Beyond the Einstein Addition Law and its Gyroscopic Thomas Precession: The Theory of Gyrogroups and Gyrovector Spaces

BEYOND THE EINSTEIN ADDITION LAW AND ITS GYROSCOPIC THOMAS PRECESSION: THE THEORY OF GYROGROUPS AND GYROVECTOR SPACES

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Preface

"I cannot define coincidence [in mathematics]. But I shall argue that coincidence can always be elevated or organized into a superstructure which performs a unification along the coincidental elements. The existence of a coincidence is strong evidence for the existence of a covering theory."

—Philip J. Davis [Dav81]

Alluding to the Thomas gyration, this book presents the Theory of gyrogroups and gyrovector spaces, taking the reader to the immensity of hyperbolic geometry that lies beyond the Einstein special theory of relativity.

Soon after its introduction by Einstein in 1905 [Ein05], special relativity theory (as named by Einstein ten years later) became overshadowed by the appearance of general relativity. Subsequently, the exposition of special relativity followed the lines laid down by Minkowski, in which the role of hyperbolic geometry is not emphasized. This can doubtlessly be explained by the strangeness and unfamiliarity of hyperbolic geometry [Bar98].

The aim of this book is to reverse the trend of neglecting the role of hyperbolic geometry in the special theory of relativity, initiated by Minkowski, by emphasizing the central role that hyperbolic geometry plays in the theory. We will find in this book that the special theory of relativity and hyperbolic geometry cross-pollinate to produce important new results:

(1) The relativistic gyroscopic precession, known as the Thomas precession, turns out to play an important role in the foundations of hyperbolic geometry, uncovering hitherto unnoticed analogies with Euclidean geometry; and

(2) guided by the analogies that hyperbolic geometry shares with Euclidean geometry, uncovered by means of the Thomas precession, we discover related analogies that the Lorentz transformation shares with the Galilei transformation. The emerging analogies reverse a trend initiated by Minkowski: The relativistic spacetime emerges as a derived concept rather than a primitive (that is, unexplained) notion.

Exploring the special theory of relativity and its structure beyond the Einstein addition law and its gyroscopic Thomas precession, we take advantage, in this book, of the idea that hyperbolic geometry governs velocities in relativity physics in the same way that Euclidean geometry governs velocities in prerelativity physics, allowing us to partially restore the classical picture. In particular, we take advantage of the result that the Einstein velocity addition of relativistically admissible velocities is a gyrocommutative gyrogroup operation in the same way that the Galilei velocity addition (that is, ordinary vector addition) is a commutative group operation. Furthermore, Einstein's addition is a gyrocommutative gyrogroup operation that admits scalar multiplication, thus giving rise to a gyrovector space. Gyrovector spaces, in turn, form the setting for hyperbolic geometry in the same way that vector spaces form the setting for Euclidean geometry.

The primary purpose of this book is, accordingly, to provide readers with a self-contained account of the two topics in the subtitle.

A gyrogroup is a grouplike algebraic structure that provides a most natural generalization of the group notion. Historically, the first gyrogroup was discovered by the author in the algebraic structure that underlies Einstein's velocity addition [Ung88a]. However, the best way to introduce the gyrogroup notion by example is offered by the Möbius transformation of the disc.

Ahlfors' book [Ahl73], *Conformal Invariants: Topics in Geometric Function Theory*, begins with the presentation of the most general Möbius transformation of the complex open unit disc $\mathbb{D}$

$$z \mapsto e^{i\theta} \frac{z_0 + z}{1 + \bar{z}_0 z}$$

which we write as

$$z \mapsto e^{i\theta} \frac{z_0 + z}{1 + \bar{z}_0 z} = e^{i\theta}(z_0 \oplus z)$$

$z_0 \in \mathbb{D}$, $\theta \in \mathbb{R}$. Suggestively, we define the Möbius addition $\oplus$ in the disc, allowing the generic Möbius transformation of the disc to be viewed as a Möbius *left gyrotranslation*

$$\boxed{z \mapsto z_0 \oplus z = \frac{z_0 + z}{1 + \bar{z}_0 z}}$$

followed by a rotation. The prefix "gyro" that we use to emphasize analogies with classical notions, stems from the Thomas gyration, which will soon become clear. The resulting Möbius addition in the disc $\mathbb{D}$ is neither commutative nor associative. To 'repair' the breakdown of commutativity in the Möbius addition we associate it with the *gyration* (or, rotation) generated by $a, b \in \mathbb{D}$,

$$\boxed{\mathrm{gyr}[a, b] = \frac{a \oplus b}{b \oplus a} = \frac{1 + a\bar{b}}{1 + \bar{a}b}}$$

giving rise to the gyrocommutative law of Möbius addition,

$$a{\oplus}b = \mathrm{gyr}[a, b](b{\oplus}a) \qquad \text{Gyrocommutative law}$$

Following the gyration definition the gyrocommutative law is not terribly surprising, but we are not finished.

Coincidentally, the gyration that repairs the breakdown of commutativity in the Möbius addition repairs the breakdown of associativity as well, giving rise to identities that capture analogies,

$$\begin{aligned}
&a{\oplus}(b{\oplus}c) = (a{\oplus}b){\oplus}\mathrm{gyr}[a, b]c && \text{Left Gyroassociative Law}\\
&(a{\oplus}b){\oplus}c = a{\oplus}(b{\oplus}\mathrm{gyr}[b, a]c) && \text{Right Gyroassociative Law}\\
&\mathrm{gyr}[a, b] = \mathrm{gyr}[a{\oplus}b, b] && \text{Left Loop Property}\\
&\mathrm{gyr}[a, b] = \mathrm{gyr}[a, b{\oplus}a] && \text{Right Loop Property}
\end{aligned}$$

The Möbius addition is thus regulated by its associated gyration so that, in fact, the Möbius addition and its associated gyration are inextricably linked.

Where there are coincidences there is significance. The emerging coincidences to which the gyration gives rise uncover an interesting algebraic structure that merits extension by abstraction, leading to the grouplike structure called a gyrogroup.

Gyrogroups are generalized groups that share remarkable analogies with groups. In full analogy with groups:

(1) Gyrogroups are classified into gyrocommutative gyrogroups and non-gyrocommutative gyrogroups.

(2) Some gyrocommutative gyrogroups admit scalar multiplication, turning them into gyrovector spaces.

(3) Gyrovector spaces, in turn, provide the setting for hyperbolic geometry in the same way that vector spaces provide the setting for Euclidean geometry, thus enabling the two geometries to be unified.

(4) Moreover, the resulting analogies shared by the motions of Euclidean geometry (that is, a commutative group of translations and a group of rotations) and the motions of hyperbolic geometry (that is, a gyrocommutative gyrogroup of left gyrotranslations and a group of rotations) induce analogies shared by the Galilei transformation and the Lorentz transformation. These analogies, in turn, enable Lorentz transformation problems, hitherto poorly understood, to be straightforwardly solved in full analogy with the respective solutions of their Galilean counterparts. A point in case is, for instance, the determination of the visible shape of relativistically moving objects in Chapter 11.

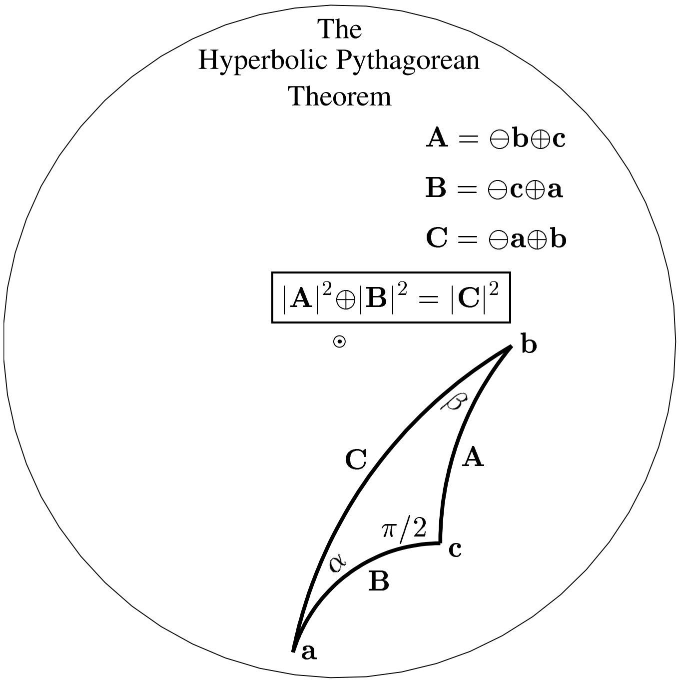

Figure 0.1. Artful Application of the Möbius Transformation in Hyperbolic Geometry, I. The hyperbolic Pythagorean theorem for Möbius right angled hyperbolic triangles in the complex unit disc $\mathbb{D}$ in a form fully analogous to its Euclidean counterpart [Ung99].

The flavor of this book is illustrated by Figs. 0.1 and 0.2. To demonstrate the ability of gyro-formalism to capture analogies:

(1) We present graphically in Fig. 0.1 the hyperbolic Pythagorean Theorem in the Möbius gyrovector plane $(\mathbb{R}^2_c, \oplus, \otimes)$, which we will study in Chapters 4 and 6; and

(2) We present graphically in Fig. 0.2 the algebra of the hyperbolic parallel transport along geodesics in the Möbius gyrovector plane $(\mathbb{R}^2_c, \oplus, \otimes)$, which we will study in Chapter 7.

Typically to the study of hyperbolic geometry as the geometry of gyrovector spaces, the hyperbolic geometry concepts shown in Figs. 0.1 and 0.2 turn out to be fully analogous to their Euclidean counterparts. In contrast, prior to the emergence of gyrogroup and gyrovector space theory:

(1) The Hyperbolic Pythagorean Theorem appeared in the literature in a form which shares no obvious analogies with its Euclidean counterpart.

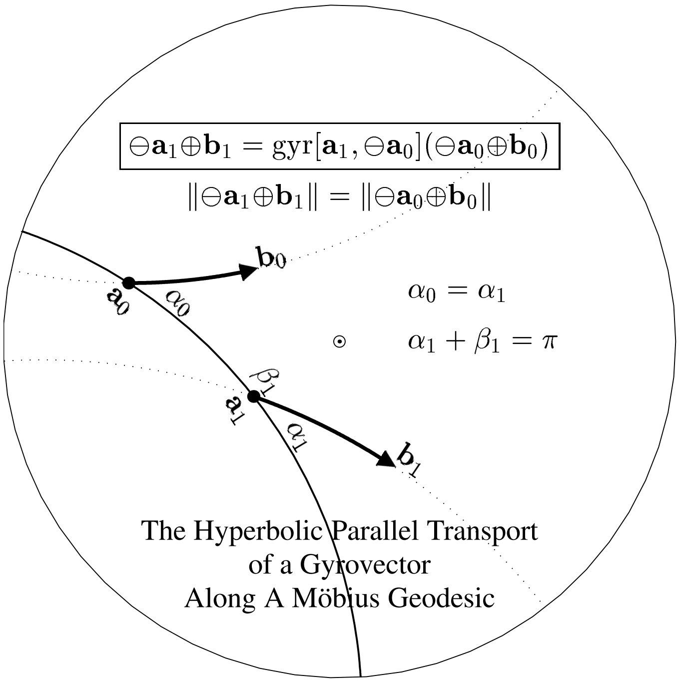

Figure 0.2. Artful Application of the Möbius Transformation in Hyperbolic Geometry, II. The Poincaré n-dimensional ball model of hyperbolic geometry turns out to be the n-dimensional Möbius gyrovector space. In Chapter 7 we will find that in gyro-formalism the nonassociative algebra of the hyperbolic parallel transport of a gyrovector $(-\mathbf{a}_0\oplus\mathbf{b}_0)$ rooted at $\mathbf{a}_0$ to the gyrovector $(-\mathbf{a}_1\oplus\mathbf{b}_1)$ rooted at $\mathbf{a}_1$ along the Möbius geodesic that links $\mathbf{a}_0$ and $\mathbf{a}_1$ in a Möbius gyrovector space $(\mathbb{V}_c, \oplus, \otimes)$ is fully analogous to the algebra of its Euclidean counterpart. The special case of $n = 2$ is shown here graphically.

(2) Parallel transport in classical hyperbolic geometry is achieved by methods of differential geometry rather than by methods of nonassociative algebra.

One of the attractive features of this book is that the prerequisites are minimal, encouraging readers to perform their own research at an early stage once they master the manipulation of analogies to which the Thomas gyration gives rise. The theory of gyrogroups and gyrovector spaces recasts hyperbolic geometry and aspires to further conquests in the immense domain of mathematical physics and hyperbolic geometry. Some gyrogroup identities are presented without proof. At the end of chapters exercises are provided to enhance practical experience. This book can be used by teachers and researchers as a source for research programs, classroom projects, and master theses (e.g., [Big94]), and as a self-study book, perhaps as part of a directed reading course. For that

Notion	Euclidean	Hyperbolic
Addition	$+$	$\oplus$
Commutativity	$\mathbf{a}+\mathbf{b}=\mathbf{b}+\mathbf{a}$	$\mathbf{a}\oplus\mathbf{b}=\mathrm{gyr}[\mathbf{a},\mathbf{b}](\mathbf{b}\oplus\mathbf{a})$
Associativity	$\mathbf{a}+(\mathbf{b}+\mathbf{c})=(\mathbf{a}+\mathbf{b})+\mathbf{c}$	$\mathbf{a}\oplus(\mathbf{b}\oplus\mathbf{c})=(\mathbf{a}\oplus\mathbf{b})\oplus\mathrm{gyr}[\mathbf{a},\mathbf{b}]\mathbf{c}$ $(\mathbf{a}\oplus\mathbf{b})\oplus\mathbf{c}=\mathbf{a}\oplus(\mathbf{b}\oplus\mathrm{gyr}[\mathbf{b},\mathbf{a}]\mathbf{c})$
Distance	$\Vert\mathbf{a}-\mathbf{b}\Vert$	$\Vert\mathbf{a}\ominus\mathbf{b}\Vert$
The Pythagorean Theorem	$\Vert\mathbf{A}\Vert^2+\Vert\mathbf{B}\Vert^2=\Vert\mathbf{C}\Vert^2$	$\Vert\mathbf{A}\Vert^2\oplus\Vert\mathbf{B}\Vert^2=\Vert\mathbf{C}\Vert^2$ in the Poincaré unit disc model
Gyrovector Transport	$-\mathbf{a}_1+\mathbf{b}_1=-\mathbf{a}_0+\mathbf{b}_0$	$\ominus\mathbf{a}_1\oplus\mathbf{b}_1=\ominus\mathbf{a}_0\oplus\mathbf{b}_0$
Parallel Transport	$-\mathbf{a}_1+\mathbf{b}_1=-\mathbf{a}_0+\mathbf{b}_0$	$\ominus\mathbf{a}_1\oplus\mathbf{b}_1=\mathrm{gyr}[\mathbf{a}_1,\ominus\mathbf{a}_0](\ominus\mathbf{a}_0\oplus\mathbf{b}_0)$
Triangle Inequality	$\Vert-\mathbf{a}+\mathbf{c}\Vert$ $\le\Vert-\mathbf{a}+\mathbf{b}\Vert+\Vert-\mathbf{b}+\mathbf{c}\Vert$	$\Vert\ominus\mathbf{a}\oplus\mathbf{c}\Vert$ $\le\Vert\ominus\mathbf{a}\oplus\mathbf{b}\Vert\oplus\Vert\ominus\mathbf{b}\oplus\mathbf{c}\Vert$
Geodesics	$\mathbf{a}+(-\mathbf{a}+\mathbf{b})t$ $-\infty<t<\infty$	$\mathbf{a}\oplus(\ominus\mathbf{a}\oplus\mathbf{b})\otimes t$ $-\infty<t<\infty$
Midpoint $(\mathbf{a},\mathbf{b})$	$\mathbf{a}+\frac{1}{2}(-\mathbf{a}+\mathbf{b})$	$\mathbf{a}\oplus\frac{1}{2}\otimes(\ominus\mathbf{a}\oplus\mathbf{b})$
Angle cosines between rays	$\cos\alpha=\frac{-\mathbf{a}+\mathbf{b}}{\Vert-\mathbf{a}+\mathbf{b}\Vert}\cdot\frac{-\mathbf{a}+\mathbf{c}}{\Vert-\mathbf{a}+\mathbf{c}\Vert}$	$\cos\alpha=\frac{\ominus\mathbf{a}\oplus\mathbf{b}}{\Vert\ominus\mathbf{a}\oplus\mathbf{b}\Vert}\cdot\frac{\ominus\mathbf{a}\oplus\mathbf{c}}{\Vert\ominus\mathbf{a}\oplus\mathbf{c}\Vert}$
Equations	$\mathbf{a}+\mathbf{x}=\mathbf{b}$ $\mathbf{x}=-\mathbf{a}+\mathbf{b}$ $\mathbf{y}+\mathbf{a}=\mathbf{b}$ $\mathbf{y}=\mathbf{b}-\mathbf{a}$	$\mathbf{a}\oplus\mathbf{x}=\mathbf{b}$ $\mathbf{x}=\ominus\mathbf{a}\oplus\mathbf{b}$ $\mathbf{y}\oplus\mathbf{a}=\mathbf{b}$ $\mathbf{y}=\mathbf{b}\ominus\mathrm{gyr}[\mathbf{b},\mathbf{a}]\mathbf{a}=\mathbf{b}\boxminus\mathbf{a}$

Table 0.1. Euclidean-Hyperbolic Analogies for the Möbius Addition $\oplus$ in gyrovector spaces

reason the exposition is as self-contained as possible, consisting of new results that are linked to well known ones by elegant novel analogies such as those we have already seen. These and some other remarkable analogies that we will study in this book are presented in Table 0.1.

We are particularly interested, in this book, in the study of Einstein's velocity addition. Like Möbius addition, Einstein's addition of relativistically admissible velocities is neither commutative nor associative. The breakdown of commutativity and associativity in Einstein's addition is not well known, since most books on relativity physics, with only a few outstanding exceptions, for instance [Foc64], [Bac77], and [SU92], present Einstein's addition for only parallel velocities, in which case it is both commutative and associative. However, like Möbius addition, Einstein's addition turns out to be a gyrocommutative gyrogroup operation. Following the breakdown of commutativity and associativity in Einstein's addition, the relativistic effect known as the Thomas precession, or gyration, comes to the rescue. This book is, accordingly, the fascinating story of the role that the Thomas gyration plays in physics, in nonassociative algebra, and in non-Euclidean geometry, and of the theory of gyrogroups and gyrovector spaces to which it gives rise. The story is of fairly recent origin, dating back to the author's work starting in the late 1980's [Ung88a].

The book is written for mathematical physicists, geometers, and algebraists, and for readers who enjoy the harmonious interplay between algebra, geometry, and physics. These include experienced researchers as well as third or fourth year students who have encountered some elements of calculus with analytic geometry, Newtonian and Einsteinian mechanics, linear algebra, and elementary group theory. The Thomas precession is a relativistic rotation that does not exist classically. The writer has long believed that the study of topics in relativity physics, non-Euclidean geometry, and nonassociative algebra, to which the Thomas precession gives rise, deserves early place in the curriculum. The concept of Thomas gyration, the abstract Thomas precession, is simple and without equal as a means of giving a unifying approach to Euclidean and non-Euclidean geometry, the geometries governing velocities in Newtonian and in Einsteinian mechanics.

Physics is a major external source of mathematical inspiration, and mathematics is the tool for understanding physics. This book provides a leisurely and elementary introduction to (i) gyrogroup theory and (ii) gyrovector space theory. The prerequisites should be met by a third year undergraduate student of physics or mathematics. It may serve as a text for physics and mathematics students and researchers, thereby highlighting Gravity Probe B, a NASA–Stanford University project led by C.W. Francis Everitt aimed at the measurement of the gyroscopic precession of gyroscopes of unprecedented accuracy in Earth orbit. NASA's Gravity Probe B (GP-B) [EFS69], [Tau97], initiated by William M. Fairbank (1917–1989) [Edi89], is a drag-free satellite carrying gyroscopes around Earth Program. Rather than studying the Thomas precession of electronic gyroscopes in atoms, NASA plans to study the Thomas precession of gyroscopes in space. It is the sensitivity of the Thomas precession to the non-Euclidean nature of the geometry of our spacetime that attracts both NASA's and our interest in measuring it, in understanding it, and in exploiting it.

Reading This Book

The main merit of this book is that it is written in an accessible language and does not require of the reader additional deep knowledge of related fields of mathematics.

The detailed description of the Thomas precession in Chapter 1 should not discourage the geometer and the algebraist from reading the remainder of the book about: (i) gyrogroups (which are generalized groups); (ii) gyrovector spaces (which are generalized vector spaces); and (iii) hyperbolic geometry, for which gyrogroups and gyrovector spaces set the stage. The remainder of the book is independent of Chapter 1. It is, however, likely that once the reader becomes familiar with the fundamental importance and elegance of the ability of *gyrations* to capture analogies, he or she will be willing to read in Chapter 1 how the abstract gyration is realized in special relativity by the Thomas precession.

The book is divided into twelve chapters.

(1) **Chapter 1** presents the history and the physical background of the introduction of the Thomas precession. The remainder of the book is independent of this chapter.

(2) **Chapter 2** presents the gyrogroup notion, modeled on Einstein's addition of relativistically admissible velocities and their Thomas precession. Elementary properties of gyrogroups are presented and analogies that gyrogroups share with groups are emphasized.

(3) **Chapter 3** extends the notion of the gyrogroup to the notion of a gyrovector space, and, in particular, presents a study of Einstein's gyrovector spaces. Elementary properties of gyrovector spaces are presented, and analogies they share with vector spaces are emphasized. It is shown that Einstein's vector spaces form the setting for the Beltrami (also known as the Klein) ball model of hyperbolic geometry in the same way that vector spaces form the setting for Euclidean geometry in any dimension.

(4) **Chapter 4** presents the study of hyperbolic geometry in terms of its underlying gyrovector space structure. Some known results are obtained in a new way, as well as some novel results, such as the new hyperbolic Pythagorean theorem.

(5) **Chapter 5** presents the Ungar gyrovector spaces, which are studied in a way similar to the study of the Einstein gyrovector spaces by replacing relativistically admissible velocities by proper relativistic admissible velocities. The resulting model of hyperbolic geometry is a whole space model, rather than a ball model (as in the case of the models of Beltrami and of Poincaré).

(6) **Chapter 6** presents the study of the Poincaré ball model of hyperbolic geometry in terms of its underlying Möbius gyrovector space structure. Some known results are obtained in a new way and some novel results, such as the hyperbolic Pythagorean theorem, emerge.

(7) **Chapter 7** presents the gyroparallelogram and the gyroderivative in hyperbolic geometry, and the parallelism in the so called cohyperbolic

geometry. The conformality of the Poincaré ball model of hyperbolic geometry makes it an attractive model for the study of parallel transport by the gyrovector space formalism.

(8) **Chapter 8** The remainder of the book is independent of this chapter, the sole purpose of which is to provide a motivational approach to readers who appreciate the importance of the $SL(2, C))$ algebra in relativity physics. Readers who wish to concentrate on the study of gyrogroups and gyrovector spaces may, therefore, skip reading this chapter and, perhaps, return to read it later on when their curiosity about links between old and new arises.

(9) **Chapter 9** presents the theory of gyrogroup extension of gyrocommutative gyrogroups, which in Chapter 10 will allow us to present and study the abstract Lorentz group.

(10) **Chapter 10** employs the results of Chapter 9 to study the abstract Lorentz group and its composition law. As opposed to the four-dimensionally covariant formalism commonly used in the literature for dealing with the applications of the Lorentz group, gyrogroup theory allows in this chapter a formalism that is fully analogous to the formalism commonly used with the Galilean group and its applications.

(11) **Chapter 11** employs the gyrogroup formalism which has been developed in this book to determine the Lorentz transformations that link any two given spacetime events. As a useful tool in quantum mechanics, the so called little Lorentz groups emerge in an elegant way in terms of the Lorentz link. Furthermore, the Lorentz link theorem allows in this chapter an extraordinarily simple and transparent derivation of the shape of relativistically moving objects. The story of the moving rigid sphere that this chapter presents is particularly fascinating. Einstein believed in 1905 that a moving sphere appears to be contracted in the direction of its motion. Following Penrose and Terrell it became widely accepted in relativity physics, since 1959, that the moving sphere does not at all appear to be contracted. The gyrogroup-theoretic techniques that are developed in this book, however, clearly tilt the balance in favor of Einstein, giving a way to calculate the visible shape of moving objects, Figs. 0.3 and 0.4, that is superior to those employed by Penrose, by Terrell, and by others.

(12) **Chapter 12** presents briefly (i) a non-standard Lorentz transformation group which is parametrized by proper, rather than coordinate, velocities; and (ii) a Möbius–Lorentz transformation group based on the Möbius addition.

Teaching From This Book

This book presents the remarkable ability of gyrogroups and gyrovector spaces to capture analogies that allow the teacher to tie the modern and unknown with the classical and familiar. As such, this book can accompany any conventional undergraduate and graduate course in (i) modern physics, (ii) non-Euclidean geometry, and (iii) abstract algebra, as part of the process of introducing novel ideas that link the old and known with the new and unfamiliar.

Gyrogroup and gyrovector space theory is still in its infancy. It is too early to write a book for teaching a course totally devoted to this area. However, the study of gyrogroups and gyrovector spaces of Einstein and of Möbius **must** be included:

(1) In any improved course on **Modern Physics**: In the same way that physics students must know that vector addition is commutative and associative, they will have to know that Einstein's vector addition of relativistically admissible velocities is, in full analogy, gyrocommutative and gyroassociative. The paradoxical breakdown of commutativity and associativity in Einstein's addition will become clear. The lost commutativity and associativity reappear as gyrocommutativity and gyroassociativity, allowing the Lorentz group to be treated in full analogy with its simpler and better understood counterpart, the Galilean group.

(2) In any improved course on **Non-Euclidean Geometry**: Geometry students will have to know that hyperbolic geometry shares remarkable analogies with Euclidean geometry, allowing the unification of the two geometries into a single super geometry, called gyro-geometry.

  (i) Vector addition with its scalar multiplication is a vector space operation, known to provide the setting for Euclidean geometry. In the same way, students should know that

  (ii) Möbius addition with its scalar multiplication is a gyrovector space operation, which provides the setting for the Poincaré ball (or, disc, in two dimensions) model of hyperbolic geometry; and

  (iii) Einstein's addition with its scalar multiplication is a gyrovector space operation, which provides the setting for the Beltrami ball (or, disc, in two dimensions) model of hyperbolic geometry; and similarly for other infinitely many isomorphic gyrovector space operations, which provide the setting for infinitely many other models of hyperbolic geometry.

(3) In improved courses on **Abstract Algebra**: The ultimate unity of mathematics and physics makes a strong case for the inclusion of the generalized groups and vector spaces, that is, gyrogroups and gyrovector spaces, (i) since they provide the mathematical model which underlies

Einstein's addition, Möbius addition, and their respective scalar multiplication, and (ii) since, moreover, they provide a theory which unifies these additions with the common vector addition. It is recommended to include in graduate abstract algebra courses the two articles [FU00] and [FU01], which are beyond the scope of this book. These articles exhibit the natural emergence of gyrogroups, both gyrocommutative and non-gyrocommutative, in group theory. This book may serve as a preparation for reading the related [SSS98] [Iss99] [KU98], but more abstract book of Sabinin [Sab99], on smooth quasigroups and loops in nonassociative algebra and differential geometry.

Guided by the analogies this book presents, and other analogies readers may discover, readers are likely to find new results in hyperbolic geometry and in relativity physics, including new gyrogroup identities. These, as well as corrections and terminology suggestions, will be gratefully received by the author, abraham_ungar@ndsu.nodak.edu, for possible inclusion in the next edition of this book, with credit to their discoverers.

The author hopes that the distinctive aspects of this book, reflected in Figs. 0.1 and 0.2 and in Table 0.1, will make it of interest to students, to instructors, and to researchers of both physics and mathematics, and that readers will approach to share the fascination that led him to write this book.

ABRAHAM A. UNGAR

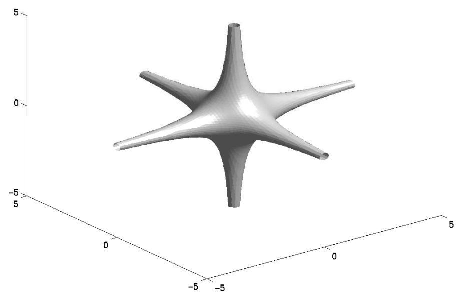

Figure 0.3. Theorem 11.21 of Chapter 11, p. 371, established by gyrogroup-theoretic techniques that are developed in this book, asserts that a stationary surface, Fig. 0.3,

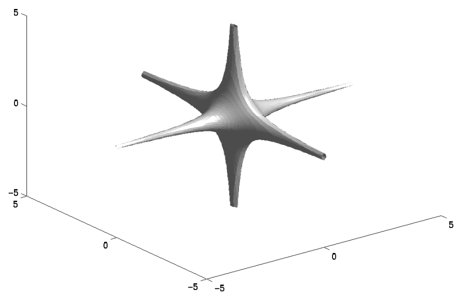

Figure 0.4. viewed from a frame relative to which it moves, is flattened in the direction of its motion, as shown here, Fig. 0.4, for the surface $x^2y^2 + y^2z^2 + z^2x^2 = 1$.

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The author is particularly indebted to Helmuth K. Urbantke for help in tracing the prehistory of gyrogroup and gyrovector space theory, as presented in [SU00], for discussing Penrose's way of understanding the Lorentz contraction, and for drawing attention to important references about the role of hyperbolic geometry in special relativity theory.

The author shares his wish to incorporate gyroscopic precession into science and mathematics education with the GP-B group of the NASA/Stanford Program to measure the precession of gyroscopes of unprecedented accuracy in Earth orbit. The hospitality during several visits with the GP-B group, led by Francis C.W Everitt, and the support of NASA's Joint Venture (JOVE) Program, Grant NAG8–1007, are acknowledged. NSF support is discussed in [Ung00b]. The numerical calculations and the graphs presented in this book were obtained by using MATLAB.

Foreword

The obscured Thomas precession of the special theory of relativity soared in 1988 [Ung88a] into prominence by deciphering the mathematical structure, called a gyrocommutative gyrogroup, that the Thomas precession encodes. The notorious Thomas precession, seemingly the ugly duckling of special relativity theory, thus became the beautiful swan of the theory of gyrogroups and gyrovector spaces.

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