v}) \ominus_{\text{E}} \mathbf{v}\|^2 \\
&= c^2 \frac{\sum_{i=1}^{n}\{c^2 - (r^2 - x_i^2)\}dx_i^2 + 2\sum_{\substack{i,j=1 \\ i<j}}^{n} x_i x_j dx_i dx_j}{(c^2 - r^2)^2}
\end{aligned}
\tag{0.61}
$$

where $r^2 = \sum_{i=1}^{n} x_i^2$. This line element, with $n = 2$, is known as the metric of the Beltrami (and Klein) disc model of hyperbolic geometry [11] [19].

9 THE BIFURCATION OF EUCLIDEAN GEOMETRY INTO TWO DUAL HYPERBOLIC GEOMETRIES

Hyperbolic geometry emerged from the denial of parallelism in Euclidean geometry. However, parallelism reappears in the novel, dual branch of hyperbolic geometry. We thus have

(1) Euclidean properties which are denied in hyperbolic geometry, but reappear in hyperbolic dual geometry, and conversely,

(2) Euclidean properties which remain valid in hyperbolic geometry, but are denied in its dual branch.

The gyro-formalism allows Euclidean and hyperbolic geometry to be unified giving rise to a super geometry called *gyro-geometry*. Euclidean geometry emerges as a degenerate case of gyro-geometry, corresponding to vanishing gyration. Consequently, hyperbolic geometry emerges in an enriched form, consisting of two hyperbolic mutually dual geometries.

(1) One of the resulting two hyperbolic mutually dual geometries turns out to be the standard hyperbolic geometry, while

(2) the other one emerges as a novel branch of hyperbolic geometry, that stores those Euclidean properties, like parallelism, that have seemingly been lost in the transition from Euclidean to non-Euclidean geometry.

Understanding the Euclidean bifurcation into two hyperbolic dual geometries is still in its infancy, but its existence is clearly indicated by Fig. 0.9.

As shown in Fig 12 of [24], and as it is well-known [7], hyperbolic triangle medians are concurrent. In contrast, hyperbolic dual triangle medians are not concurrent, thus observing a Euclidean property which is valid in hyperbolic geometry but it is invalid in hyperbolic dual geometry. A quite different situation is encountered with the triangle angle sum, which is smaller than π in

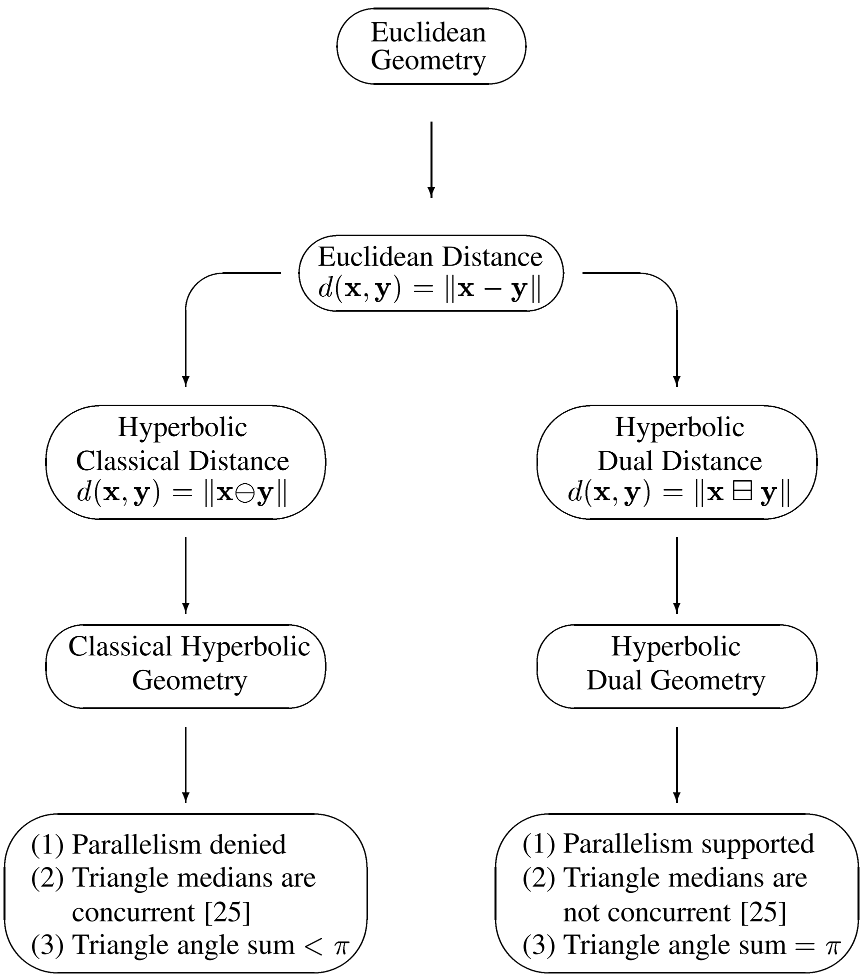

Figure 0.9. The bifurcation of Euclidean geometry into two hyperbolic mutually dual geometries.

hyperbolic geometry, but equals π in hyperbolic dual geometry, Theorem 16. Here we observe, again, a Euclidean property which is not valid in hyperbolic geometry, but reappears in hyperbolic dual geometry. Thus, some Euclidean properties that have seemingly been lost in the transition of Euclidean geometry to non-Euclidean geometry reappear in hyperbolic dual geometry.

Evidently, Euclidean geometry bifurcates into two hyperbolic dual geometries one of which is the standard hyperbolic geometry while the other one is novel. The Euclidean geometry bifurcation into two hyperbolic dual geometries is illustrated schematically in Fig. 0.9, starting from the bifurcation of the Euclidean metric into the two hyperbolic mutually dual metrics.

The bifurcation of Euclidean geometry naturally accompanies a corresponding bifurcation of its underlying axioms. A system of axioms for n-dimensional Euclidean geometry that merits special attention for its elegance and simplicity is the one proposed by Tarski since the 1920s, and recently studied by Givnat [20] and by Pambuccian [12]. It would be, therefore, interesting to explore the bifurcation of Tarski's system of axioms that corresponds to the Euclidean geometry bifurcation into the two mutually dual branches of hyperbolic geometry.

Since hyperbolic geometry underlies relativistic mechanics in the same way that Euclidean geometry underlies classical mechanics [5] [8] [15], the use of Thomas precession in hyperbolic geometry is expected [14]. Non vanishing Thomas precession, in turn, is the relativistic effect that turns velocity groups to velocity gyrogroups. Moreover, the bifurcation of Euclidean geometry in its transition to non-Euclidean geometry into two hyperbolic dual geometries may accompany a similar bifurcation of classical mechanics in its transition to relativistic mechanics giving rise to a relativistic dual mechanics theory. In this relativistic theory, for instance, the Newtonian uniform acceleration may bifurcate into the two relativistic dual uniform accelerations (0.27) and (0.29).

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