The Gyro-Structure Of The Density Matrix For Mixed State Qubits

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ABSTRACT Density matrices for mixed state qubits, parametrized by the Bloch vector in the open unit ball of the Euclidean 3-space, are well known in quantum computation theory. We bring the seemingly structureless set of all these density matrices under the umbrella of gyrovector spaces, where the Bloch vector is treated as a hyperbolic vector, called a gyrovector. Gyrovector spaces are mathematical objects that form the setting for the hyperbolic geometry of Bolyai and Lobachevski just as vector spaces form the setting for Euclidean geometry. It is thus interesting, in geometric quantum computation, to realize that the set of all qubit density matrices has rich structure with strong link to hyperbolic geometry. Concrete examples for the use of the gyro-structure to derive new, interesting identities for qubit density matrices are presented.

1 INTRODUCTION

A *qubit* is a two state quantum system completely described by the qubit density matrix $\rho(\mathbf{v})$,

$$\rho(\mathbf{v}) = \tfrac{1}{2}\begin{pmatrix} 1+v_3 & v_1 - iv_2 \\ v_1 + iv_2 & 1 - v_3 \end{pmatrix} \tag{1}$$

parametrized by the vector $\mathbf{v} = (v_1, v_2, v_3) \in \mathbb{B}^3$, where $\mathbb{B}^3$ is the open unit ball of the Euclidean 3-space $\mathbb{R}^3$. The parameter $\mathbf{v}$ is known as the Bloch vector [1] and, accordingly, the ball $\mathbb{B}^3$ is called the Bloch ball. The determinant of $\rho(\mathbf{v})$ reveals the link with special relativity,

$$det(\rho(\mathbf{v})) = \frac{1}{4\gamma_{\mathbf{v}}^2} \tag{2}$$

where $\gamma_{\mathbf{v}}$ is the Lorentz gamma factor of special relativity, given by the equation

$$\gamma_{\mathbf{v}} = \frac{1}{\sqrt{1 - \|\mathbf{v}\|^2}} \tag{3}$$

$\mathbf{v} \in \mathbb{B}^3$, where the vacuum speed of light is normalized to unity, $c = 1$. It satisfies the *gamma identity*

$$\gamma_{\mathbf{u} \oplus_{\mathrm{E}} \mathbf{v}} = \gamma_{\mathbf{u}} \gamma_{\mathbf{v}} (1 + \mathbf{u} \cdot \mathbf{v}) \tag{4}$$

for all $\mathbf{u}, \mathbf{v} \in \mathbb{B}^3$, where $\oplus_{\mathrm{E}}$ is the Einstein addition in $\mathbb{B}^3$ of relativistically admissible velocities in special relativity theory.

Despite the important role that the qubit density matrix $\rho(\mathbf{v})$ plays in quantum mechanics, is seems to be almost structureless. Indeed, the only structure commonly considered relevant is the convex structure of the set

$$D = \{\rho(\mathbf{v}) : \mathbf{v} \in \mathbb{B}^3\} \tag{5}$$

Additionally, while exploring connections between qubit density matrices and special relativity [2], Peres points out that the qubit density matrix has no covariant transformation law.

Accordingly, the aim of this article is to uncover the nonassociative algebraic rich structure that underlies the set D of all qubit density matrices, linking it to the Poincaré ball model of the hyperbolic geometry of Bolyai and Lobachevski. The algebraic structure, called a gyrovector space, turns out to be the same vector space-like structure that underlies the Einstein addition of relativistically admissible velocities in special relativity, uncovered in [3, 4].

The ability of the algebraic structure of D that we uncover in this article to capture novel identities and a strong link with hyperbolic geometry is demonstrated by several examples in Section 5. One of the elegant identities that we obtain, for instance, is

$$\rho_{\mathbf{u}} \rho_{\mathbf{v}}^2 \rho_{\mathbf{u}} = tr(\rho_{\mathbf{u}} \rho_{\mathbf{v}}^2 \rho_{\mathbf{u}}) \rho_{2 \otimes (\mathbf{u} \oplus \mathbf{v})} \tag{6}$$

$\rho_{\mathbf{v}} = \rho(\mathbf{v})$, etc., where $\oplus$ and $\otimes$ are Möbius addition and scalar multiplication in the Möbius gyrovector space structure of the ball, presented in Sections 2 and 3. This identity expresses a matrix product of several qubit density matrices as a single density matrix with a positive coefficient, Theorem 6 in Section 5. Furthermore, it reveals the strong link between the qubit density matrix and the Poincaré ball model of the hyperbolic geometry of Bolyai and Lobachevski, as we will demonstrate in this article.

Einstein addition of relativistically admissible velocities is neither commutative nor associative. As such, it is not a group operation. The seemingly structureless Einstein addition of relativistically admissible 3-velocities was therefore replaced by 4-velocities in Minkowski's reformulation of Einstein's special relativity. However, following [4] it is now known that Einstein addition is not structureless since the relativistic effect called Thomas precession comes to the rescue, turning the Einstein addition into a gyrocommutative gyrogroup operation. Furthermore, Einstein addition admits scalar multiplication, turning itself into a gyrovector addition. The prefix "gyro" that we use to emphasize analogies with classical concepts stems from the Thomas gyration. The latter, in turn, is the extension of the Thomas precession by abstraction.

Einstein addition and Möbius addition are *gyrovector space* isomorphic operations, (97), studied in [4]. Einstein gyrovector spaces form the setting for the Beltrami ball model of hyperbolic geometry just as vector spaces form the setting for the standard model of Euclidean geometry. Similarly, Möbius gyrovector spaces form the setting for the Poincaré ball model of hyperbolic geometry, as indicated in Figs. 1 and 2 of Section 3 in this article. The theory of gyrogroups and gyrovector spaces is presented in [4]. However, an elegant introductory approach to the gyrogroup (gyrovector space) notion is presented in Section 2 (Section 3) for readers of this article who are not familiar with the new concept. The abstract gyrogroup turns out to be a rich, nonassociative algebraic structure that regulates, for instance, the Einstein addition, the Möbius addition, and the qubit density matrix that we study in this article. A brief history of the emergence of the gyrogroup concept and its application in relativity physics is presented in [5, pp. 141 – 142].

The Lorentz transformation of special relativity was studied by Einstein in terms of 3-velocity addition, called Einstein addition. An attempt to study it in terms of a new addition, called Möbius addition, is presented in [6] with a remark in [7]. These studies, along with recently discovered connections with the qubit density matrix of quantum computation [8, 9, 10], led us to the discovery of the gyrovector space structure that the qubit density matrix possesses. The qubit density matrix is commonly parametrized by the so called *Bloch vector*. We demonstrate that the Bloch vector is in fact a gyrovector rather than a vector. Accordingly, we called it in [9] the Bloch gyrovector.

The seemingly abstract results of the hyperbolic geometry of Bolyai and Lobachevski found applications in physics upon Einstein's introduction of the special theory of relativity in 1905 [11, 12, 13, 14], as was pointed out later by Varičak, [15, 16, 17, 18], whose work has been cited by Pauli [19]. Walter [20, p. 121] points out that Varičak had to admit in 1924 [21, p. 80] that the adaption of ordinary vector algebra for use in hyperbolic space was just not possible. However, following [4] it is now known that Thomas gyration is tailor made for the introduction of vectors into hyperbolic geometry, where they are called gyrovectors. Not unexpectedly, thus, the Bloch vector of quantum computation theory turns out to be a gyrovector rather than a vector.

In Section 2 we motivate the gyrogroup notion and its gyrocommutative branch by considering key features of the Möbius self-transformation of the open complex unit disc. In Section 3 we present the scalar multiplication that Möbius addition admits, giving rise to Möbius gyrovector spaces. These, in turn, form the setting for the Poincaré ball model of the hyperbolic geometry of Bolyai and Lobachevski [4], as we indicate in Figs. 1 and 2. In Section

4 we uncover the Möbius gyrovector space that regulates the set of all qubit density matrices. In Section 5 we use results of gyrovector space theory from [4] to uncover new identities for qubit density matrices. The Bloch vector that parametrizes the qubit density matrix clearly turns out in the concrete examples of Section 5 to be a gyrovector, that is, a hyperbolic vector in the three-dimensional Möbius gyrovector space or, equivalently, in the Poincaré ball model of hyperbolic geometry.

2 THE MÖBIUS ADDITION AND ITS GYROGROUP STRUCTURE

The most general Möbius transformation of the complex open unit disc $\mathbb{D} = \{z : |z| < 1\}$ in the complex z-plane $\mathbb{C}$ [22, 23],

$$z \mapsto e^{i\theta}\frac{a+z}{1+\overline{a}z} = e^{i\theta}(a \oplus z) \tag{7}$$

induces the Möbius addition $\oplus$ in the disc, allowing the Möbius transformation of the disc to be viewed as a Möbius left *gyrotranslation*

$$z \mapsto a \oplus z = \frac{a+z}{1+\overline{a}z} \tag{8}$$

followed by a rotation. Here $\theta \in \mathbb{R}$ is a real number, $a, z \in \mathbb{D}$, and $\overline{a}$ is the complex conjugate of a. Möbius addition $\oplus$ is thus a consequence of the polar decomposition (7) of the Möbius transformation group of the disc. It is neither commutative nor associative. The breakdown of commutativity in Möbius addition is "repaired" by the introduction of gyration,

$$\mathrm{gyr} : \mathbb{D} \times \mathbb{D} \to Aut(\mathbb{D}, \oplus) \tag{9}$$

given by the equation

$$\mathrm{gyr}[a, b] = \frac{a \oplus b}{b \oplus a} = \frac{1+a\overline{b}}{1+\overline{a}b} \tag{10}$$

where $Aut(\mathbb{D}, \oplus)$ is the automorphism group of the groupoid $(\mathbb{D}, \oplus)$. We recall that a groupoid $(G, +)$ is a nonempty set G with a binary operation $+$, and an automorphism of a groupoid $(G, +)$ is a bijective self-map of the groupoid G that respects its binary operation $+$. The set of all automorphisms of a groupoid $(G, +)$ forms a group under bijection composition, denoted $Aut(G, +)$.

The elegant *gyrocommutative law* of Möbius addition $\oplus$ that follows from the definition of gyr in (10),

$$a \oplus b = \mathrm{gyr}[a, b](b \oplus a) \tag{11}$$

is not terribly surprising, but we are not finished.

Coincidentally, the gyration $\mathrm{gyr}[a, b]$ that repairs the breakdown of the commutative law of $\oplus$ in (11), repairs the breakdown of the associative law of $\oplus$ as well, giving rise to the respective *left and right gyroassociative laws*

$$\begin{aligned} a\oplus(b\oplus z) &= (a\oplus b)\oplus\mathrm{gyr}[a, b]z \\ (a\oplus b)\oplus z &= a\oplus(b\oplus\mathrm{gyr}[b, a]z) \end{aligned} \tag{12}$$

for all $a, b, z\in\mathbb{D}$.

The unimodular complex number $\mathrm{gyr}[a, b]$ in (10) involves division of complex numbers, an operation that cannot be extended to vector spaces. We therefore note that, by means of (12), it can also be written as

$$\mathrm{gyr}[a, b]z = \ominus(a\oplus b)\oplus\{a\oplus(b\oplus z)\} \tag{13}$$

where $\ominus a = -a$ for all $a\in\mathbb{D}$. In (13) we have an expression for $\mathrm{gyr}[a, b]$ that, unlike (10), admits extension to vector spaces, (15), (26).

Guided by analogies with groups we take the (gyrocommutative and the) gyroassociative laws and other key features of the Möbius addition as a model of a (gyrocommutative) gyrogroup, obtaining the following two definitions.

Definition 1 (Gyrogroups). *The groupoid $(G, \oplus)$ is a gyrogroup if its binary operation satisfies the following axioms. In G there is at least one element, 0, called a left identity, satisfying*

(G1) $\quad 0\oplus a = a$ $\qquad$ *Left Identity*

for all $a\in G$. There is an element $0\in G$ satisfying axiom $(G1)$ such that for each a in G there is an element $\ominus a$ in G, called a left inverse of a, satisfying

(G2) $\quad \ominus a\oplus a = 0$ $\qquad$ *Left Inverse*

Moreover, for any $a, b, z\in G$ there exists a unique element $\mathrm{gyr}[a, b]z\in G$ such that

(G3) $\quad a\oplus(b\oplus z) = (a\oplus b)\oplus\mathrm{gyr}[a, b]z$ $\qquad$ *Left Gyroassociative Law*

If $gyr[a, b]$ denotes the map $gyr[a, b] : G \to G$ given by $z \mapsto gyr[a, b]z$ then

(G4) $\quad \mathrm{gyr}[a, b]\in Aut(G, \oplus)$ $\qquad$ *Gyroautomorphism*

and $gyr[a, b]$ is called the Thomas gyration, or the gyroautomorphism of G, generated by $a, b\in G$. The operation $\mathrm{gyr} : G \times G \to Aut(G, \oplus)$ is called the gyrooperation of G. Finally, the gyroautomorphism $gyr[a, b]$ generated by any $a, b\in G$ satisfies

(G5) $\quad \mathrm{gyr}[a, b] = \mathrm{gyr}[a\oplus b, b]$ $\qquad$ *Left Loop Property*

The gyrogroup axioms are classified into three classes.

(1) The first pair of axioms, $(G1)$ and $(G2)$, are reminiscent of the group axioms;

(2) The last pair of axioms, $(G4)$ and $(G5)$, are the gyrooperation axioms; and

(3) The middle axiom, $(G3)$, is a hybrid axiom linking the two pairs of axioms in (1) and (2).

As in group theory, we use the notation

$$a\ominus b = a\oplus(\ominus b) \tag{14}$$

In full analogy with groups, gyrogroups are classified into gyrocommutative and non-gyrocommutative gyrogroups. The definition of gyrocommutativity in gyrogroups follows.

Definition 2 (Gyrocommutative Gyrogroups). *The gyrogroup $(G, \oplus)$ is gyrocommutative if for all $a, b{\in}G$*

(G6) $\quad a \oplus b = \mathrm{gyr}[a, b](b \oplus a) \qquad$ *Gyrocommutative Law*

Gyrogroups, both gyrocommutative and non-gyrocommutative, abound in group theory as evidenced from [24] and [25].

An introduction to the theory of gyrogroups is presented in [4], where the Möbius disc groupoid $(\mathbb{D}, \oplus)$ and its generalization to the ball of any real inner product space turn out to be gyrocommutative gyrogroups. An important theorem in gyrogroup theory allows the representation of the gyrogroup gyrooperator, gyr, in terms of its gyrogroup operation $\oplus$ in a gyrogroup $(G, \oplus)$,

$$\mathrm{gyr}[a, b]z = \ominus(a\oplus b)\oplus\{a\oplus(b\oplus z)\} \tag{15}$$

$a, b, z{\in}G$, in accordance with (13).

Since gyrogroup summation is nonassociative, we define general gyrogroup summations by successive left summations as, for instance,

$$\sum_{k=n}^{1} \oplus g_k = g_1\oplus(g_2\oplus \ldots \oplus(g_{n-2}\oplus(g_{n-1}\oplus g_n)) \ldots) \tag{16}$$

in a gyrogroup $(G, \oplus)$. One should note that the index k in (16) runs backwards from n to 1 and, accordingly, the summation runs from g_n to g_1.

Möbius addition is not restricted to the open complex unit disc. In the following definition we extend it to the open unit ball B of any real inner product space.

Definition 3 (The Möbius Addition). *Let $\mathbb{V}$ be a real inner product space, and let $\mathbb{B} = \{\mathbf{v}\in\mathbb{V} : \|\mathbf{v}\| < 1\}$ be the open unit ball of $\mathbb{V}$. Möbius addition $\oplus$ in the ball $\mathbb{B}$ is a binary operation in $\mathbb{B}$ given by the equation*

$$\mathbf{u}\oplus\mathbf{v} = \frac{(1 + 2\mathbf{u}\cdot\mathbf{v} + \|\mathbf{v}\|^2)\mathbf{u} + (1 - \|\mathbf{u}\|^2)\mathbf{v}}{1 + 2\mathbf{u}\cdot\mathbf{v} + \|\mathbf{u}\|^2\|\mathbf{v}\|^2} \tag{17}$$

where $\cdot$ and $\|\cdot\|$ are the inner product and norm that the ball $\mathbb{B}$ inherits from its space $\mathbb{V}$.

To justify calling $\oplus$ in Definition 3 a Möbius addition we note in (20) that $\oplus$ of (17) in two dimensions, when $\mathbb{V} = \mathbb{R}^2$ and $\mathbb{B} = \mathbb{B}^2$, is identifiable with $\oplus$ of (8). Identifying vectors in $\mathbb{R}^2$ with complex numbers in the usual way,

$$\mathbf{u} = (u_1, u_2) = u_1 + iu_2 = u \tag{18}$$

the inner product and the norm in $\mathbb{R}^2$ become the real numbers

$$\begin{aligned} \mathbf{u}\cdot\mathbf{v} &= Re(\bar{u}v) = \frac{\bar{u}v + u\bar{v}}{2} \\ \|\mathbf{u}\| &= |u| \end{aligned} \tag{19}$$

where $\bar{u}$ is the complex conjugate of u.

Under the translation (19) of elements of the open unit disc $\mathbb{B}^2$ of $\mathbb{R}^2$ to elements of the open complex unit disc $\mathbb{D}$, the Möbius addition (17) in the open unit disc $\mathbb{B}^2$ of $\mathbb{R}^2$ takes the form

$$\begin{aligned} \mathbf{u}\oplus\mathbf{v} &= \frac{(1 + 2\mathbf{u}\cdot\mathbf{v} + \|\mathbf{v}\|^2)\mathbf{u} + (1 - \|\mathbf{u}\|^2)\mathbf{v}}{1 + 2\mathbf{u}\cdot\mathbf{v} + \|\mathbf{u}\|^2\|\mathbf{v}\|^2} \\ &= \frac{(1 + \bar{u}v + u\bar{v} + |v|^2)u + (1 - |u|^2)v}{1 + \bar{u}v + u\bar{v} + |u|^2|v|^2} \\ &= \frac{(1 + u\bar{v})(u + v)}{(1 + \bar{u}v)(1 + u\bar{v})} \\ &= \frac{u + v}{1 + \bar{u}v} \\ &= u\oplus v \end{aligned} \tag{20}$$

for all $\mathbf{u}, \mathbf{v}\in\mathbb{B}^2$ and all $u, v\in\mathbb{D}$. In (20) we recover the Möbius addition $\oplus$ in the open unit disc $\mathbb{D}$ of $\mathbb{C}$, (8). Hence, the Möbius addition in Definition 3 is the natural extension of the Möbius addition (8) from the disc to the ball. More about the extension is found in [26, 27]. Furthermore, like Möbius addition in the complex disc, Möbius addition in the ball $\mathbb{B}$ of any real inner product space $\mathbb{V}$ is a gyrocommutative gyrogroup operation, giving rise to the Möbius ball gyrogroup $(\mathbb{B}, \oplus)$.

For parallel vectors in $\mathbb{B}$, $\mathbf{u}\|\mathbf{v}$, Möbius addition takes the simpler form

$$\mathbf{u}\oplus\mathbf{v} = \frac{\mathbf{u}+\mathbf{v}}{1+\mathbf{u}\cdot\mathbf{v}}, \qquad \mathbf{u}\|\mathbf{v} \tag{21}$$

which is also the Möbius addition in the open unit ball $\mathbb{B}^1 = (-1, 1)$ of the real line $\mathbb{R}$. Thus, for instance, for all $\mathbf{u}, \mathbf{v}\in\mathbb{B}$ we have

$$\|\mathbf{u}\|\oplus\|\mathbf{v}\| = \frac{\|\mathbf{u}\|+\|\mathbf{v}\|}{1+\|\mathbf{u}\|\|\mathbf{v}\|} \tag{22}$$

In this article we apply the Möbius addition only in three dimensions. We therefore note that Möbius addition in the ball $\mathbb{B}^3$ of $\mathbb{R}^3$ can be written in terms of the cross product as

$$\mathbf{u}\oplus\mathbf{v} = \frac{1+\mathbf{u}\cdot\mathbf{v}-(\mathbf{u}\times\mathbf{v})\times}{(1+\mathbf{u}\cdot\mathbf{v})^2+(\mathbf{u}\times\mathbf{v})^2}(\mathbf{u}+\mathbf{v}) \tag{23}$$

$\mathbf{u}, \mathbf{v}\in\mathbb{B}^3$. The three components of $\mathbf{u}$ and $\mathbf{v}$ are

$$\begin{aligned} \mathbf{u} &= (u_1, u_2, u_3) \\ \mathbf{v} &= (v_1, v_2, v_3) \end{aligned} \tag{24}$$

and, accordingly, the three components of $\mathbf{u}\times\mathbf{v}$ are

$$\begin{aligned} \mathbf{u}\times\mathbf{v} &= ((\mathbf{u}\times\mathbf{v})_1, (\mathbf{u}\times\mathbf{v})_2, (\mathbf{u}\times\mathbf{v})_3) \\ &= (u_2v_3 - u_3v_2, -u_1v_3 + u_3v_1, u_1v_2 - u_2v_1) \end{aligned} \tag{25}$$

According to (15), the gyrooperation gyr of the Möbius gyrogroup $(\mathbb{B}^3, \oplus)$ is given by the equation

$$\mathrm{gyr}[\mathbf{u}, \mathbf{v}]\mathbf{z} = \ominus(\mathbf{u}\oplus\mathbf{v})\oplus\{\mathbf{u}\oplus(\mathbf{v}\oplus\mathbf{z})\} \tag{26}$$

where $\mathrm{gyr}[\mathbf{u}, \mathbf{v}]$ is an automorphism of $\mathbb{B}^3$, called the gyroautomorphism generated by $\mathbf{u}, \mathbf{v}\in\mathbb{B}^3$.

As shown in [26], the Möbius gyroautomorphism (26) can be written in terms of the cross product as

$$\mathrm{gyr}[\mathbf{u}, \mathbf{v}] = I + 2\frac{-(1+\mathbf{u}\cdot\mathbf{v})+(\mathbf{u}\times\mathbf{v})\times}{(1+\mathbf{u}\cdot\mathbf{v})^2+(\mathbf{u}\times\mathbf{v})^2}(\mathbf{u}\times\mathbf{v})\times \tag{27}$$

$\mathbf{u}, \mathbf{v}\in\mathbb{B}^3$, I being the identity map of $\mathbb{B}^3$.

3 MÖBIUS ADDITION AND HYPERBOLIC GEOMETRY

To reveal the connection between Möbius addition and hyperbolic geometry we present the scalar multiplication that it admits, turning it into the Möbius gyrovector space. We will find that Möbius gyrovector spaces form the setting for the Poincaré ball model of the hyperbolic geometry of Bolyai and Lobachevski just as vector spaces form the setting for the standard model of Euclidean geometry.

Definition 4 (Möbius Scalar Multiplication). *Let* $(\mathbb{B},\oplus)$ *be a Möbius gyrogroup. The Möbius scalar multiplication* $r\otimes\mathbf{v} = \mathbf{v}\otimes r$ *in* $\mathbb{B}$ *is given by the equation*

$$\begin{aligned} r\otimes\mathbf{v} &= \frac{(1+\|\mathbf{v}\|)^r - (1-\|\mathbf{v}\|)^r}{(1+\|\mathbf{v}\|)^r + (1-\|\mathbf{v}\|)^r}\frac{\mathbf{v}}{\|\mathbf{v}\|} \\ &= \tanh(r\,\tanh^{-1}\|\mathbf{v}\|)\frac{\mathbf{v}}{\|\mathbf{v}\|} \end{aligned} \tag{28}$$

where $r\in\mathbb{R}$*,* $\mathbf{v}\in\mathbb{B}$*,* $\mathbf{v}\neq\mathbf{0}$*; and* $r\otimes\mathbf{0}=\mathbf{0}$.

As an example we present the Möbius half,

$$\tfrac{1}{2}\otimes\mathbf{v} = \frac{\gamma_{\mathbf{v}}}{1+\gamma_{\mathbf{v}}}\mathbf{v} \tag{29}$$

so that, accordingly,

$$\begin{aligned} 2\otimes(\tfrac{1}{2}\otimes\mathbf{v}) &= 2\otimes\frac{\gamma_{\mathbf{v}}}{1+\gamma_{\mathbf{v}}}\mathbf{v} \\ &= \frac{\gamma_{\mathbf{v}}}{1+\gamma_{\mathbf{v}}}\mathbf{v}\oplus\frac{\gamma_{\mathbf{v}}}{1+\gamma_{\mathbf{v}}}\mathbf{v} \\ &= \mathbf{v} \end{aligned} \tag{30}$$

The Möbius addition $\oplus$ and its associated scalar multiplication $\otimes$ possess the following properties. For any positive integer n and for all $r, r_1, r_2\in\mathbb{R}$ and $\mathbf{a},\mathbf{b},\mathbf{u},\mathbf{v}\in\mathbb{B}$,

(V1)	$n\otimes\mathbf{v} = \mathbf{v}\oplus\,\ldots\,\oplus\mathbf{v}$	(n terms)
(V2)	$(r_1+r_2)\otimes\mathbf{v} = r_1\otimes\mathbf{v}\oplus r_2\otimes\mathbf{v}$	Scalar Distributive Law
(V3)	$(r_1 r_2)\otimes\mathbf{v} = r_1\otimes(r_2\otimes\mathbf{v})$	Scalar Associative Law
(V4)	$r\otimes(r_1\otimes\mathbf{v}\oplus r_2\otimes\mathbf{v}) = r\otimes(r_1\otimes\mathbf{v})\oplus r\otimes(r_2\otimes\mathbf{v})$	Monodistributive Law
(V5)	$\dfrac{\lvert r\rvert\otimes\mathbf{v}}{\|r\otimes\mathbf{v}\|} = \dfrac{\mathbf{v}}{\|\mathbf{v}\|}$	Scaling Property

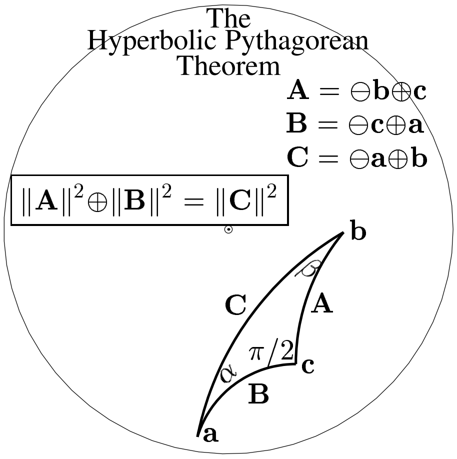

Figure 1. The Hyperbolic Pythagorean Theorem in the Poincaré ball model of hyperbolic geometry and, equivalently, in the Möbius gyrovector space $(\mathbb{B}, \oplus, \otimes)$ [4, 28, 29].

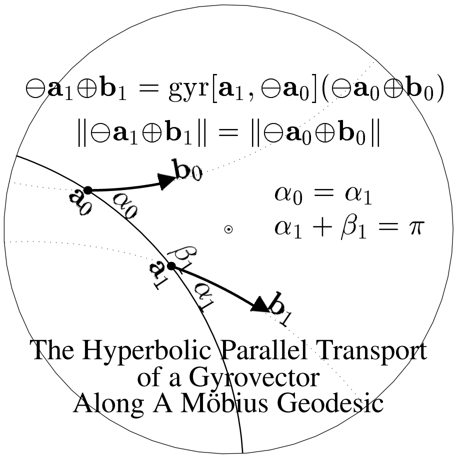

Figure 2. The Hyperbolic Parallel Transport in the Poincaré ball model of hyperbolic geometry and, equivalently, in the Möbius gyrovector space $(\mathbb{B}, \oplus, \otimes)$ [4].

(V6)	$\|r\otimes\mathbf{v}\| = \|r\|\otimes\|\mathbf{v}\|$	Homogeneity Property
(V7)	$\|\mathbf{u}\oplus\mathbf{v}\| \le \|\mathbf{u}\|\oplus\|\mathbf{v}\|$	Gyrotriangle inequality
(V8)	$\text{gyr}[\mathbf{a}, \mathbf{b}](r\otimes\mathbf{v}) = r\otimes\text{gyr}[\mathbf{a}, \mathbf{b}]\mathbf{v}$	Gyroautomorphism

We may note that, ambiguously, $\oplus$ in $(V7)$ is a binary operation in $\mathbb{B}$, (17), and in $\mathbb{B}^1$, (22). In view of $(V2)$, the condition in $(V1)$ can be relaxed to $1\otimes\mathbf{v} = \mathbf{v}$. A Möbius gyrogroup $(\mathbb{B}, \oplus)$ with scalar multiplication $\otimes$ is called a Möbius gyrovector space, $(\mathbb{B}, \oplus, \otimes)$.

A Möbius gyrovector space is a metric space with metric given by the distance function

$$d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u}\ominus\mathbf{v}\| \tag{31}$$

called the Poincaré – Möbius metric of the ball [4, p. 177] since, in the complex disc $\mathbb{D}$, it is known as the Poincaré metric [4, p. 170].

The infinitesimal version of the Poincaré – Möbius metric (31) turns out to be the Riemannian line element of the Poincaré ball model of hyperbolic geometry, as explained in [4, Sec. 6.2].

The unique geodesic in a Möbius gyrovector space $(\mathbb{B}, \oplus, \otimes)$ that passes through the points $\mathbf{u}, \mathbf{v} \in \mathbb{B}$ is represented by the parametric equation with parameter $t\in\mathbb{R}$,

$$\mathbf{u}\oplus(\ominus\mathbf{u}\oplus\mathbf{v})\otimes t \tag{32}$$

in full analogy with its Euclidean counterpart. We therefore call it a Möbius gyroline. Several Möbius gyrolines are shown in Figs. 1 and 2. The Möbius gyroline (32) passes through the point $\mathbf{u}$ at "time" $t = 0$ and through the point $\mathbf{v}$ at "time" $t = 1$. Möbius gyroline segments form, for instance, triangles in the Poincaré ball model of hyperbolic geometry. Hyperbolic angles between gyrolines are defined in [4] in a way fully analogous to their Euclidean definition. The analogies between hyperbolic geometry and Euclidean geometry that the gyroformalism capture allow the two geometries to be unified. Thus, for instance, following the advent of the theory of gyrogroups and gyrovector spaces in [4]:

(1) A novel hyperbolic Pythagorean theorem that, unlike the customary one, shares visual analogies with the Euclidean Pythagorean theorem emerges, Fig. 1; and

(2) The algebra of the hyperbolic parallel transport, that shares analogies with its Euclidean parallel transport, is uncovered, Fig. 2.

It is clear from Figs. 1 and 2 that gyrovectors are hyperbolic vectors since they share remarkable analogies with vectors in Euclidean geometry. The echo of the intrinsic beauty, elegance, and power of the theory of gyrogroups and gyrovector spaces, their gyro-formalism, and their ability to capture analogies with classical results [4] is reflected in Figs. 1 and 2. The usefulness of bringing the density matrix for mixed state qubits under the umbrella of gyrovector spaces, where the Bloch vector is treated as a gyrovector [9], is thus evident.

4 THE DENSITY MATRIX FOR MIXED STATE QUBITS

All the postulates of quantum mechanics can be reformulated in terms of the density matrix language [1, p. 99]. The generic density matrix for a mixed state qubit turns out to be the 2×2 *qubit density matrix* $\rho(\mathbf{v})$ [1, p. 105],

$$\rho(\mathbf{v}) = \tfrac{1}{2}(\mathbf{1} + \boldsymbol{\sigma}\cdot\mathbf{v}) = \tfrac{1}{2}\begin{pmatrix} 1+v_3 & v_1 - iv_2 \\ v_1 + iv_2 & 1 - v_3 \end{pmatrix} \tag{33}$$

$\mathbf{v} = (v_1, v_2, v_3)\in\mathbb{B}^3$, where $\mathbf{1}$ is the unit matrix and $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ are the Pauli matrices in vector notation [10, 30, 31]. The qubit density matrix (33) obeys the constraints of unit trace $(tr(\rho(\mathbf{v})) = 1)$ and positivity $(det(\rho(\mathbf{v})) > 0)$; see, for instance, [32].

The *qubit* is, thus, a two state quantum system completely described by the qubit density matrix $\rho(\mathbf{v})$, parametrized by the vector $\mathbf{v}\in\mathbb{B}^3$. The vector $\mathbf{v} = (v_1, v_2, v_3)\in\mathbb{B}^3$ that parametrizes the qubit density matrix (33) is known as the *Bloch vector* in the ball $\mathbb{B}^3$. Accordingly, the open unit ball $\mathbb{B}^3$ of the Euclidean 3-space $\mathbb{R}^3$ is called the Bloch ball.

We will find in this section that the Bloch vector behaves like a hyperbolic vector, described in Figs. 1 and 2, rather than a Euclidean one. Hence, we rename the Bloch vector in Section 5 and in [9], calling it the *Bloch gyrovector*.

Remarkable identities linking the qubit density matrices $\rho(\mathbf{v})$ to the Möbius gyrovector space $(\mathbb{B}^3, \oplus, \otimes)$ of their Bloch vector parameter $\mathbf{v} \in \mathbb{B}^3$ are presented. As such, it is found in Section 5 that the Bloch vector deserves the title "gyrovector" rather than "vector". It is suggested that on first reading readers may skip the remainder of this section and continue reading Section 5.

Let $\rho(\mathbf{u})$ and $\rho(\mathbf{v})$ be two qubit density matrices generated by the two Bloch vectors $\mathbf{u}, \mathbf{v} \in \mathbb{B}^3$, and let $\rho(\mathbf{u}\oplus\mathbf{v})$ be the qubit density matrix generated by the Möbius sum $\mathbf{u}\oplus\mathbf{v}$ of $\mathbf{u}$ and $\mathbf{v}$ (23). Furthermore, let $R(\mathbf{u}, \mathbf{v})$ be the 2×2 matrix given by the equation

$$R(\mathbf{u}, \mathbf{v}) = \rho^{-1}(\mathbf{u}\oplus\mathbf{v})\rho(\mathbf{u})\rho(\mathbf{v}) \tag{34}$$

where we use the notation $\rho^{-1}(\mathbf{v}) = (\rho(\mathbf{v}))^{-1}$ and, more generally, $\rho^r(\mathbf{v}) = (\rho(\mathbf{v}))^r$, $r \in \mathbb{R}$. The matrix $R(\mathbf{u}, \mathbf{v})$ has elegant form,

$$R(\mathbf{u}, \mathbf{v}) = \frac{1}{2} \begin{pmatrix} 1 + \mathbf{u}\cdot\mathbf{v} + i(\mathbf{u}\times\mathbf{v})_3 & (\mathbf{u}\times\mathbf{v})_2 + i(\mathbf{u}\times\mathbf{v})_1 \\ -(\mathbf{u}\times\mathbf{v})_2 + i(\mathbf{u}\times\mathbf{v})_1 & 1 + \mathbf{u}\cdot\mathbf{v} - i(\mathbf{u}\times\mathbf{v})_3 \end{pmatrix} \tag{35}$$

as well as elegant determinant,

$$4det(R(\mathbf{u}, \mathbf{v})) = (1 + \mathbf{u}\cdot\mathbf{v})^2 + (\mathbf{u}\times\mathbf{v})^2 \tag{36}$$

It turns out to have the form

$$R(\mathbf{u}, \mathbf{v}) = \begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix} \in PSU(2) \tag{37}$$

for some complex numbers a and b, with a non-zero determinant.

The matrix $R(\mathbf{u}, \mathbf{v})$ is a positively scaled unitary unimodular matrix in the sense that it can be written as a positive number, $\sqrt{|a|^2 + |b|^2}$, times a unitary unimodular matrix. It is, thus, an element of the group $PSU(2) = (0, \infty) \times SU(2)$ of all 2×2 positively scaled unitary unimodular matrices, a group isomorphic to the multiplicative group of nonzero quaternions [33]. Moreover, it enjoys the following two remarkable features,

$$tr(R(\mathbf{u}, \mathbf{v})) = 2tr(\rho(\mathbf{u})\rho(\mathbf{v})) \tag{38}$$

and

$$R(\mathbf{u}, \mathbf{v}) = \bar{R}^t(\mathbf{v}, \mathbf{u}) \tag{39}$$

for all $\mathbf{u}, \mathbf{v} \in \mathbb{B}^3$, $\bar{R}^t$ being the transpose, complex conjugate of R.

The special unitary matrix $R_s(\mathbf{u},\mathbf{v}) \in SU(2),\ \mathbf{u},\mathbf{v} \in \mathbb{B}^3$, associated with $R(\mathbf{u},\mathbf{v})$ is

$$\begin{aligned} R_s(\mathbf{u},\mathbf{v}) &= \frac{R(\mathbf{u},\mathbf{v})}{\sqrt{det(R(\mathbf{u},\mathbf{v}))}} \\ &= \frac{\begin{pmatrix} 1+\mathbf{u}\cdot\mathbf{v}+i(\mathbf{u}\times\mathbf{v})_3 & (\mathbf{u}\times\mathbf{v})_2+i(\mathbf{u}\times\mathbf{v})_1 \\ -(\mathbf{u}\times\mathbf{v})_2+i(\mathbf{u}\times\mathbf{v})_1 & 1+\mathbf{u}\cdot\mathbf{v}-i(\mathbf{u}\times\mathbf{v})_3 \end{pmatrix}}{\sqrt{(1+\mathbf{u}\cdot\mathbf{v})^2+(\mathbf{u}\times\mathbf{v})^2}} \end{aligned} \tag{40}$$

It satisfies the functional equations

$$R_s(\mathbf{u},\mathbf{v}) = R_s(-\mathbf{u},-\mathbf{v}) \tag{41}$$

$$R_s^{-1}(\mathbf{u},\mathbf{v}) = R_s(\mathbf{v},\mathbf{u}) \tag{42}$$

and

$$R_s(\mathbf{u}\oplus\mathbf{v},\mathbf{v}) = R_s(\mathbf{u},\mathbf{v}) \tag{43}$$

$$R_s(\mathbf{u},\mathbf{v}\oplus\mathbf{u}) = R_s(\mathbf{u},\mathbf{v}) \tag{44}$$

for all $\mathbf{u},\mathbf{v}\in\mathbb{B}^3$, thus exhibiting strong link with Möbius addition. Furthermore, if $\mathbf{u}$ and $\mathbf{v}$ are parallel, $\mathbf{u}\|\mathbf{v}$, then

$$R_s(\mathbf{u},\mathbf{v}) = I, \qquad \mathbf{u}\|\mathbf{v} \tag{45}$$

I being the identity 2×2 matrix.

The denominator in (40),

$$F(\mathbf{u},\mathbf{v}) = \sqrt{(1+\mathbf{u}\cdot\mathbf{v})^2+(\mathbf{u}\times\mathbf{v})^2} > 0 \tag{46}$$

$\mathbf{u},\mathbf{v}\in\mathbb{B}^3$, the square of which appears in the denominator of Möbius addition law (23), can be written as

$$F(\mathbf{u},\mathbf{v}) = \frac{\gamma_{\mathbf{u}\oplus\mathbf{v}}}{\gamma_{\mathbf{u}}\gamma_{\mathbf{v}}} \tag{47}$$

where $\gamma_{\mathbf{v}}$ is the Lorentz gamma factor of special relativity, (3), as we see from (2), (34) and (36).

Furthermore, it satisfies the following cocycle functional equation and normalization conditions [4, p. 289]

$$\begin{gathered} F(\mathbf{u},\mathbf{v}\oplus\mathbf{w})F(\mathbf{v},\mathbf{w}) = F(\mathbf{v}\oplus\mathbf{u},\mathbf{w})F(\mathbf{u},\mathbf{v}) \\ F(\mathbf{u},\mathbf{0}) = F(\mathbf{0},\mathbf{v}) = 1 \end{gathered} \tag{48}$$

Coincidentally, the cocycle equation (48) with an ordinary addition, +, rather than Möbius addition, $\oplus$, arises in several branches of mathematics, as pointed out by B. R. Ebanks and C. T. Ng [34]. It is thus interesting to realize that with Möbius addition $\oplus$ replacing the ordinary vector addition, the well known cocycle equation appears in the study of gyrogroups [35], in general, and in the present study of the qubit density matrix, in particular.

The matrices $R(\mathbf{u},\mathbf{v})$, $\mathbf{u},\mathbf{v}\in\mathbb{B}^3$, play a role in the definition of the gyrooperation gyr of the gyrocommutative gyrogroup of qubit density matrices.

We define the gyration $\mathrm{gyr}[\rho(\mathbf{u}),\rho(\mathbf{v})]$ generated by the qubit density matrices $\rho(\mathbf{u})$ and $\rho(\mathbf{v})$ in terms of its effects on a qubit density matrix $\rho(\mathbf{w})$ by the equation

$$\mathrm{gyr}[\rho(\mathbf{u}),\rho(\mathbf{v})]\rho(\mathbf{w}) = R(\mathbf{u},\mathbf{v})\rho(\mathbf{w})R^{-1}(\mathbf{u},\mathbf{v}) \tag{49}$$

$\mathbf{u},\mathbf{v},\mathbf{w}\in\mathbb{B}^3$, where we use the notation $R^{-1}(\mathbf{u},\mathbf{v}) = (R(\mathbf{u},\mathbf{v}))^{-1}$.

The right hand side of (49) is interesting. It is in a form that describes the evolution of a closed quantum system by the unitary operator $R_s(\mathbf{u},\mathbf{v})$, (40), according to [1, p. 99].

Also the left hand side of (49) is interesting. It satisfies the identity

$$\mathrm{gyr}[\rho(\mathbf{u}),\rho(\mathbf{v})]\rho(\mathbf{w}) = \rho(\mathrm{gyr}[\mathbf{u},\mathbf{v}]\mathbf{w}) \tag{50}$$

for all $\mathbf{u},\mathbf{v},\mathbf{w}\in\mathbb{B}^3$. Interested readers may verify that (50) follows from (49) by computer algebra.

Finally, we define a binary operation $\odot$ on the set

$$D = \{\rho(\mathbf{v}) :\ \mathbf{v}\in\mathbb{B}^3\} \tag{51}$$

of all mixed state qubit density matrices by the equation

$$\rho(\mathbf{u})\odot\rho(\mathbf{v}) = \rho(\mathbf{u}\oplus\mathbf{v}) \tag{52}$$

for all $\mathbf{u},\mathbf{v}\in\mathbb{B}^3$. Identity (50) relates the gyrooperation of $(D,\odot)$, generated by $\rho(\mathbf{u}),\rho(\mathbf{v})\in D$, (49), to the gyrooperator of the parameter space $(\mathbb{B}^3,\oplus)$ of D, generated by $\mathbf{u},\mathbf{v}\in\mathbb{B}^3$, (15).

The groupoid $(D,\odot)$ involves the Bloch vector parameter space $\mathbb{B}^3$, and its composition law is given in (52) by parameter composition in $\mathbb{B}^3$. The parameter composition in $\mathbb{B}^3 = (\mathbb{B}^3,\oplus)$, in turn, is given by Möbius addition $\oplus$. The groupoid $(D,\odot)$ turns out to be a gyrocommutative gyrogroup, which is an isomorphic copy of the the Möbius gyrogroup $(\mathbb{B}^3,\oplus)$. The identity element of the gyrocommutative gyrogroup D is $\rho(\mathbf{0})$, where $\mathbf{0}\in\mathbb{B}^3$ is the zero vector of the Euclidean 3-space $\mathbb{R}^3$. The gyrogroup inverse $\rho^{\ominus 1}(\mathbf{v})$ of $\rho(\mathbf{v})\in(D,\odot)$ is $\rho^{\ominus 1}(\mathbf{v}) = \rho(\ominus\mathbf{v})$ where $\ominus\mathbf{v} = -\mathbf{v}$ in $\mathbb{B}^3$. Indeed, by (52) we have,

$$\begin{aligned}\rho^{\ominus 1}(\mathbf{v})\odot\rho(\mathbf{v}) &= \rho(\ominus\mathbf{v})\odot\rho(\mathbf{v})\\ &= \rho(\ominus\mathbf{v}\oplus\mathbf{v})\\ &= \rho(\mathbf{0})\end{aligned} \tag{53}$$

Hence, for instance, the gyration effects in (49) can also be written, according to (26) and (53), as

$$\mathrm{gyr}[\rho(\mathbf{u}),\rho(\mathbf{v})]\rho(\mathbf{w}) = \rho(\ominus\mathbf{u}\ominus\mathbf{v})\odot\{\rho(\mathbf{u})\odot(\rho(\mathbf{v})\odot\rho(\mathbf{w}))\} \tag{54}$$

as interested readers may verify by computer algebra.

The relationship between the generic qubit density matrix $\rho_{\mathbf{v}} \in D$ and its Bloch gyrovector parameter $\mathbf{v}\in\mathbb{B}^3$ is described by the following commutative diagram.

$$\begin{array}{ccccccc}
\mathbf{v} & \xrightarrow{\mathbf{u}\oplus} & \mathbf{u}\oplus\mathbf{v} & \xrightarrow{\mathrm{gyr}[\mathbf{v},\mathbf{u}]} & \mathbf{v}\oplus\mathbf{u} & \xrightarrow{\mathbf{w}\oplus} & \\
\Big\downarrow{\scriptstyle\rho} & & \Big\downarrow{\scriptstyle\rho} & & \Big\downarrow{\scriptstyle\rho} & & \\
\rho_{\mathbf{v}} & \xrightarrow{\rho_{\mathbf{u}}\odot} & \rho_{\mathbf{u}}\odot\rho_{\mathbf{v}} & \xrightarrow{\mathrm{gyr}[\rho_{\mathbf{v}},\rho_{\mathbf{u}}]} & \rho_{\mathbf{v}}\odot\rho_{\mathbf{u}} & \xrightarrow{\rho_{\mathbf{w}}\odot} &
\end{array}$$

$$\begin{array}{ccc}
\mathbf{w}\oplus(\mathbf{v}\oplus\mathbf{u}) & \xrightarrow{=} & (\mathbf{w}\oplus\mathbf{v})\oplus\mathrm{gyr}[\mathbf{w},\mathbf{v}]\mathbf{u} \\
\Big\downarrow{\scriptstyle\rho} & & \Big\downarrow{\scriptstyle\rho} \\
\rho_{\mathbf{w}}\odot(\rho_{\mathbf{v}}\odot\rho_{\mathbf{u}}) & \xrightarrow{=} & (\rho_{\mathbf{w}}\odot\rho_{\mathbf{v}})\odot\mathrm{gyr}[\rho_{\mathbf{w}},\rho_{\mathbf{v}}]\rho_{\mathbf{u}}
\end{array} \tag{55}$$

Relationships for qubit density matrices with Einstein addition, rather than Möbius addition, are presented in [8], and will be encountered again in (96).

Density matrices for pure state qubits are given by (33) with a Bloch vector $\mathbf{v}\in\partial\mathbb{B}^3$ on the boundary of the Bloch ball $\mathbb{B}^3$, that is, $\|\mathbf{v}\| = 1$ [1, p. 100]. The qubit composition law (52) for $\mathbf{u},\mathbf{v}\in\mathbb{B}^3$ remains valid on the closure $\bar{\mathbb{B}}^3$ of $\mathbb{B}^3$ with one exception. The composition $\rho(\mathbf{v})\odot\rho(-\mathbf{v})$ is undefined for all $\mathbf{v}\in\partial\mathbb{B}^3$.

5 THE BLOCH GYROVECTOR

Identity (34) can be presented as the polar decomposition,

$$\rho(\mathbf{u})\rho(\mathbf{v}) = \rho(\mathbf{u}\oplus\mathbf{v})R(\mathbf{u},\mathbf{v}) = F(\mathbf{u},\mathbf{v})\rho(\mathbf{u}\oplus\mathbf{v})R_s(\mathbf{u},\mathbf{v}) \tag{56}$$

demonstrating that, in general, the matrix product of two qubit density matrices is not equivalent to a qubit density matrix but, rather, to a qubit density matrix preceded, or followed, by a positively scaled unitary unimodular matrix. Indeed, Identity (56), in which the positively scaled unitary unimodular matrix $R(\mathbf{u},\mathbf{v})\in PSU(2)$ precedes the qubit density matrix $\rho(\mathbf{u}\oplus\mathbf{v})$, is associated with the polar decomposition

$$\rho(\mathbf{u})\rho(\mathbf{v}) = R(\mathbf{u},\mathbf{v})\rho(\mathbf{v}\oplus\mathbf{u}) = F(\mathbf{u},\mathbf{v})R_s(\mathbf{u},\mathbf{v})\rho(\mathbf{v}\oplus\mathbf{u}) \tag{57}$$

in which the positively scaled unitary unimodular matrix $R(\mathbf{u},\mathbf{v})\in PSU(2)$ follows the qubit density matrix $\rho(\mathbf{v}\oplus\mathbf{u})$.

The polar decomposition (57) generalizes the one in (7). We may note that the qubit density matrix on the right hand side of (56) is parametrized by the Möbius sum $\mathbf{u}\oplus\mathbf{v}$ of Bloch gyrovectors, while the qubit density matrix on the right hand side of (57) is parametrized by the Möbius sum $\mathbf{v}\oplus\mathbf{u}$ of Bloch gyrovectors. Each of the right hand sides of (56) and (57) presents a so called *polar decomposition group*. The use of polar decomposition groups to obtain gyrocommutative gyrogroups is uncovered in [36], where it is called the *exact decomposition*. With the advent of gyrolanguage (see fn. 36 in [37]), it became the gyrosemidirect product group [38], in full analogy with the customary semidirect product group. Thus, for instance,

(1) The Lorentz group is the gyrosemidirect product group of the gyrogroup of Einsteinian velocities (under Einstein addition) and a group of rotations, fully analogous to

(2) The Galilei group, which is the semidirect product group of the group of Newtonian velocities (under the common vector addition) and a group of rotations [4].

The use of a polar decomposition to uncover a gyrocommutative gyrogroup can be found, for instance, in [39].

In general, the matrix product of several qubit density matrices is not equivalent to a single qubit density matrix, as we see from Identities (56) and (57). However, there are special cases in which the matrix product of several qubit density matrices is equivalent to a single qubit density matrix with a positive coefficient. Each of these cases corresponds to a *symmetric matrix product of qubit density matrices* that we define below. These cases clearly exhibit the nature of the Bloch vector as a gyrovector in the Möbius gyrovector space $(\mathbb{B}^3,\oplus,\otimes)$ rather than a vector in the open unit ball $\mathbb{B}^3$ of the Euclidean 3-space $\mathbb{R}^3$.

Definition 5. *Let*

$$\prod_{k=1}^{n}\rho^{r_k}(\mathbf{v}_k)=\rho^{r_1}(\mathbf{v}_1)\rho^{r_2}(\mathbf{v}_2)\dots\rho^{r_{n-1}}(\mathbf{v}_{n-1})\rho^{r_n}(\mathbf{v}_n) \tag{58}$$

be the matrix product of n *qubit density matrices parametrized by the* n *Bloch gyrovectors* $\mathbf{v}_1,\mathbf{v}_2,\dots,\mathbf{v}_{n-1},\mathbf{v}_n\in\mathbb{B}^3$*, and raised to the respective powers* $r_1,r_2,\dots,r_n\in\mathbb{R}$*.*

A reversion of the matrix product (58) *is the matrix product*

$$\prod_{k=n}^{1}\rho^{r_k}(\mathbf{v}_k)=\rho^{r_n}(\mathbf{v}_n)\rho^{r_{n-1}}(\mathbf{v}_{n-1})\dots\rho^{r_2}(\mathbf{v}_2)\rho^{r_1}(\mathbf{v}_1) \tag{59}$$

A matrix product, ρ_s, of qubit density matrices is symmetric if it identically equal to its own reversion,

$$\rho_s = \prod_{k=1}^{n} \rho^{r_k}(\mathbf{v}_k) = \prod_{k=n}^{1} \rho^{r_k}(\mathbf{v}_k) \tag{60}$$

Let $\rho_s = \prod_{k=1}^{n} \rho^{r_k}(\mathbf{w}_k)$ be a symmetric matrix product of qubit density matrices. Its Bloch gyrovector $\mathbf{w}$ (or, equivalently, the Bloch gyrovector $\mathbf{w}$ that it possesses) is given by the equation, (16),

$$\mathbf{w} = \sum_{k=n}^{1} \oplus (r_k \otimes \mathbf{w}_k) \tag{61}$$

in the Möbius gyrovector space $(\mathbb{B}^3, \oplus, \otimes)$.

Examples illustrating Definition 5 follow. The reversion of

$$\rho_1 = \rho(\mathbf{u})\rho(\mathbf{v})\rho(\mathbf{w}) \tag{62}$$

is

$$\rho_2 = \rho(\mathbf{w})\rho(\mathbf{v})\rho(\mathbf{u}) \tag{63}$$

The matrix product ρ_1 is, in general, different from its reversion ρ_2 so that it is not symmetric. The matrix product

$$\rho_3 = \rho(\mathbf{u})\rho(\mathbf{v})\rho(\mathbf{u}) \tag{64}$$

is symmetric, possessing the Bloch gyrovector

$$\mathbf{w} = \mathbf{u} \oplus (\mathbf{v} \oplus \mathbf{u}) \tag{65}$$

Similarly, the matrix product

$$\rho_3 = \rho(\mathbf{u})\rho(\mathbf{v})\rho(\mathbf{v})\rho(\mathbf{u}) \tag{66}$$

is symmetric, possessing the Bloch gyrovector

$$\mathbf{w} = \mathbf{u} \oplus (2 \otimes \mathbf{v} \oplus \mathbf{u}) = 2 \otimes (\mathbf{u} \oplus \mathbf{v}) \tag{67}$$

calculated by the nonassociative algebra of gyrogroups [4, Theorem 3.13].

Theorem 6. *Let ρ_s be a symmetric matrix product of qubit density matrices, possessing the Bloch gyrovector $\mathbf{w} \in \mathbb{B}^3$. Then*

$$\rho_s = tr(\rho_s)\rho(\mathbf{w}) \tag{68}$$

The proof of Theorem 6, which is beyond the scope of this article, is derived from results in gyrogroup theory presented in [4], like Theorem 2.35 in [4], details of which will be presented elsewhere.

Theorem 6 states that up to a positive coefficient, $tr(\rho_s)$, a symmetric matrix product of qubit density matrices, ρ_s, is equivalent to a single qubit density matrix $\rho(\mathbf{w})$ parametrized by the Bloch gyrovector $\mathbf{w}$ that it possesses. The latter, in turn, is a gyrovector in the Möbius gyrovector space $(\mathbb{B}^3, \oplus, \otimes)$ generated by the operations $\oplus$ and $\otimes$ in $\mathbb{B}^3$, as (61) indicates.

When convenient, we may use the standard notation

$$\rho_{\mathbf{v}} = \rho(\mathbf{v}) \tag{69}$$

for the 2×2 *density matrix* $\rho(\mathbf{v})$ in (33).

Illustrative examples of results that follow from Theorem 6 are now presented.

Example 1: Let

$$\rho_s = \rho_{\mathbf{v}}^n \tag{70}$$

$n\in\mathbb{R}, n \neq \mathbf{0}$. By Definition 5, ρ_s is symmetric, possessing the Bloch gyrovector, (61),

$$\mathbf{w} = \overbrace{\mathbf{v} \oplus \cdots \oplus \mathbf{v}}^{n \text{ terms}} = n\otimes\mathbf{v} \tag{71}$$

Hence, by Theorem 6,

$$\boxed{\rho_{\mathbf{v}}^n = tr(\rho_{\mathbf{v}}^n)\rho_{n\otimes\mathbf{v}}} \tag{72}$$

for all $\mathbf{v}\in\mathbb{B}^3$ and $n\in\mathbb{R}$. Identity (72) is remarkable, expressing any real power of a qubit density matrix as a qubit density matrix with a positive coefficient.

Let $\mathbf{u} = n\otimes\mathbf{v}$ so that $\mathbf{v} = (1/n)\otimes\mathbf{u}$ in $\mathbb{B}^3$. Then, by (72),

$$\rho_{\mathbf{u}} = \frac{\rho_{(1/n)\otimes\mathbf{u}}^n}{tr(\rho_{(1/n)\otimes\mathbf{u}}^n)} \tag{73}$$

for all $n\in\mathbb{R}$, $n \neq 0$. Hence, for $m = 1/n$ (73) can be written as

$$\rho_{\mathbf{u}} = \frac{\rho_{m\otimes\mathbf{u}}^{1/m}}{tr(\rho_{m\otimes\mathbf{u}}^{1/m})} \tag{74}$$

for all $\mathbf{u}\in\mathbb{B}^3$ and $m\in\mathbb{R}$, $m \neq 0$.

Renaming m and $\mathbf{u}$ as n and $\mathbf{v}$, (74) can be written as

$$\rho_{\mathbf{v}}^n = \frac{\rho_{n\otimes\mathbf{v}}}{(tr(\rho_{n\otimes\mathbf{v}}^{1/n}))^n} \tag{75}$$

Finally, comparing (72) and (75) we have,

$$tr(\rho_{\mathbf{v}}^{n})(tr(\rho_{n\otimes\mathbf{v}}^{1/n}))^{n} = 1 \tag{76}$$

for all $\mathbf{v}\in\mathbb{B}^3$ and $n\in\mathbb{R}$, $n \neq 0$.

Identity (72) of Example 1 can be used to determine by inspection the matrix $\rho_{\mathbf{v}}^{r}$ and its trace $tr(\rho_{\mathbf{v}}^{r})$ for all $r\in\mathbb{R}$. Let

$$\begin{aligned} A &= 1 + \|\mathbf{v}\| \\ B &= 1 - \|\mathbf{v}\| \end{aligned} \tag{77}$$

$\mathbf{v}\in\mathbb{B}^3$. Then, it follows from (33) and (28) that

$$\rho_{r\odot\mathbf{v}} = \frac{1}{A^r + B^r}\frac{1}{2\|\mathbf{v}\|}M \tag{78}$$

where M is the matrix

$$M = \begin{pmatrix} (A^r + B^r)\|\mathbf{v}\| + (A^r - B^r)v_3 & (A^r - B^r)(v_1 - iv_2) \\ (A^r - B^r)(v_1 + iv_2) & (A^r + B^r)\|\mathbf{v}\| - (A^r - B^r)v_3 \end{pmatrix} \tag{79}$$

$\mathbf{v} \neq \mathbf{0}$, so that from (72) and by inspection,

$$tr(\rho_{\mathbf{v}}^{r}) = \frac{A^r + B^r}{2^r} \tag{80}$$

and

$$\begin{aligned} \rho_{\mathbf{v}}^{r} &= tr(\rho_{\mathbf{v}}^{r})\rho_{r\otimes\mathbf{v}} \\ &= \frac{A^r + B^r}{2^r}\frac{1}{A^r + B^r}\frac{1}{2\|\mathbf{v}\|}M \\ &= \frac{1}{2^{r+1}\|\mathbf{v}\|}M \end{aligned} \tag{81}$$

Finally, (72) can be extended to a power of any real number, obtaining

$$\begin{aligned} \rho_{\mathbf{v}}^{r} &= tr(\rho_{\mathbf{v}}^{r})\rho_{r\otimes\mathbf{v}} \\ &= \frac{A^r + B^r}{2^r}\rho_{r\otimes\mathbf{v}} \end{aligned} \tag{82}$$

for all $\mathbf{v}\in\mathbb{B}^3$ and $r\in\mathbb{R}$.

Example 2: Let

$$\rho_s = \rho_{\mathbf{u}}\rho_{\mathbf{v}}^2\rho_{\mathbf{u}} \tag{83}$$

By Definition 5, ρ_s is symmetric, possessing the Bloch gyrovector, (61),

$$\begin{aligned}\mathbf{w} &= \mathbf{u}\oplus(\mathbf{v}\oplus(\mathbf{v}\oplus\mathbf{u}))\\ &= \mathbf{u}\oplus(2\otimes\mathbf{v}\oplus\mathbf{u})\\ &= 2\otimes(\mathbf{u}\oplus\mathbf{v})\end{aligned} \tag{84}$$

In (84) we use identities from [4], that is, the left gyroassociative law of gyrogroup theory, and [4, Theorem 3.13].

Hence, by Theorem 6,

$$\begin{aligned}\rho_{\mathbf{u}}\rho_{\mathbf{v}}^2\rho_{\mathbf{u}} &= tr(\rho_{\mathbf{u}}\rho_{\mathbf{v}}^2\rho_{\mathbf{u}})\rho_{\mathbf{w}}\\ &= tr(\rho_{\mathbf{u}}\rho_{\mathbf{v}}^2\rho_{\mathbf{u}})\rho_{2\otimes(\mathbf{u}\oplus\mathbf{v})}\end{aligned} \tag{85}$$

But, by (72),

$$\rho_{2\otimes(\mathbf{u}\oplus\mathbf{v})} = \frac{\rho_{\mathbf{u}\oplus\mathbf{v}}^2}{tr(\rho_{(\mathbf{u}\oplus\mathbf{v})}^2)} \tag{86}$$

and by (76),

$$\frac{1}{tr(\rho_{(\mathbf{u}\oplus\mathbf{v})}^2)} = (tr(\rho_{2\otimes(\mathbf{u}\oplus\mathbf{v})}^{1/2}))^2 \tag{87}$$

Hence, (85) can be written as

$$\rho_{\mathbf{u}}\rho_{\mathbf{v}}^2\rho_{\mathbf{u}} = tr(\rho_{\mathbf{u}}\rho_{\mathbf{v}}^2\rho_{\mathbf{u}})(tr(\rho_{2\otimes(\mathbf{u}\oplus\mathbf{v})}^{1/2}))^2\rho_{\mathbf{u}\oplus\mathbf{v}}^2 \tag{88}$$

Example 3: We wish to calculate $[tr(\sqrt{\rho_{\mathbf{v}}})]^2$ for later reference. By (76) and (29) we have

$$\begin{aligned}[tr(\sqrt{\rho_{\mathbf{v}}})]^2 &= [tr(\rho_{(1/2)\otimes\mathbf{v}}^2)]^{-1}\\ &= [tr(\rho_{\gamma_{\mathbf{v}}\mathbf{v}/(1+\gamma_{\mathbf{v}})}^2)]^{-1}\\ &= \frac{1+\gamma_{\mathbf{v}}}{\gamma_{\mathbf{v}}}\end{aligned} \tag{89}$$

where the extreme right hand side of (89) is calculated by straightforward algebra.

Example 4: Let

$$\rho_s = \rho_{\mathbf{u}}^m\rho_{\mathbf{v}}^{2n}\rho_{\mathbf{u}}^m \tag{90}$$

By Definition 5, ρ_s is symmetric, possessing the Bloch gyrovector, (61),

$$\begin{aligned}\mathbf{w} &= m\otimes\mathbf{u}\oplus((2n)\otimes\mathbf{v}\oplus m\otimes\mathbf{u})\\ &= m\otimes\mathbf{u}\oplus(2\otimes(n\otimes\mathbf{v})\oplus m\otimes\mathbf{u})\\ &= 2\otimes(m\otimes\mathbf{u}\oplus n\otimes\mathbf{v})\end{aligned} \tag{91}$$

as, similarly, in (83) and (84).

Hence, by Theorem 6,

$$\begin{aligned}\frac{\rho_{\mathbf{u}}^{m}\rho_{\mathbf{v}}^{2n}\rho_{\mathbf{u}}^{m}}{tr(\rho_{\mathbf{u}}^{m}\rho_{\mathbf{v}}^{2n}\rho_{\mathbf{u}}^{m})} &= \rho(m\otimes\mathbf{u}\oplus((2n)\otimes\mathbf{v}\oplus m\otimes\mathbf{u})) \\ &= \rho(m\otimes\mathbf{u}\oplus(2\otimes(n\otimes\mathbf{v})\oplus m\otimes\mathbf{u})) \\ &= \rho(2\otimes(m\otimes\mathbf{u}\oplus n\otimes\mathbf{v})) \\ &= \frac{\rho_{m\otimes\mathbf{u}\oplus n\otimes\mathbf{v}}^{2}}{tr(\rho_{m\otimes\mathbf{u}\oplus n\otimes\mathbf{v}}^{2})}\end{aligned} \tag{92}$$

Square rooting both sides of (92) we have

$$\sqrt{\rho_{\mathbf{u}}^{m}\rho_{\mathbf{v}}^{2n}\rho_{\mathbf{u}}^{m}} = \sqrt{\frac{tr(\rho_{\mathbf{u}}^{m}\rho_{\mathbf{v}}^{2n}\rho_{\mathbf{u}}^{m})}{tr(\rho_{m\otimes\mathbf{u}\oplus n\otimes\mathbf{v}}^{2})}}\,\rho_{m\otimes\mathbf{u}\oplus n\otimes\mathbf{v}} \tag{93}$$

Tracing both sides of (93) we have

$$tr(\sqrt{\rho_{\mathbf{u}}^{m}\rho_{\mathbf{v}}^{2n}\rho_{\mathbf{u}}^{m}}) = \sqrt{\frac{tr(\rho_{\mathbf{u}}^{m}\rho_{\mathbf{v}}^{2n}\rho_{\mathbf{u}}^{m})}{tr(\rho_{m\otimes\mathbf{u}\oplus n\otimes\mathbf{v}}^{2})}} \tag{94}$$

so that, finally, by squaring both sides of (94) we obtain the identity

$$\Big[tr(\sqrt{\rho_{\mathbf{u}}^{m}\rho_{\mathbf{v}}^{2n}\rho_{\mathbf{u}}^{m}})\Big]^{2} = \frac{tr(\rho_{\mathbf{u}}^{m}\rho_{\mathbf{v}}^{2n}\rho_{\mathbf{u}}^{m})}{tr(\rho_{m\otimes\mathbf{u}\oplus n\otimes\mathbf{v}}^{2})} \tag{95}$$

for all $\mathbf{u},\mathbf{v}\in\mathbb{B}^3$ and $m,n\in\mathbb{R}$.

When $m=n=1/2$, (95) gives rise to an identity for the so called *Bures fidelity* $\mathcal{F}(\rho_{\mathbf{u}},\rho_{\mathbf{v}})$ [9], [1, p. 409], also known as Uhlmann's transition probability for mixed qubit states [40]. Thus,

$$
\begin{aligned}
\mathcal{F}(\rho_{\mathbf{u}},\rho_{\mathbf{v}}) &\overset{(1)}{=\!=\!=} \left[tr(\sqrt{\sqrt{\rho_{\mathbf{u}}}\,\rho_{\mathbf{v}}\sqrt{\rho_{\mathbf{u}}}}\ )\right]^2 \\
&\overset{(2)}{=\!=\!=} \frac{tr(\sqrt{\rho_{\mathbf{u}}}\,\rho_{\mathbf{v}}\sqrt{\rho_{\mathbf{u}}}\,)}{tr(\rho^2_{(1/2)\otimes\mathbf{u}\oplus(1/2)\otimes\mathbf{v}})} \\
&\overset{(3)}{=\!=\!=} \left[tr(\sqrt{\rho_{2\otimes((1/2)\otimes\mathbf{u}\oplus(1/2)\otimes\mathbf{v})}}\ )\right]^2 tr(\sqrt{\rho_{\mathbf{u}}}\,\rho_{\mathbf{v}}\sqrt{\rho_{\mathbf{u}}}) \\
&\overset{(4)}{=\!=\!=} \left[tr(\sqrt{\rho_{\mathbf{u}\oplus_{E}\mathbf{v}}}\ )\right]^2 tr(\sqrt{\rho_{\mathbf{u}}}\,\rho_{\mathbf{v}}\sqrt{\rho_{\mathbf{u}}}) \\
&\overset{(5)}{=\!=\!=} \frac{1+\gamma_{\mathbf{u}\oplus_{E}\mathbf{v}}}{\gamma_{\mathbf{u}\oplus_{E}\mathbf{v}}}\,tr(\sqrt{\rho_{\mathbf{u}}}\,\rho_{\mathbf{v}}\sqrt{\rho_{\mathbf{u}}}) \\
&\overset{(6)}{=\!=\!=} \tfrac{1}{2}\tfrac{1+\gamma_{\mathbf{u}\oplus_{E}\mathbf{v}}}{\gamma_{\mathbf{u}\oplus_{E}\mathbf{v}}}(1+\mathbf{u}\cdot\mathbf{v}) \\
&\overset{(7)}{=\!=\!=} \tfrac{1}{2}\tfrac{1+\gamma_{\mathbf{u}\oplus_{E}\mathbf{v}}}{\gamma_{\mathbf{u}\oplus_{E}\mathbf{v}}}\tfrac{\gamma_{\mathbf{u}\oplus_{E}\mathbf{v}}}{\gamma_{\mathbf{u}}\gamma_{\mathbf{v}}} \\
&=\!=\!= \tfrac{1}{2}\tfrac{1+\gamma_{\mathbf{u}\oplus_{E}\mathbf{v}}}{\gamma_{\mathbf{u}}\gamma_{\mathbf{v}}}
\end{aligned}
\tag{96}
$$

for all $\mathbf{u},\mathbf{v}\in\mathbb{B}^3$.

The equalities in (96) are numbered for the respective explanation below.

(1) presents the definition of Bures fidelity,
(2) follows from (95) with $m = n = 1/2$,
(3) follows from tracing both sides of (75) with $n = 2$,
(4) follows from the isomorphism (97) between $\oplus = \oplus_{M}$ and $\oplus_{E}$,
(5) follows from (89),
(6) follows from (100),
(7) follows from (4).

Let $\oplus = \oplus_{M}$ and $\oplus_{E}$ be, respectively, the Möbius and Einstein addition. These two gyrovector space operations turn out to be isomorphic. The gyrovector space isomorphism between these two operations, mentioned in the derivation of (96), is given by each of the following two equivalent identities in the ball

$\mathbb{B}^3$ [4, p. 276],

$$\begin{aligned}\mathbf{u}\oplus_{\scriptscriptstyle E}\mathbf{v} &= 2\otimes(\tfrac{1}{2}\otimes\mathbf{u}\oplus_{\scriptscriptstyle M}\tfrac{1}{2}\otimes\mathbf{v})\\ \mathbf{u}\oplus_{\scriptscriptstyle M}\mathbf{v} &= \tfrac{1}{2}\otimes(2\otimes\mathbf{u}\oplus_{\scriptscriptstyle E}2\otimes\mathbf{v})\end{aligned} \tag{97}$$

Identity (96) expresses Bures fidelity in terms of Einstein addition as in [9],

$$\mathcal{F}(\rho_{\mathbf{u}},\rho_{\mathbf{v}}) = \tfrac{1}{2}\,\frac{1+\gamma_{\mathbf{u}\oplus_{\scriptscriptstyle E}\mathbf{v}}}{\gamma_{\mathbf{u}}\gamma_{\mathbf{v}}} \tag{98}$$

From (96) and (4) we obtain the elegant, obvious identity

$$tr(\sqrt{\rho_{\mathbf{u}}}\,\rho_{\mathbf{v}}\sqrt{\rho_{\mathbf{u}}}) = \tfrac{1}{2}\tfrac{\gamma_{\mathbf{u}\oplus_{\scriptscriptstyle E}\mathbf{v}}}{\gamma_{\mathbf{u}}\gamma_{\mathbf{v}}} = \tfrac{1}{2}(1+\mathbf{u}\cdot\mathbf{v}) \tag{99}$$

Indeed,

$$tr(\sqrt{\rho_{\mathbf{u}}}\,\rho_{\mathbf{v}}\sqrt{\rho_{\mathbf{u}}}) = tr(\rho_{\mathbf{u}}\rho_{\mathbf{v}}) = \frac{1+\mathbf{u}\cdot\mathbf{v}}{2} \tag{100}$$

Substituting (4) in (98) we have, by (3),

$$\begin{aligned}\mathcal{F}(\rho_{\mathbf{u}},\rho_{\mathbf{v}}) &= \tfrac{1}{2}\left\{1+\mathbf{u}\cdot\mathbf{v}+\frac{1}{\gamma_{\mathbf{u}}\gamma_{\mathbf{v}}}\right\}\\ &= \tfrac{1}{2}\left\{1+\mathbf{u}\cdot\mathbf{v}+\sqrt{1-\|\mathbf{u}\|^2}\sqrt{1-\|\mathbf{v}\|^2}\right\}\end{aligned} \tag{101}$$

thus recovering a well known identity for the Bures fidelity [41], and a link with Einstein's special relativity theory concealed in (4).

In Examples $1-4$ we uncover the tip of the giant iceberg of the rich gyroalgebra that qubit density matrices possess in terms of their Bloch gyrovector parameter in the Möbius gyrovector space $(\mathbb{B}^3,\oplus,\otimes)$. The parameter Möbius gyrovector space, in turn, forms the setting for the Poincaré ball model of the hyperbolic geometry of Bolyai and Lobachevski, indicated in Figs. 1 and 2 and studied in [4]. By demonstrating that the qubit density matrices of quantum theory are regulated by a gyrovector space and its hyperbolic geometry, we have completed the task we faced in this article.

Exploiting the convex structure of the space of qubit density matrices, the evolution of a density matrix $\rho(\mathbf{u})$ into another density matrix $\rho(\mathbf{v})$ through continuously successive density matrices is demonstrated by the *convex sum*

$$(1-t)\rho(\mathbf{u})+t\rho(\mathbf{v}), \qquad 0\le t\le 1 \tag{102}$$

At "time" $t=0$ (102) gives the density matrix $\rho(\mathbf{u})$ and at "time" $t=1$ (102) gives the density matrix $\rho(\mathbf{v})$ [42, Sec. 2.5.1].

Following the discovery in this article of the gyro-structure of the space of qubit density matrices and the link to hyperbolic geometry to which it gives rise, it is more natural from the geometric viewpoint to exploit the rich gyro-structure and its hyperbolic geometry rather than the convex structure. Thus, exploiting the hyperbolic geodesic curve (32) of the Poincaré ball model of hyperbolic geometry, we generate the evolution of a density matrix $\rho(\mathbf{u})$ into another density matrix $\rho(\mathbf{v})$ through continuously successive density matrices by the parameter transport along a geodesic,

$$\rho(\mathbf{u}\oplus(\ominus\mathbf{u}\oplus\mathbf{v})\otimes t), \qquad 0 \le t \le 1 \tag{103}$$

At "time" $t = 0$ (103) gives the density matrix $\rho(\mathbf{u})$ and at "time" $t = 1$ (103) gives the density matrix $\rho(\mathbf{v})$. As t evolves from $t = 0$ to $t = 1$, the Bloch gyrovector parameter of the qubit density matrix in (103) travels from $\mathbf{u}$ to $\mathbf{v}$ along the geodesic segment joining the points $\mathbf{u}$ and $\mathbf{v}$ of the Möbius gyrovector space $(\mathbb{B}^3, \oplus, \otimes)$.

The parallel transport, Fig. 2, of a Bloch gyrovector $\mathbf{v}$ along a closed path in $\mathbb{B}^3$ results in a hyperbolic geometric angle defect, studied in [4, Chap. 7]. The study of the resulting geometric phase of the evolution of the associated density matrix $\rho(\mathbf{v})$ could be of interest in geometric quantum computation [43].

Following the presentation of its power and elegance, it remains to harness the gyro-structure of the qubit density matrix for work in geometric quantum computation [43].

ACKNOWLEDGMENTS Much of the insight presented in this article was obtained by useful discussions with Jing-Ling Chen and Helmuth K. Urbantke.

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