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Thomas Precession:
Its Underlying Gyrogroup Axioms
and Their Use in
Hyperbolic Geometry and Relativistic Physics

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ABSTRACT

Gyrogroup theory and its applications is introduced and explored, exposing the fascinating interplay between Thomas precession of special relativity theory and hyperbolic geometry. The *abstract Thomas precession,* called *Thomas gyration,* gives rise to grouplike objects called *gyrogroups* [A.A. Ungar, Amer. J. Physics, **59,** 824 (1991)] the underlying axioms of which are presented. The prefix *gyro,* extensively used in terms like *gyrogroups, gyroassociative and gyrocommutative laws, gyroautomorphisms* and *gyrosemidirect products,* stems from their underlying abstract Thomas gyration. Thomas gyration is tailor made for hyperbolic geometry. In a similar way that commutative groups underlie vector spaces, gyrocommutative gyrogroups underlie gyrovector spaces. Gyrovector spaces, in turn, provide a most natural setting for hyperbolic geometry in full analogy with vector spaces that provide the setting for Euclidean geometry. As such, their applicability to relativistic physics and its spacetime geometry is obvious.

1. Introduction - Hyperbolic Geometry Becomes Coordinate by means of Thomas Precession

Thomas precession of special theory of relativity (STR) has reached a milestone in 1988 following the discovery of the mathematical regularity that it stores.[1–2] Resulting links with hyperbolic geometry now follow, giving rise to coordinate hyperbolic geometry analogous to coordinate Euclidean geometry in the sense indicated by Eqs. (1.3) and (1.4) below. Links between hyperbolic geometry and Thomas precession are not unexpected.[3] Unexpectedly, however, the recent exposition of the symmetries concealed in Thomas precession places it centrally in the foundations of hyperbolic geometry.

Hyperbolic geometry preceded by the study of the hyperbolic functions, which share analogies with the circular (or, trigonometric) functions. The study of the hyperbolic functions began when it was noticed that the area under the circle is given by $\int\sqrt{a^2-x^2}dx$ whereas the area under the hyperbola is given by $\int\sqrt{x^2-a^2}dx$. Since the two differ by the multiplicative constant $i=\sqrt{-1}$, and since the area under the circle can be expressed by circular functions (letting $x=a\cos\theta$), it was natural to change the circular functions into hyperbolic ones. It was Johann Heinrich Lambert (1728-1777) who elaborated the subject of hyperbolic functions which he designated by $\sinh x$, $\cosh x$, etc.

In full analogy with rotations of the Euclidean plane, expressed by circular functions, the hyperbolic functions enable one to express *pseudo rotations* of the Minkowski plane, where the pseudo angle of the pseudo rotation is called *rapidity.* The pseudo rotations parametrized by rapidities are known in STR as *Lorentz transformations.*[4] For this reason the term "non-Euclidean geometry" of Bolyai and Lobachevsky is often changed to "hyperbolic geometry".[5]

Further analogies shared by Euclidean and hyperbolic geometries are exposed in this article by means of properties of a peculiar relativistic rotation called Thomas precession, described in Fig. 1.1. It is the vector space structure underlying Euclidean geometry which allows us to study Euclidean geometry by means of coordinates. In order to develop analogous coordinate hyperbolic geometry we will employ unexpected connections with Thomas precession and uncover a grouplike structure, called a *gyrogroup,* to which the symmetries hidden in Thomas precession give rise. The gyrogroup concept, in turn, will lead us to the hyperbolic counterpart of the vector space notion that we call a *gyrovector space.* The resulting coordinate hyperbolic geometry will allow us, for instance, to describe geodesics relative to natural metrics in full analogy with the description of the Euclidean straight lines, which are the geodesics relative to the common metric of a vector space given by its norm.

Gyrogroups emerged in 1988,[1] following the discovery that Thomas precession of STR stores an algebraic structure that turns the groupoid (a groupoid is a non-empty set with a binary operation) of all relativistically admissible velocities with their Einstein’s velocity addition law into a grouplike object called a gyrogroup. The abstract gyrogroup then followed by abstraction in which the abstract Thomas precession, called Thomas gyration, plays a central role. Due to their interdisciplinary character, spreading from abstract algebra and non-Euclidean geometry to mathematical physics, gyrogroups have extensively been studied,[1, 2, 6–22] making their way into the mainstream.[23 – 25] For recent studies about Thomas precession see Refs. 26 – 28.

The relationship between gyrogroups and structures of interest in geometry and in non-associative algebra like K-loops, Bruck loops, Bol loops and quasigroups as well as relevant literature,[29–34] has been presented in the Introduction in Ref. 35.

Prerelativity velocities, the elements of the Euclidean 3-space $\mathbb{R}^3$, form a commutative group $(\mathbb{R}^3,+)$ under ordinary vector addition, +. In contrast, relativistically admissible velocities, the elements of the c-ball $\mathbb{R}_c^3$ of $\mathbb{R}^3$,

$$\mathbb{R}_c^3 = \left\{ \mathbf{v}\in\mathbb{R}^3 : \|\mathbf{v}\| < c \right\},$$

do not form a group under Einstein's velocity addition $\oplus$; the groupoid $(\mathbb{R}_c^3, \oplus)$ is not a group since its binary operation $\oplus$ is nonassociative. Einstein's relativistic velocity addition $\oplus$ (for velocities that are not necessarily parallel) is given by the equation[(36)]

$$\mathbf{u}\oplus\mathbf{v} = \frac{1}{1+\mathbf{u}\cdot\mathbf{v}/c^2}\left\{\mathbf{u}+\frac{1}{\gamma_{\mathbf{u}}}\mathbf{v}+\frac{1}{c^2}\frac{\gamma_{\mathbf{u}}}{1+\gamma_{\mathbf{u}}}(\mathbf{u}\cdot\mathbf{v})\mathbf{u}\right\}, \qquad \mathbf{u},\mathbf{v}\in\mathbb{R}_c^3 \tag{1.1}$$

where c is an arbitrary fixed constant that represents the speed of light in empty space, and where $\gamma_{\mathbf{u}}$ is the Lorentz factor given by the equation

$$\gamma_{\mathbf{u}} = \frac{1}{\sqrt{1-\frac{\|\mathbf{u}\|^2}{c^2}}}, \qquad \mathbf{u}\in\mathbb{R}_c^3.$$

Einstein's binary operation $\oplus$ in $\mathbb{R}_c^3$ is neither commutative nor associative. Since groups measure symmetry, it seems that mathematical symmetry has been lost in the progress from prerelativity to relativity velocities. It has been discovered in 1988 that the seemingly lost symmetries are in fact concealed in Thomas precession.[(1)] Following the discovery of the symmetries hidden in Thomas precession it became possible to extend Thomas precession by abstraction to the so called *Thomas gyration.* In order to describe the symmetries concealed in Thomas precession, let the 3-space rotation known in relativity theory as the Thomas precession generated by $\mathbf{u},\mathbf{v}\in\mathbb{R}_c^3$ be denoted by $\mathrm{gyr}[\mathbf{u};\mathbf{v}]$ and viewed as an automorphism of the groupoid $(\mathbb{R}_c^3, \oplus)$. The physical meaning of the rotation $\mathrm{gyr}[\mathbf{u};\mathbf{v}]$ is presented graphically in Fig 1.1 below, in Fig. 1 of Ref. 1 and in Figs. 2 and 3 of Ref. 14.

Thomas precession is expressible in terms of Einstein's velocity addition (1.1). The Thomas rotation (or, precession) generated by $\mathbf{u},\mathbf{v}\in\mathbb{R}^3$ of some $\mathbf{x}\in\mathbb{R}^3$ is given by the equation

$$\mathrm{gyr}[\mathbf{u};\mathbf{v}]\mathbf{x} = -(\mathbf{u}\oplus\mathbf{v})\oplus(\mathbf{u}\oplus(\mathbf{v}\oplus\mathbf{x})) \tag{1.2}$$

Thomas precessions $\mathrm{gyr}[\mathbf{u};\mathbf{v}]$ are peculiar, distinctive rotations. The aim of the present article is, accordingly, to expose and employ the mathematical regularity that Thomas precession stores.

To describe the Thomas rotation (or, precession) $\mathrm{gyr}[\mathbf{u};\mathbf{v}]$ generated by any two relativistically admissible velocities $\mathbf{u}$ and $\mathbf{v}$ let us consider three inertial frames Σ, Σ' and Σ'', Fig. 1.1, which were coincident at time $t=0$. If (i) Σ'' moves relative to Σ' with velocity $\mathbf{v}\in\mathbb{R}_c^2$ *without* rotation, and (ii) Σ' moves relative to Σ with velocity $\mathbf{u}\in\mathbb{R}_c^2$ *without* rotation, Fig. 1.1 (a), then (iii) Σ'' moves relative to Σ (iv) with velocity $\mathbf{u}\oplus\mathbf{v}$ *preceded* by the Thomas rotation $\mathrm{gyr}[\mathbf{u};\mathbf{v}]$ generated by $\mathbf{u}$ and $\mathbf{v}$ or, equivalently, (v) with velocity $\mathbf{v}\oplus\mathbf{u}$ *followed* by the Thomas rotation $\mathrm{gyr}[\mathbf{u};\mathbf{v}]$, Fig. 1.1 (b).

FIGURE 1.1

The paradox presented by the bizarre and counterintuitive noncommutativity and nonassociativity of Einstein's addition, $\oplus$, of non-collinear velocities stems from the fact that in STR relative velocities and relative orientations between inertial frames are coupled.[8] The paradoxical question of whether the "right" velocity of Σ'' relative to Σ in Fig. 1.1 is $\mathbf{u}\oplus\mathbf{v}$ or $\mathbf{v}\oplus\mathbf{u}$ makes no sense in STR since relations between inertial frames cannot be determined by relative velocities alone. They are determined by both relative velocities and relative orientations which, unlike their Galilean counterparts, cannot be decoupled unless Thomas precession is invoked.

Since relative velocities and relative orientations are decoupled in Galilei relativity, in order to expose analogies shared by Galilei relativity and by Einstein's relativity it would be useful to decouple relative velocities and relative orientations in STR as well. It turns out that these can indeed be decoupled in STR by means of Thomas precession, resulting in the group $SO(3)$ of orientations and in the *gyrogroup* $(\mathbb{R}_c^3, \oplus)$ of relativistically admissible velocities that we call Einstein gyrogroup. Accordingly, the Lorentz group emerges as the *gyrosemidirect* product of the gyrogroup $(\mathbb{R}_c^3, \oplus)$ and the group $SO(3)$ with full analogy with the Galilean group which is known to be the semidirect product of the group $(\mathbb{R}^3, +)$ and the group $SO(3)$. The semidirect product structure of the Galilean transformation group allows time and space in Galilean spacetime to be decoupled. Similarly, the gyrosemidirect product structure of the Lorentz transformation group allows time and space in relativistic spacetime to be decoupled.[37]

The key features of Einstein gyrogroup are abstracted in this article, and the axioms underlying the abstract gyrogroup are presented. The resulting abstract Thomas precession emerges as an automorphism called *Thomas gyration,* giving rise to our extensive use of the prefix "gyro" in terms like *gyrogroups, gyroassociative and gyrocommutative laws, gyroautomorphisms* and *gyrosemidirect products.*

Merging the "gyroterminology" with the "morphism" mathematical terminology was proposed by the physicist Helmuth Urbantke who wrote in a personal communication: "While giving a seminar about your work, the word *gyromorphism* instead of precession came over my lips. Since it ties in with the many morphisms the mathematicians love, it might appeal to you" (fn. 36 in Ref. 14).

The analogies shared by groups and gyrogroups suggest the extension of gyrocommutative gyrogroups into gyrovector spaces by the introduction of scalar multiplication, $\odot$, in addition to the gyrogroup operation $\oplus$. These, in turn, prove useful in hyperbolic geometry. The dramatic analogies shared by vector spaces and gyrovector spaces indicate that gyrovector spaces provide the natural setting for hyperbolic geometry in full analogy with vector spaces, that provide the natural setting for Euclidean geometry. In particular, the theory of gyrovector spaces unifies some aspects of the coordinate geometry of both Euclidean and hyperbolic geometries. Thus for instance, the parametric expression

$$\mathbf{a}+t(\mathbf{b}-\mathbf{a}) \tag{1.3}$$

that parametrizes by $t\in\mathbb{R}$ the unique straight line passing through $\mathbf{a}$ and $\mathbf{b}$ in a Euclidean space shares remarkable analogies with the parametric expression

$$\mathbf{a}\oplus t\odot(-\mathbf{a}\oplus\mathbf{b}) \tag{1.4}$$

that parametrizes the unique geodesic passing through $\mathbf{a}$ and $\mathbf{b}$ in the Klein disk model of a hyperbolic geometry, as we will see in Fig. 7.1 of Section 7 and as shown in Refs. 35, 38. Thus, our development of gyrovector spaces that underlie hyperbolic geometry turns it into a coordinate hyperbolic geometry.

The commutative-associative laws of the binary operation + in $\mathbb{R}^3$ that have seemingly been lost in the transition from ordinary vector addition, +, in $\mathbb{R}^3$ to Einstein's binary operation, $\oplus$, in $\mathbb{R}_c^3$ reappear as the following *gyrocommutative-gyroassociative* laws. For all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}_c^3$,

$\mathbf{u} \oplus \mathbf{v} = \mathrm{gyr}[\mathbf{u}; \mathbf{v}](\mathbf{v} \oplus \mathbf{u})$	Gyrocommutative Law
$\mathbf{u} \oplus (\mathbf{v} \oplus \mathbf{w}) = (\mathbf{u} \oplus \mathbf{v}) \oplus \mathrm{gyr}[\mathbf{u}; \mathbf{v}]\mathbf{w}$	Left Gyroassociative Law
$(\mathbf{u} \oplus \mathbf{v}) \oplus \mathbf{w} = \mathbf{u} \oplus (\mathbf{v} \oplus \mathrm{gyr}[\mathbf{v}; \mathbf{u}]\mathbf{w})$	Right Gyroassociative Law

In fact, while the two gyroassociative laws are a relatively new discovery,(1) the gyrocommutative law, though not recognized as such, is well known and appeared in *early literature* on STR.(39) Thomas precession $\mathrm{gyr}[\mathbf{u}; \mathbf{v}]$ is the unique rotation of $\mathbb{R}^3$ about its origin that takes the vector $\mathbf{v} \oplus \mathbf{u} \in \mathbb{R}_c^3 \subset \mathbb{R}^3$ into the vector $\mathbf{u} \oplus \mathbf{v} \in \mathbb{R}_c^3 \subset \mathbb{R}^3$ by a rotation about an axis perpendicular to both $\mathbf{u}$ and $\mathbf{v}$ through an angle $< \pi$.(40)

As opposed to the early literature, modern texts on STR either ignore Thomas precession or treat it as an isolated phenomenon. The present exposition of the central role played by Thomas precession places it in the foundations of hyperbolic geometry. The fact that Thomas precession is not only the rotation that repairs the commutative law of the composition of relativistically admissible velocities,(39) but also the rotation that repairs its associative law as well,(1) is just one of those amazing coincidences with which gyrogroup theory abounds. We will describe in this article the various distance functions and geodesics to which Thomas precession gives rise in several models of hyperbolic geometry.

The appearance of grouplike symmetries in Einstein's binary operation in terms of Thomas precession suggests to define a grouplike object, called a *gyrogroup,* endowed with abstract Thomas precession, called *Thomas gyration,* that stores grouplike symmetries.(14) To develop an abstract gyrogroup theory in a way analogous to abstract group theory, Thomas precession is abstracted into an automorphism called *Thomas gyration,* that satisfies certain conditions. The resulting axiomatic approach to gyrogroups has been explored and various attempts have been presented, in which the above gyrocommutative law and gyroassociative laws (left and right) play a key role.(6, 9)

Originally, gyrogroups were defined in a narrow sense, allowing only gyrocommutative gyrogroups since it was believed that, unlike commutativity in group theory, the gyrocommutative law plays indispensable role in gyrogroup theory, that is, in the abstraction of an algebraic structure stored in Thomas precession.(9) Indeed, the key role played by the *left loop property*

$$\mathrm{gyr}[\mathbf{u}; \mathbf{v}] = \mathrm{gyr}[\mathbf{u} \oplus \mathbf{v}; \mathbf{v}]$$

or, equivalently, by its right counterpart

$$\mathrm{gyr}[\mathbf{u}; \mathbf{v}] = \mathrm{gyr}[\mathbf{u}; \mathbf{v} \oplus \mathbf{u}]$$

was not fully appreciated in Ref. 9 and, as a result, the forerunners of gyrocommutative gyrogroups (called WACGs in Ref. 9) were found to be loops, while the forerunners of non-gyrocommutative gyrogroups (called WAGs in Ref. 9) were found to violate a basic loop-axiom concerning the solvability of an equation.(34) However, a recently discovered concrete example,(41) shows that the approach taken in Ref. 9 can (and, hence, should) be improved. We therefore (i) propose in this article an extended gyrogroup definition that allows non-gyrocommutative gyrogroups as well; and (ii) present the (reduced) system of axioms that underlie the gyrogroup concept in the extended sense.

Being guided by new concrete examples, analogies shared by groups and relativity grouplike objects, and the needs of space-time physics, we find it necessary to extend the gyrogroup notion by deleting one of its underlying axioms, the gyrocommutative law. It is hoped that this is not inconvenient at this early stage of the study of gyrogroups, which are now finding their way into mainstream literature.[23–25] Following the new, extended gyrogroup notion, gyrogroups are now classified as (i) *gyrocommutative gyrogroups* [identical with (a) K-loops, (b) Bruck loops, and with (c) "gyrogroups in the narrow sense", i.e., gyrogroups as defined prior to the present publication]; and (ii) *non-gyrocommutative gyrogroups,* in full analogy with the classification of groups into commutative (Abelian) and noncommutative ones.

The new system of gyrogroup axioms is presented in Section 2, following which six examples of interesting gyrogroups are presented. Some elementary results emerging from the gyrogroup axioms are presented in Section 3. Every gyrogroup is extendible to a group called a *gyrosemidirect* product group, presented in Section 4. In the special case when Thomas gyration vanishes, the gyrosemidirect product group reduces to the well known semidirect product group in group theory. Some other results emerging from the gyrogroup axioms, obtained by means of the gyrosemidirect product group, are then presented in Section 5.

An alternative definition of gyrogroups, equivalent to the gyrogroup definition in Definition 2.2, is given in Definition 5.2. Scalar multiplication in three gyrocommutative gyrogroups is presented in Section 6 giving rise to gyrovector spaces. The relevance of resulting gyrovector spaces to hyperbolic geometry and relativistic physics is indicated in Sect. 7. Finally, the abstract Lorentz-boost gyrogroup underlying the abstract Lorentz group is discussed in Sect. 8.

2. Gyrogroups - Axioms and Examples

Gyrogroup theory is obtained by abstraction from the set of all relativistically admissible velocities of STR with their Einstein's relativistic velocity composition law and their Thomas precession. Consequently, a gyrogroup is a grouplike groupoid in which the associative law has been relaxed by means of Thomas gyration into the so called left and right gyroassociative laws. Following Definition 2.2 of gyrogroups we will see in Example 2.2 (Example 2.4) that the set of all relativistically admissible velocities (the set of all proper velocities) with their Einstein's velocity (proper velocity) addition law and Thomas precession forms a gyrogroup, in full analogy with the group formed by classical velocities.

DEFINITION 2.1 *(Binary operations, groupoids and their automorphism groups)* A *binary operation,* +, in a non-empty set, S, is a mapping $+: S \times S \to S$. We will use the common notation $s_1 + s_2$ for $+(s_1, s_2)$, $s_1, s_2 \in S$. A groupoid $(S, +)$ is a non-empty set S with a binary operation +. An automorphism of the groupoid $(S, +)$ is a bijection of S that respects the binary operation + in S. The set of all automorphisms of $(S, +)$ forms a group denoted by $\mathrm{Aut}(S, +)$. The identity automorphism is denoted by I.

DEFINITION 2.2 *(Gyrogroups)* A groupoid $(G, +)$ is a *gyrogroup* if its binary operation satisfies the following axioms. In G there is at least one element, 0, called a *left identity,* satisfying

$(G1)$ $\quad 0 + a = a$ $\qquad$ Left Identity

for all $a \in G$. There is an element $0 \in G$ satisfying axiom $(G1)$ such that for each a in G there is an x in G, called a *left inverse* of a, satisfying

$(G2)$ $\quad x + a = 0$ $\qquad$ Left Inverse

Moreover, for any $a, b, z \in G$ there exists a unique element $\mathrm{gyr}[a; b]z \in G$ such that

$(G3)$ $\quad a + (b + z) = (a + b) + \mathrm{gyr}[a; b]z$ $\qquad$ Left Gyroassociative Law

If $\mathrm{gyr}[a; b]$ denotes the map $\mathrm{gyr}[a; b]: G \to G$ given by $z \mapsto \mathrm{gyr}[a; b]z$ then

$(G4)$ $\quad \mathrm{gyr}[a; b] \in \mathrm{Aut}(G, +)$ $\qquad$ Gyroautomorphism

and $\mathrm{gyr}[a; b]$ is called the *Thomas gyration,* or the *gyroautomorphism of* G*, generated by* $a, b \in G$. Finally, the gyroautomorphism $\mathrm{gyr}[a; b]$ generated by any $a, b \in G$ satisfies

$(G5)$ $\quad \mathrm{gyr}[a; b] = \mathrm{gyr}[a + b; b]$ $\qquad$ Left Loop Property

The gyrogroup $(G, +)$ is *gyrocommutative* if for all $a, b \in G$

$(G6)$ $\quad a + b = \mathrm{gyr}[a; b](b + a)$ $\qquad$ Gyrocommutative Law

The action of the gyroautomorphism $\mathrm{gyr}[a;b]$ on G is called *gyration.* The set of all gyroautomorphisms of a gyrogroup $(G,+)$ need not form a group. A *gyroautomorphism group* of $(G,+)$ is any subgroup (not necessarily the smallest one) of $\mathrm{Aut}(G,+)$ containing all the gyroautomorphisms of $(G,+)$.

Several examples of gyrogroups follow. In these examples the unique inverse (see Theorem 3.1) of an element a of a gyrogroup $(G, \oplus)$ is $-a$ and $a \ominus b$ denotes $a \oplus (-b)$ in G.

EXAMPLE 2.1 (The Disk Gyrogroup; D-Gyrogroup, in short) In this example we realize the abstract gyrogroup by the complex disk D_c,

$$D_c = \left\{ z \in \mathbb{C} : |z| < c \right\},$$

and its *pure* Möbius transformations (i.e., its Möbius transformations without rotations). These are the transformations

$$z \mapsto \frac{a+z}{1+\bar{a}z/c^2} := a \oplus_D z \tag{2.1}$$

of D_c, where $\mathbb{C}$ is the complex plane, c is an arbitrary fixed positive constant, and $\bar{a}$ is the complex conjugate of a.

The pair $(D_c, \oplus_D)$ forms a gyrocommutative gyrogroup whose Thomas gyration generated by any $a, b \in D_c$ is given by the unimodular complex number

$$\mathrm{gyr}[a;b] = \frac{a \oplus_D b}{b \oplus_D a} = \frac{c^2 + a\bar{b}}{c^2 + \bar{a}b}. \tag{2.2}$$

Thomas gyration $\mathrm{gyr}[a;b]$ in (2.2) is thus a rotation of the disk D_c about its origin. The fact that the complex number $\mathrm{gyr}[a;b]$ in (2.2) satisfies the gyrocommutative law (G6) in $(D_c, \oplus_D)$ is trivially obvious. Unexpectedly, however, it satisfies the gyroassociative law (left and right) and the loop property (left and right) that have been mentioned in the Introduction as well.

Without loss of generality one may select $c=1$. However, we refrain from exploiting the simplicity offered by $c=1$ and retain c as a free, but fixed, parameter in order to allow reduction of the disk gyrogroup $(D_c, \oplus_D)$ into the group $(\mathbb{C},+)$ of the complex plane under ordinary addition by letting $c \to \infty$. This remark applies to the other examples in this section as well.

The disk gyrogroup $(D_c, \oplus_D)$ has been studied in Ref. 17 and extended to balls in Ref. 22.

EXAMPLE 2.2 (The Standard Relativity-Velocity Gyrogroup or, Einstein Gyrogroup; E-Gyrogroup, in short) In this example we realize the abstract gyrogroup by the *standard relativity velocity gyrogroup.* Let $(V_\infty, +, \cdot)$ be a real or complex inner product space, and let

$$V_c = \left\{ \mathbf{v} \in V_\infty : \|\mathbf{v}\| < c \right\}$$

be its open c-ball, c being an arbitrary fixed positive constant. The inner product $<\mathbf{u},\mathbf{v}>$ in V_∞ is denoted by $\mathbf{u}\cdot\mathbf{v}$ or $\mathbf{v}\cdot\mathbf{u}$ if V_∞ is real, and by $\mathbf{u}\cdot\bar{\mathbf{v}}$ or $\bar{\mathbf{v}}\cdot\mathbf{u}$ if V_∞ is complex (we make no attempt to define $\bar{\mathbf{v}}$ alone). To avoid unnecessary repetitions, if V_∞ is real then $\mathbf{u}\cdot\bar{\mathbf{v}} = \mathbf{u}\cdot\mathbf{v}$. Under this notation, the groupoid *induced* by the real or complex inner product space $(V_\infty, +, \cdot)$ is the pair $(V_c, \oplus_E)$ where the binary operation $\oplus_E$ in V_c is given by the abstract relativistic velocity addition

$$\mathbf{u} \oplus_E \mathbf{v} = \frac{1}{1+\bar{\mathbf{u}}\cdot\mathbf{v}/c^2}\left\{\mathbf{u} + \frac{1}{\gamma_{\mathbf{u}}}\mathbf{v} + \frac{1}{c^2}\frac{\gamma_{\mathbf{u}}}{1+\gamma_{\mathbf{u}}}(\bar{\mathbf{u}}\cdot\mathbf{v})\mathbf{u}\right\}, \qquad \mathbf{u},\mathbf{v}\in V_c\ , \tag{2.3a}$$

$\gamma_{\mathbf{u}}$ being the well known Lorentz factor of STR (written here for a possibly complex $\mathbf{u}$),

$$\gamma_{\mathbf{u}} = \frac{1}{\sqrt{1-\frac{\bar{\mathbf{u}}\cdot\mathbf{u}}{c^2}}} \tag{2.3b}$$

satisfying, with $\oplus = \oplus_E$,

$$\gamma_{\mathbf{u}\oplus\mathbf{v}} = \gamma_{\mathbf{u}}\gamma_{\mathbf{v}}\left|1 + \frac{\bar{\mathbf{u}}\cdot\mathbf{v}}{c^2}\right| . \tag{2.3c}$$

The real counterpart of identity (2.3c), Eq. (8.13), is well known.[42] The pair $(V_c, \oplus_E)$ forms a gyrogroup called the *standard (real or complex) relativity velocity gyrogroup* induced by the (real or complex) inner product space $(V_\infty, +, \cdot)$. Its binary operation and Thomas gyration specialize respectively to Einstein's velocity addition law and to Thomas precession of STR when the abstract inner product space V_∞ is realized by the Euclidean three-space $\mathbb{R}^3$. Since (2.3a) is an obvious generalization of Einstein's relativistic velocity composition law for velocities in $\mathbb{R}_c^3$, we call the standard relativity velocity gyrogroup $(V_c, \oplus_E)$ *Einstein gyrogroup* or, in short, E-gyrogroup.

The gyroautomorphism operation gyr of Einstein gyrogroup $(V_c, \oplus_E)$ is given by the equation

$$\mathrm{gyr}[\mathbf{u};\mathbf{v}]\mathbf{z} = -(\mathbf{u}\oplus_E\mathbf{v}) \oplus_E \{\mathbf{u}\oplus_E(\mathbf{v}\oplus_E\mathbf{z})\} \tag{2.3d}$$

for all $\mathbf{u},\mathbf{v},\mathbf{z}\in V_c$, and it reduces to the common Thomas precession of STR when V_∞ is realized by $\mathbb{R}^3$. It can be shown that the gyroautomorphisms $\mathrm{gyr}[\mathbf{u};\mathbf{v}]$ in (2.3d) generated by any $\mathbf{u},\mathbf{v}\in V_c$ are isometries of V_∞.

Elements of the set V_c that underlies Einstein gyrogroup $(V_c, \oplus_E)$ inherit the norm of V_∞, and the pair (V_c, d_E) is a metric space with the distance function

$$d_E(\mathbf{x},\mathbf{y}) = \|\mathbf{x}\ominus_E\mathbf{y}\| \tag{2.3e}$$

as one can verify by employing (2.3c), the Cauchy-Schwarz inequality, and Theorem 5.8 of Section 5.

EXAMPLE 2.3 (The Non-standard Relativity-Velocity Gyrogroup or, Möbius Gyrogroup; M-Gyrogroup, in short) In this example we realize the abstract gyrogroup by a *non-standard relativity velocity gyrogroup.* Let $(V_\infty, +, \cdot)$ be a real inner product space, and let V_c be its open c-ball, c being an arbitrary fixed positive constant. The binary operation in V_c, $\oplus_M$, is given by the equation

$$\mathbf{u}\oplus_M\mathbf{v} = \frac{1}{1+\frac{2}{c^2}\mathbf{u}\cdot\mathbf{v}+\frac{1}{c^4}\|\mathbf{u}\|^2\|\mathbf{v}\|^2}\left\{(1+\frac{2}{c^2}\mathbf{u}\cdot\mathbf{v}+\frac{1}{c^2}\|\mathbf{v}\|^2)\mathbf{u} + (1-\frac{1}{c^2}\|\mathbf{u}\|^2)\mathbf{v}\right\} \tag{2.4a}$$

representing a non-standard relativistic velocity addition. The pair $(V_c, \oplus_M)$ forms a gyrogroup called the *non-standard* relativity velocity gyrogroup induced by the real inner product space $(V_\infty, +, \cdot)$. Since the binary operation $\oplus_M$ results from the abstraction of Möbius transformation,[22] the non-standard relativity velocity gyrogroup $(V_c, \oplus_M)$ is called *Möbius gyrogroup* or, in short, M-gyrogroup.

The Lorentz factor $\gamma_{\mathbf{v}}$ in a Möbius gyrogroup is given by the equation

$$\gamma_{\mathbf{v}} = \frac{1}{\sqrt{1-\frac{\|\mathbf{v}\|^2}{c^2}}}, \tag{2.4b}$$

satisfying, with $\oplus = \oplus_M$,

$$\gamma_{\mathbf{u}\oplus\mathbf{v}} = \gamma_{\mathbf{u}}\gamma_{\mathbf{v}}\sqrt{1+\frac{2}{c^2}\mathbf{u}\cdot\mathbf{v}+\frac{1}{c^4}\|\mathbf{u}\|^2\|\mathbf{v}\|^2}. \tag{2.4c}$$

The gyroautomorphism operation gyr of Möbius gyrogroup $(V_c, \oplus_M)$ is given by the equation

$$\mathrm{gyr}[\mathbf{u};\mathbf{v}]\mathbf{z} = -(\mathbf{u}\oplus_M\mathbf{v}) \oplus_M \{\mathbf{u}\oplus_M(\mathbf{v}\oplus_M\mathbf{z})\} \tag{2.4d}$$

for all $\mathbf{u},\mathbf{v},\mathbf{z}\in V_c$. It can be shown that the gyroautomorphisms $\mathrm{gyr}[\mathbf{u};\mathbf{v}]$ in (2.4d) generated by any $\mathbf{u},\mathbf{v}\in V_c$ are isometries of V_∞.

Elements of the set V_c that underlies Möbius gyrogroup $(V_c, \oplus_M)$ inherit the norm of V_∞, and the pair (V_c, d_M) is a metric space with the distance function

$$d_M(\mathbf{x},\mathbf{y}) = \|\mathbf{x}\ominus_M\mathbf{y}\| \tag{2.4e}$$

as one can verify by employing (2.4c), the Cauchy-Schwarz inequality, and Theorem 5.8 of Section 5. Details are given in Ref. 22. The generalized Möbius transformation (2.4a) is known in the literature on Möbius groups. It is related to the map $x \to T_a(x)$ that Ahlfors advocated to use as a standard conformal map of the ball B^n on itself; one can readily show that $T_a(x) = -a\oplus_M x$.[43] However, the grouplike structure (gyrogroup) underlying the binary operation $\oplus_M$ to which the map $T_a(x)$ gives rise has gone unnoticed.

REMARK 2.1 *(Concerning Examples 2.2 and 2.3)* When the abstract space V_∞ is realized by the real line $\mathbb{R}$ the c-ball V_c specializes to the interval $\mathbb{R}_c = (-c, c)$ in which the two binary operations of Einstein and of Möbius are coincident, $\oplus_E = \oplus_M$, and are given by

$$u\oplus v = \frac{u+v}{1+\frac{uv}{c^2}}, \qquad u, v\in\mathbb{R}_c \tag{2.5}$$

for both $\oplus = \oplus_E$ and $\oplus = \oplus_M$.

EXAMPLE 2.4 *(The Relativity-Proper-Velocity Gyrogroup or, the Whole-Space Gyrogroup; W-Gyrogroup, in short)* In this example we realize the abstract gyrogroup by a gyrogroup whose underlying vector space is the whole of the space rather than its ball. Let $(V, +, \cdot)$ be a real inner product space, and let $\oplus_W$ be a binary operation in V given by the equation

$$\mathbf{u}\oplus_W\mathbf{v} = \mathbf{u}+\mathbf{v}+\left\{\frac{\beta_{\mathbf{u}}}{1+\beta_{\mathbf{u}}}\frac{\mathbf{u}\cdot\mathbf{v}}{c^2}+\frac{1-\beta_{\mathbf{v}}}{\beta_{\mathbf{v}}}\right\}\mathbf{u} \tag{2.6a}$$

where

$$\beta_{\mathbf{v}} = \frac{1}{\sqrt{1+\frac{\|\mathbf{v}\|^2}{c^2}}}, \tag{2.6b}$$

satisfying, with $\oplus = \oplus_W$,

$$\frac{1}{\beta_{\mathbf{u}\oplus\mathbf{v}}} = \frac{1}{\beta_{\mathbf{u}}}\frac{1}{\beta_{\mathbf{v}}} + \frac{\mathbf{u}\cdot\mathbf{v}}{c^2} \tag{2.6c1}$$

or, equivalently,

$$\beta_{\mathbf{u}\oplus\mathbf{v}} = \frac{\beta_{\mathbf{u}}\beta_{\mathbf{v}}}{1+\beta_{\mathbf{u}}\beta_{\mathbf{v}}\mathbf{u}\cdot\mathbf{v}/c^2}. \tag{2.6c2}$$

The gyroautomorphism operation gyr of the whole-space gyrogroup is given by the equation

$$\mathrm{gyr}[\mathbf{u};\mathbf{v}]\mathbf{z} = -(\mathbf{u}\oplus_W\mathbf{v})\oplus_W\{\mathbf{u}\oplus_W(\mathbf{v}\oplus_W\mathbf{z})\} \tag{2.6d}$$

for all $\mathbf{u},\mathbf{v},\mathbf{z}\in V$.

Elements of the set V that underlies the whole-space gyrogroup $(V, \oplus_W)$ possess the norm of the inner product space V, and the pair (V, d_W) is a metric space with the distance function

$$d_W(\mathbf{x},\mathbf{y}) = \|\mathbf{x}\ominus_W\mathbf{y}\| \tag{2.6e}$$

as one can verify by employing (2.6c1), the Cauchy-Schwarz inequality, and Theorem 5.8 of Section 5.

We may remark that unlike Example 2.2 where the underlying inner product space V_∞ is allowed to be complex, a similar complexification of the gyrogroups in Examples 2.3 and 2.4 does not result in a gyrogroup.

When the underlying vector spaces V_∞ and V in Examples 2.2-2.4 are coincident (and hence real), the three binary operations in these examples are equivalent in the following sense.

Möbius and Einstein gyrogroup operations are related by the equations

$$\begin{aligned}\mathbf{u}\oplus_M\mathbf{v} &= \frac{1}{2}\odot(2\odot\mathbf{u}\oplus_E 2\odot\mathbf{v})\\ \mathbf{u}\oplus_E\mathbf{v} &= 2\odot(\frac{1}{2}\odot\mathbf{u}\oplus_M\frac{1}{2}\odot\mathbf{v})\end{aligned} \tag{2.7}$$

where $2\odot$ is the "two" bijection of each of the two gyrogroups $(V_c, \oplus_E)$ and $(V_c, \oplus_M)$ taking $\mathbf{v}\in V_c$ into $2\odot\mathbf{v}=\mathbf{v}\oplus_E\mathbf{v}=\mathbf{v}\oplus_M\mathbf{v}$, whose inverse is the "half" bijection taking $\mathbf{v}\in V_c$ into its half,

$$\frac{1}{2}\odot\mathbf{v} = \frac{\gamma_v}{1+\gamma_v}\mathbf{v},$$

satisfying $(1/2)\odot\mathbf{v}\oplus_E(1/2)\odot\mathbf{v}=(1/2)\odot\mathbf{v}\oplus_M(1/2)\odot\mathbf{v}=\mathbf{v}$.

The map $2\odot$ of the unit disk, which by Eq. (2.1) is

$$2\odot z = \frac{2z}{1+|z|^2} \tag{2.8}$$

is well known in the literature (but not as a scalar product by 2), being the map that takes the Poincaré disk model of hyperbolic geometry into the Klein disk model of hyperbolic geometry.[44]

Einstein gyrogroup and the whole-space gyrogroup operations are related by the equation

$$\mathbf{u} \oplus_W \mathbf{v} \;=\; f^{-1}(f(\mathbf{u}) \oplus_E f(\mathbf{v})) \tag{2.9a}$$

where $f : (V, \oplus_W) \to (V_c, \oplus_E)$ is given by

$$f(\mathbf{v}) \;=\; \frac{\mathbf{v}}{\sqrt{1+\dfrac{\|\mathbf{v}\|^2}{c^2}}} \;=\; \beta_{\mathbf{v}}\mathbf{v}\,, \qquad \mathbf{v}\in V\,, \tag{2.9b}$$

whose inverse is

$$f^{-1}(\mathbf{v}) \;=\; \frac{\mathbf{v}}{\sqrt{1-\dfrac{\|\mathbf{v}\|^2}{c^2}}} \;=\; \gamma_{\mathbf{v}}\mathbf{v}\,, \qquad \mathbf{v}\in V_c\,. \tag{2.9c}$$

The elements $\mathbf{v}\in V_c$ represent relativistically admissible velocities in STR. Their images, $f^{-1}(\mathbf{v}) = \gamma_{\mathbf{v}}\mathbf{v}$, that comprise the elements of V, represent *proper velocities* in STR, in the following sense.[45]

The relativistically admissible velocity $\mathbf{v}\in V_c$ of an object moving uniformly from a point **a** to a point **b** is the distance **s** traveled by the object divided by the travel time t of the motion as measured by two synchronized clocks placed at **a** and **b**,

$$\mathbf{v} \;=\; \frac{\mathbf{s}}{t}$$

The velocity **v** is clearly synchrony dependent since it is measured by two distant, synchronized clocks. Different agreements on the synchronization of distant clocks result in different values for the velocity of uniformly moving objects.[46] In contrast, the proper velocity $\gamma_{\mathbf{v}}\mathbf{v}\in V$ of the object is the distance **s** divided by the *proper time* interval of the motion, $\sqrt{t^2 - \mathbf{s}^2/c^2}$, as measured by a clock that moves with the object, so that clock synchronization is not needed. Thus, the proper velocity $\gamma_{\mathbf{v}}\mathbf{v}$ of an object moving uniformly with relativistically admissible velocity $\mathbf{v}\in V_c$ is $\gamma_{\mathbf{v}}\mathbf{v}\in V$ given by the equation

$$\gamma_{\mathbf{v}}\,\mathbf{v} \;=\; \frac{\mathbf{s}}{\sqrt{t^2 - \mathbf{s}^2/c^2}}$$

While one-way velocities $\mathbf{v}\in V_c$ in STR are synchrony dependent, their associated proper velocities $\gamma_{\mathbf{v}}\mathbf{v}\in V$ are synchrony-free.[46] Moreover, in the relativistic correction $p = m\gamma_{\mathbf{v}}\mathbf{v}$ of the classical momentum equation $p = m\mathbf{v}$ the Lorentz factor $\gamma_{\mathbf{v}}$ is commonly, but wrongly, considered as a mass correction correcting the classical mass m into its relativistic counterpart $m\gamma_{\mathbf{v}}$, m being the *rest mass.* Indeed, several authors have demonstrated that the relativistic mass obscures the elegance and clarity brought to STR by the four-vector formulation.[47, 45] In particular, Brehme pointed out that the relativistic mass is both undeserved and misleading.[48]

In fact, the Lorentz factor $\gamma_{\mathbf{v}}$ in the relativistic momentum equation $p = m\gamma_{\mathbf{v}}\mathbf{v}$ is a correction of **v** rather than m, converting the synchrony dependent one-way velocity $\mathbf{v}\in V_c$ into its *synchrony-free* proper velocity $\gamma_{\mathbf{v}}\mathbf{v}\in V$. The physical significance of synchrony-free expressions in STR has been studied and emphasized by Winnie and, later again, by Ungar.[49,46] It is therefore interesting to realize that like relativistically admissible velocities, also proper velocities form a gyrogroup. The use of this gyrogroup in the study of relativistic uniform acceleration is presented graphically in Section 7.

EXAMPLE 2.5 *(The Complex Lorentz-Boost Gyrogroup)* Let $(V_c, \oplus_E)$ be Einstein gyrogroup, Example 2.2, and let

$$F(\mathbf{u},\mathbf{v}) = 1 + \frac{\bar{\mathbf{u}}\cdot\mathbf{v}}{c^2}$$

be a *cocycle form* in Einstein gyrogroup satisfying the cocycle identity

$$F(\mathbf{u},\mathbf{v}\oplus_E\mathbf{w})F(\mathbf{v},\mathbf{w}) = F(\mathbf{v}\oplus_E\mathbf{u},\mathbf{w})F(\mathbf{u},\mathbf{v}).$$

It gives rise to the extended gyrogroup $(\mathbb{C}^+\times V_c, \oplus_B)$ of pairs $(t,\mathbf{v})$ where $\mathbb{C}^+$ is the right half of the complex plane $\mathbb{C}$ consisting of all complex numbers with positive real part.[41] The gyrogroup operation in $\mathbb{C}^+\times V_c$ is given by

$$(s,\mathbf{u})\oplus_B(t,\mathbf{v}) = (F(\mathbf{u},\mathbf{v})st,\ \mathbf{u}\oplus_E\mathbf{v}).$$

EXAMPLE 2.6 (The Abstract Lorentz-Boost Gyrogroup) This example follows from the previous one by abstraction. Let $(G, \oplus)$ be an abstract gyrogroup and let $\rho_v = \rho(v)$ be the abstract Lorentz factor where $\rho : G \to \mathbb{R}^+$ is any map from G into the positive ray $\mathbb{R}^+$ of the real line $\mathbb{R}$ which is (i) invariant under automorphisms of G so that, in particular, it is invariant under gyrations, that is,

$$\rho(\mathrm{gyr}[u;v]w) = \rho(w) \tag{2.10}$$

for any $u, v, w \in G$; and which is (ii) normalized by the condition $\rho_0 = 1$. Furthermore, for the sake of simplicity we assume $\rho(-w) = \rho(w)$. Then, the set of pairs $\mathbb{R}^+\times(G, \oplus)$ forms a gyrogroup whose gyrogroup operation, written multiplicatively, is given by

$$\begin{pmatrix} s \\ u \end{pmatrix}\begin{pmatrix} t \\ v \end{pmatrix} = \begin{pmatrix} \dfrac{\rho_{u\oplus v}}{\rho_u\,\rho_v}\,st \\ u \oplus v \end{pmatrix} \tag{2.11}$$

The identity element of $\mathbb{R}^+\times(G, \oplus)$ is clearly $(1, 0)$, and the inverse of $(t, v)^t$ is $(\rho_{\mathbf{v}}^2 t^{-1}, -v)^t$ where exponent t denotes transposition.

If the norm of $(t, v)^t$ is defined by

$$\|(t,v)^t\| = t\,\rho_v^{-1} \tag{2.12}$$

then the norm of a product is the product of the norms,

$$\left\|\begin{pmatrix} s \\ u \end{pmatrix}\begin{pmatrix} t \\ v \end{pmatrix}\right\| = \left\|\begin{pmatrix} s \\ u \end{pmatrix}\right\| \left\|\begin{pmatrix} t \\ v \end{pmatrix}\right\| \tag{2.13}$$

and the subset

$$B = \left\{ \begin{pmatrix} \rho_v \\ v \end{pmatrix} :\ v \in G \right\} \subset \mathbb{R}^+\times(G, \oplus) \tag{2.14}$$

forms a subgyrogroup of the gyrogroup $\mathbb{R}^+\times(G, \oplus)$ consisting of all elements of $\mathbb{R}^+\times(G, \oplus)$ with norm 1. The gyromultiplication (i.e., gyroaddition, written multiplicatively) in B is given by gyromultiplication in $\mathbb{R}^+\times(G, \oplus)$, Eq. (2.11),

$$\begin{pmatrix}\rho_u\\ u\end{pmatrix}\begin{pmatrix}\rho_v\\ v\end{pmatrix} = \begin{pmatrix}\rho_{u\oplus v}\\ u\oplus v\end{pmatrix} \tag{2.15}$$

and the inverse of $(\rho_v, v)^t$ is $(\rho_v, -v)^t$.

Thomas gyration in $\mathbb{R}^+\times(G,\oplus)$ is given in terms of its application to any $(p,w)^t\in\mathbb{R}^+\times(G,\oplus)$ by

$$\mathrm{gyr}\Big[\begin{pmatrix}s\\ u\end{pmatrix};\begin{pmatrix}t\\ v\end{pmatrix}\Big]\begin{pmatrix}p\\ w\end{pmatrix} = \Big(\begin{pmatrix}s\\ u\end{pmatrix}\begin{pmatrix}t\\ v\end{pmatrix}\Big)^{-1}\Big(\begin{pmatrix}s\\ u\end{pmatrix}\Big(\begin{pmatrix}t\\ v\end{pmatrix}\begin{pmatrix}p\\ w\end{pmatrix}\Big)\Big) = \begin{pmatrix}p\\ \mathrm{gyr}[u;v]w\end{pmatrix} \tag{2.16}$$

and, accordingly, Thomas gyration in B is given by

$$\mathrm{gyr}\Big[\begin{pmatrix}\rho_u\\ u\end{pmatrix};\begin{pmatrix}\rho_v\\ v\end{pmatrix}\Big]\begin{pmatrix}\rho_w\\ w\end{pmatrix} = \begin{pmatrix}\rho_w\\ \mathrm{gyr}[u;v]w\end{pmatrix} = \begin{pmatrix}\rho_{\mathrm{gyr}[u;v]w}\\ \mathrm{gyr}[u;v]w\end{pmatrix} \tag{2.17}$$

where the last equality follows from the invariance of ρ under gyrations, Eq. (2.10).

The subgyrogroup B of the gyrogroup $\mathbb{R}^+\times(G,\oplus)$, is called the abstract Lorentz gyrogroup, and its elements are called abstract Lorentz-boosts. We will see in Sect. 8 that when the abstract gyrogroup $(G,\oplus)$ with an abstract Lorentz factor ρ is realized by Einstein gyrogroup $(V_c,\oplus_E)$ with the common Lorentz factor γ of STR, B reduces to the gyrogroup of the common Lorentz-boosts (that is, Lorentz transformations without rotations) in STR. This, in turn, gives rise to the common Lorentz group of STR by means of the gyrosemidirect product (see Sect. 3) of B with a rotation group.

The gyrogroups in Examples 2.1-2.4 and 2.6 as well as the real counterpart of the gyrogroup in Example 2.5 are all gyrocommutative. In contrast, the complex Lorentz-boost gyrogroup ($\mathbb{C}^+\times V_c,\ \oplus_B$) of Example 2.5 is non-gyrocommutative.

Since a gyrogroup $(G,+)$ involves a gyroautomorphism operation, gyr, it seems on first glance that it would be appropriate to characterize gyrogroups by triples $(G,+,\mathrm{gyr})$ rather than pairs $(G,+)$. However, we will see in Eq. (ix) of Theorem 3.1 that the gyroautomorphisms $\mathrm{gyr}[x;y]$, $x,y\in G$, of a gyrogroup $(G,+)$ are uniquely determined by the gyrogroup operation +, as has already been indicated in Eqs. (2.3d), (2.4b), (2.6d) and (2.16)-(2.17).

The hyperbolic (or, Lobachevskian) structure of the velocity space geometry in STR, now recognized in Example 2.2 as a gyrogroup, had been established in the early years of STR by Varićak and others. The scientific activity of Vladimir Varićak (1865-1942), professor of mathematics at the University of Zagreb, was mainly in non-Euclidean geometry and its applications to Einstein's theory of relativity. His work has been cited in Wolfgang Pauli's ‘‘Relativitatstheorie’’.[50] In fact, all the gyrogroups in Examples 2.1-2.6 are linked with various models of hyperbolic geometry as we will see in the sequel. Remarkably, the whole-space gyrogroup of proper velocities in Example 2.4 gives a new model of hyperbolic geometry whose underlying set is the whole of a vector space rather than its half space or its ball as in standard models.

The usefulness and elegance of proper velocities in STR, as well as analogies they share with prerelativity velocities is emphasized by Fraundorf,[45] and again, but in a totally different context, in Section 7. The term *proper velocity* is somewhat misleading, as Moore has recently pointed out.[51] A more appropriate term would be *traveler's velocity* as measured by *traveler's time*, that is, by time shown on a clock that moves with the traveler. Both Fraundorf results and ours indicate the superiority of the study of STR by means of proper velocities rather than ordinary ones. Fraundorf approach to this conclusion and our approach are, however, totally different. Fraundorf studies STR in 1+1 spacetime dimensions where Thomas precession effects do not exist (since all velocities are parallel) while we study STR in 1+n dimensions, n being finite or infinite, where Thomas gyration plays a central role. It is thus interesting to realize that two totally different approaches indicate that in terms of analogies shared by relativistic and prerelativistic physics, the study of STR by means of proper velocities is simpler and more elegant.

Gyrogroup structures are not rare. In fact, Friedman and Ungar have shown that one of the two factors in some Cartan decompositions of Lie groups is a gyrocommutative gyrogroup (the other factor is a group).[20] The three gyrogroups in Examples 2.2-2.4, (i) the E-gyrogroup, (ii) the M-gyrogroup, and (iii) the W-gyrogroup, are equivalent to each other when they share their underlying vector spaces and its balls. The E-M gyrogroup equivalence has been shown in Eqs. (2.7), and the E-W gyrogroup equivalence has been shown in Eqs. (2.9). The latter equivalence has a profound implication on Function Theory in the Unit Ball in $\mathbb{C}^n$.[52]

Following Rudin,[52] a basic family of automorphisms of the ball B of any complex Hilbert space in Function Theory in the Unit Ball is $\phi_{\mathbf{a}}$, $a \in B$, given by (Ref. 52 p. 25),

$$\phi_{\mathbf{a}}(-\mathbf{z}) \;=\; \frac{1}{1+\bar{\mathbf{a}}\cdot\mathbf{z}} \left\{ [\,1+\frac{\bar{\mathbf{a}}\cdot\mathbf{z}}{\|\mathbf{a}\|^2}\,(1-\sqrt{1-\|\mathbf{a}\|^2})]\,\mathbf{a} + \sqrt{1-\|\mathbf{a}\|^2}\,\mathbf{z} \right\} \tag{2.18}$$

which is, in fact, Einstein's addition law (2.3a) in disguise,[18]

$$\phi_{\mathbf{a}}(-\mathbf{z}) \;=\; \mathbf{a} \oplus_E \mathbf{z} \tag{2.19}$$

Rudin mentions in Section 2.2.9 of Ref. 52 that automorphisms of B in a two dimensional complex inner product space $\mathbb{C}^2$ have already been studied by Poincaré in 1907. He however emphasizes that he does not know when they were first written down for $n > 2$ dimensions. Being now identified as complex relativistically admissible velocity additions given by Einstein's addition law (2.3a), it seems that it was Einstein's 1905 paper[53] that triggered interest in the automorphisms $\phi_{\mathbf{a}}$ of unit balls.

In gyrogroup terms the automorphisms $\phi_{\mathbf{a}}$ of the ball B are left gyrotranslations of the ball, Eq. (2.19), where the ball itself is an Einstein gyrogroup. Function Theory in the Unit Ball can thus be identified with a branch of gyrogroup theory, that is, with the theory of E-gyrogroups and their hyperbolic geometry. Moreover, since E-gyrogroups are equivalent to corresponding W-gyrogroups, the theory of an E-gyrogroup of a ball is equivalent to the theory of the W-gyrogroup of the whole of the space containing the ball. Hence, the term *Function Theory in the Unit Ball* is no longer justified being equivalent to Function Theory in the whole of the space containing the Unit Ball. Undoubtedly, following the publication of the present article, it would be more appropriate and less misleading to refer Function Theory in the Unit Ball to as a branch of gyrogroup theory.

3. Some Elementary Gyrogroup Theorems

THEOREM 3.1 Let $(G,+)$ be any gyrogroup and let I be the identity automorphism of G. Then, in G, for any elements $a, b, c, z \in G$ we have:

(i) If $a+b = a+c$ then $b=c$ (Left Cancellation Law).

(ii) $\mathrm{gyr}[0;a]=I$ for any left identity 0 in G.

(iii) $\mathrm{gyr}[x;a]=I$ for any left inverse x of a in G.

(iv) $\mathrm{gyr}[a;a]=I$

(v) There is a left identity which is a right identity.

(vi) There is only one left identity.

(vii) Every left inverse is a right inverse.

(viii) There is only one left inverse of a.

(ix) $\mathrm{gyr}[a;b]z = -(a+b) + \{a+(b+z)\}$

Proof

(i) Let x be a left inverse of a corresponding to a left identity, 0, in G. We have $x+(a+b) = x+(a+c)$. By left gyroassociativity, $(x+a)+\mathrm{gyr}[x;a]b = (x+a)+\mathrm{gyr}[x;a]c$. Since 0 is a left identity, $\mathrm{gyr}[x;a]b = \mathrm{gyr}[x;a]c$. Since automorphisms are bijective, $b=c$.

(ii) By left gyroassociativity we have for any left identity 0 of G, $a+x = 0+(a+x) = (0+a)+\mathrm{gyr}[0;a]x = a+\mathrm{gyr}[0;a]x$. By left cancellation we then have $x=\mathrm{gyr}[0;a]x$ for all $x \in G$ so that $\mathrm{gyr}[0;a]=I$.

(iii) By the left loop property and by (ii) above we have $\mathrm{gyr}[x;a]=\mathrm{gyr}[x+a;a]=\mathrm{gyr}[0;a]=I$.

(iv) Follows from an application of the left loop property to (ii) above.

(v) Let x be a left inverse of a corresponding to a left identity, 0, of G. Then, by left gyroassociativity and (iii) above, $x+(a+0) = (x+a)+\mathrm{gyr}[x;a]0=0+0=0=x+a$. Hence, by left cancellation, $a+0=a$ for all $a \in G$ so that 0 is a right identity.

(vi) Suppose 0 and $0'$ are two left identities, one of which, say 0, is also a right identity. Then $0=0'+0=0'$.

(vii) Let x be a left inverse of a. Then $x+(a+x) = (x+a)+\mathrm{gyr}[x;a]x = 0+x=x=x+0$, by left gyroassociativity, (G2), (iii), (v), and (vi) above. By left cancellation we have $a+x=0$ so that x is a right inverse of a.

(viii) Suppose x and y are left inverses of a. By (vii) above, they are also right inverses, so $a+x=0=a+y$. By left cancellation, $x=y$.

(ix) Follows from an application of a left cancellation to the left gyroassociative law (G3).

THEOREM 3.2 Any three elements a, b, c of a gyrogroup $(G, +)$ satisfy the identities

$$\mathrm{gyr}[a; b+c]\,\mathrm{gyr}[b; c] = \mathrm{gyr}[a+b; \mathrm{gyr}[a; b]c]\,\mathrm{gyr}[a; b] \tag{3.1}$$

$$\mathrm{gyr}[a; -\mathrm{gyr}[a; b]b]\,\mathrm{gyr}[a; b] = I \tag{3.2}$$

and

$$\mathrm{gyr}[-a; a+b]\,\mathrm{gyr}[a; b] = I \tag{3.3}$$

Proof By two successive applications of the left gyroassociative law in two different ways we obtain the following two chains of equations for all $a, b, c, x \in G$,

$$\begin{aligned} a+(b+(c+x)) &= a+((b+c)+\mathrm{gyr}[b; c]x) \\ &= (a+(b+c)) + \mathrm{gyr}[a; b+c]\,\mathrm{gyr}[b; c]x \end{aligned}$$

and

$$\begin{aligned} a+(b+(c+x)) &= (a+b) + \mathrm{gyr}[a; b](c+x) \\ &= (a+b) + (\mathrm{gyr}[a; b]c + \mathrm{gyr}[a; b]x) \\ &= ((a+b) + \mathrm{gyr}[a; b]c) + \mathrm{gyr}[a+b; \mathrm{gyr}[a; b]c]\,\mathrm{gyr}[a; b]x \\ &= (a+(b+c)) + \mathrm{gyr}[a+b; \mathrm{gyr}[a; b]c]\,\mathrm{gyr}[a; b]x \end{aligned}$$

By comparing the extreme right-hand sides of these two chains of equations, and by employing the left cancellation law we obtain (3.1).

In the special case when $c = -b$ (3.1) reduces to

$$I = \mathrm{gyr}[a+b; -\mathrm{gyr}[a; b]b]\,\mathrm{gyr}[a; b]$$

from which (3.2) follows by applying the left loop property and the left gyroassociative law,

$$\begin{aligned} I &= \mathrm{gyr}[(a+b)-\mathrm{gyr}[a; b]b; -\mathrm{gyr}[a; b]b]\,\mathrm{gyr}[a; b] \\ &= \mathrm{gyr}[a+(b-b); -\mathrm{gyr}[a; b]b]\,\mathrm{gyr}[a; b] \\ &= \mathrm{gyr}[a; -\mathrm{gyr}[a; b]b]\,\mathrm{gyr}[a; b] \end{aligned}$$

thus verifying (3.2).

To verify (3.3) let us consider the special case of (3.1) when $b = -a$,

$$\mathrm{gyr}[a; -a+c]\,\mathrm{gyr}[-a; c] = \mathrm{gyr}[0; \mathrm{gyr}[a; -a]c]\,\mathrm{gyr}[a; -a] = I$$

Replacing a by $-a$ and c by b we obtain (3.3) and the proof is complete.

THEOREM 3.3 For any two elements a, b of a gyrogroup G and any automorphism A of G

$$A\,\mathrm{gyr}[a; b] = \mathrm{gyr}[Aa; Ab]A \tag{3.4}$$

and

$$A\,\mathrm{gyr}^{-1}[a; b] = \mathrm{gyr}^{-1}[Aa; Ab]A \tag{3.5}$$

where $\mathrm{gyr}^{-1}[a; b]$ is the automorphism inverse to $\mathrm{gyr}[a; b]$.

Proof For any three elements $a, b, x \in (G, +)$ and any automorphism $A \in \mathrm{Aut}(G, +)$ we have by the left gyroassociative law,

$$\begin{aligned}(Aa + Ab) + A\,\mathrm{gyr}[a; b]x &= A((a+b) + \mathrm{gyr}[a; b]x) \\ &= A(a + (b + x)) \\ &= Aa + (Ab + Ax) \\ &= (Aa + Ab) + \mathrm{gyr}[Aa; Ab]Ax\,.\end{aligned}$$

Hence, by the left cancellation law we have

$$A\,\mathrm{gyr}[a; b]x = \mathrm{gyr}[Aa; Ab]Ax$$

for all $x \in G$ thus verifying (3.4). The second identity in the Theorem follows from the first one by automorphism inversion, and the proof is complete.

As an application of Theorem 3.3 we prove Theorems 3.4 and 5.11 below.

THEOREM 3.4 Let a, b be any two elements of a gyrogroup $(G, +)$ and let $A \in \mathrm{Aut}(G)$ be an automorphism of G. Then

$$\mathrm{gyr}[a; b] = \mathrm{gyr}[Aa; Ab]$$

if and only if the automorphisms A and $\mathrm{gyr}[a; b]$ commute.

Proof If $\mathrm{gyr}[Aa; Ab] = \mathrm{gyr}[a; b]$ then, by Theorem 3.3, the automorphisms $\mathrm{gyr}[a; b]$ and A commute. Conversely, if $\mathrm{gyr}[a; b]$ and A commute then by Theorem 3.3 $\mathrm{gyr}[Aa; Ab] = A\,\mathrm{gyr}[a; b]A^{-1} = \mathrm{gyr}[a; b]$, as desired.

As a simple, but useful, consequence of Theorem 3.4 we note the identity

$$\mathrm{gyr}[\mathrm{gyr}[a; b]a\,;\, \mathrm{gyr}[a; b]b] = \mathrm{gyr}[a; b]$$

4. The Gyrosemidirect Product Group

The gyrogroup axioms allow any gyrogroup to be embedded as a subset of an associated group (by means of the left gyroassociative law) and to reduce group manipulations into gyrogroup ones (by means of the left loop property). Gyrogroup theory can thus be understood by means of group theory. The associated groups are called *gyrosemidirect product groups.*

DEFINITION 4.1 *(Gyrosemidirect product groups)* Let $G=(G,+)$ be a gyrogroup, let $\mathrm{Aut}(G)=\mathrm{Aut}(G,+)$ be the automorphism group of G, and let $\mathrm{Aut}_0(G)$ be any gyroautomorphism group of G, that is, a subgroup of $\mathrm{Aut}(G)$ containing all the gyroautomorphisms $\mathrm{gyr}[a;b]$ of G. The *gyrosemidirect product group*

$$G \times \mathrm{Aut}_0(G) \tag{4.1}$$

between the gyrogroup G and the group $\mathrm{Aut}_0(G)$ is a group of pairs (x,X), where $x\in G$ and $X\in\mathrm{Aut}_0(G)$, with group operation given by the *gyrosemidirect product*

$$(x,X)(y,Y) \;=\; (x+Xy,\, \mathrm{gyr}[x;Xy]XY)\,. \tag{4.2}$$

By viewing G and $\mathrm{Aut}_0(G)$, respectively, as a gyrogroup of pairs (x,I) and a group of pairs $(0,X)$, where I is the identity automorphism of G and 0 is the identity element of G, we (i) identify the group $\mathrm{Aut}_0(G)$ with a subgroup of $G\times\mathrm{Aut}_0(G)$; and (ii) identify the gyrogroup G with a subset of $G\times\mathrm{Aut}_0(G)$ which is normalized by $\mathrm{Aut}_0(G)$, that is, $A\in\mathrm{Aut}_0(G)$ and $g\in G \Rightarrow AgA^{-1}\in G$. Indeed, $AgA^{-1} = (0,A)(g,I)(0,A^{-1}) = (0+Ag,\mathrm{gyr}[0;Ag]AA^{-1}) = (Ag,I) = Ag\in G$.

It is anticipated in Definition 4.1 that the gyrosemidirect product group (4.1) is a group with group operation given by the gyrosemidirect product (4.2). To verify that the gyrosemidirect product (4.2) is indeed a group operation we view the elements (x,X) of $G\times\mathrm{Aut}_0(G)$ as bijections of G according to the equation

$$(x,X)g \;=\; x+Xg$$

for any $x,g\in G$ and $X\in\mathrm{Aut}_0(G)$, where the identity bijection is the pair $(0,I)$. To show that the elements of $G\times\mathrm{Aut}_0(G)$ indeed act bijectively on G one may show that, by Theorem 3.1 (iii) and by the left loop property, $(-X^{-1}x, X^{-1})$ is a right inverse of $(x,X)\in G\times\mathrm{Aut}_0(G)$. As such, the set $G\times\mathrm{Aut}_0(G)$ forms a group of bijections with group operation given by bijection composition,

$$\begin{aligned}(x,X)(y,Y)g &= (x,X)(y+Yg)\\ &= x+X(y+Yg)\\ &= x+(Xy+XYg)\\ &= (x+Xy)+\mathrm{gyr}[x;Xy]XYg\\ &= (x+Xy,\mathrm{gyr}[x;Xy]XY)g\,.\end{aligned}$$

The bijection composition turns out to be the gyrosemidirect product (4.2) in $G\times\mathrm{Aut}_0(G)$ thus verifying that the gyrosemidirect product (4.2) is indeed a group operation in the gyrosemidirect product group (4.1). We may note that the bijection composition law results from an application of the left gyroassociative law. It is this law that allows the extension of any gyrogroup into a group, as we remarked in the first paragraph of this section.

In the special case when Thomas gyration vanishes, $\mathrm{gyr}[x;y]=I$ for all $x,y\in G$, the gyrosemidirect product (4.2) between a gyrogroup and a group reduces to the common semidirect product

$$(x,X)(y,Y) \;=\; (x+Xy,\; XY). \tag{4.3}$$

between two groups.

In the following theorem we will use the obvious notation $a+(-b)=a-b$ in gyrogroups.

THEOREM 4.1 Let $(G,+)$ be a gyrogroup, let $a,b\in G$ be any two given elements of G, and let $R\in\mathrm{Aut}(G,+)$ be any given automorphism of $(G,+)$. Then, the unique solution of the equation

$$x \;+\; a \;=\; b \tag{4.4a}$$

in G for the unknown x is

$$x \;=\; b - \mathrm{gyr}[b;a]a\,, \tag{4.4b}$$

and the unique solution of the equation

$$R \;=\; \mathrm{gyr}[b;Xa]X \tag{4.5a}$$

in $\mathrm{Aut}(G,+)$ for the unknown $X\in\mathrm{Aut}(G,+)$ is

$$X \;=\; \mathrm{gyr}[b;-Ra]R \tag{4.5b}$$

Proof Let

$$G_s \;=\; G\times\mathrm{Aut}(G)$$

be the gyrosemidirect product of a gyrogroup $G=(G,+)$ and its automorphism group $\mathrm{Aut}(G)=\mathrm{Aut}(G,+)$.

The two equations in G_s

$$(x,X)(a,A) \;=\; (b,B) \tag{4.6}$$

and

$$(x,X) \;=\; (b,B)(-A^{-1}a,A^{-1}) \tag{4.7}$$

are equivalent since $(-A^{-1}a,A^{-1}) = (a,A)^{-1}$ is the inverse of (a,A) in G_s.

Expanding the gyrosemidirect product in (4.6) we have

$$\begin{aligned}(b,B) &= (x,X)(a,A)\\ &= (x+Xa,\,\mathrm{gyr}[x;Xa]XA)\end{aligned}$$

implying

$$b \;=\; x \;+\; Xa \tag{4.8}$$

and, by means of the left loop property,

$$
\begin{aligned}
B &= \mathrm{gyr}[x\,; Xa\,]\,XA \\
&= \mathrm{gyr}[x + Xa\,; Xa\,]\,XA \\
&= \mathrm{gyr}[b\,; Xa\,]\,XA\,.
\end{aligned} \tag{4.9}
$$

It is the application of the left loop property in (4.9) that allows us to reduce a gyrogroup manipulation into a group one, as we remarked in the first paragraph of this section. The expansion of the gyrosemidirect product in (4.6) thus gives rise to the system (4.10) below of an equation in G and an equation in $\mathrm{Aut}(G)$,

$$
\begin{aligned}
x + Xa &= b \\
R &= \mathrm{gyr}[b\,; Xa\,]\,X
\end{aligned} \tag{4.10}
$$

where $R = BA^{-1}$. Similarly, expanding the gyrosemidirect product in (4.7) and denoting $R = BA^{-1}$ we have

$$
\begin{aligned}
(x\,, X\,) &= (b\,, B\,)\,(-A^{-1}a\,, A^{-1}) \\
&= (b - BA^{-1}a\,, \mathrm{gyr}[b\,; -BA^{-1}a\,]\,BA^{-1}\,) \\
&= (b - Ra\,, \mathrm{gyr}[b\,; -Ra\,]\,R\,)
\end{aligned}
$$

thus giving rise to the system (4.11) below of an equation in G and an equation in $\mathrm{Aut}(G)$,

$$
\begin{aligned}
x &= b - Ra \\
X &= \mathrm{gyr}[b\,; -Ra\,]\,R
\end{aligned} \tag{4.11}
$$

Since (4.6) and (4.7) are equivalent, and since (i) (4.10) is equivalent to (4.6) and (ii) (4.11) is equivalent to (4.7), the systems (4.10) and (4.11) are equivalent. The second equation in each of the two equivalent systems (4.10) and (4.11) is independent of x. These are

$$
R = \mathrm{gyr}[b\,; Xa\,]\,X \tag{4.12}
$$

$$
X = \mathrm{gyr}[b\,; -Ra\,]\,R \tag{4.13}
$$

and, being equivalent, can be considered as a solution of one another: (4.13) is a solution of (4.12) for the unknown X, and (4.12) is a solution of (4.13) for the unknown R. Moreover, subject to the equivalence between (4.12) and (4.13) also the first equation in each of the systems (4.10) and (4.11) are equivalent. Thus, subject to the relationship between R and X in (4.12)-(4.13), the following equations (4.14) and (4.15) in G are equivalent,

$$
x + X\,a = b \tag{4.14}
$$

$$
x = b - R\,a\,, \tag{4.15}
$$

that is, one of them is valid if and only if the other one is valid. Clearly, (4.15) can be viewed as the unique solution of (4.14) for the unknown x, where X and R in (4.14)-(4.15) are interrelated by (4.12)-(4.13). Particularly, for $X = I$ (4.4b) is the unique solution of (4.4a). The proof is thus complete.

It is useful to define a second binary operation, $\boxplus$, in a gyrogroup $(G\,, +)$ by the equation

$$
a \boxplus b = a + \mathrm{gyr}[a\,; -b\,]\,b \tag{4.16}
$$

so that

$$a \boxminus b \;=\; a \boxplus (-b) \;=\; a - \mathrm{gyr}[a\,;b\,]\,b \tag{4.17}$$

in terms of which the solution (4.4b) of equation (4.4a) can be written as $x = b \boxminus a$. The importance of the "*square*" *binary operation* $\boxplus$ in $(G,+)$ lies in the fact that it gives rise to two *right cancellation laws.* By Eqs. (4.4a), (4.4b) and (4.17) we have

$$(b \boxminus a) + a \;=\; b\,. \tag{4.18a}$$

A slightly different right cancellation law is given by the equation

$$(b - a) \boxplus a \;=\; b \tag{4.18b}$$

which follows from the left loop property and the left gyroassociative law,

$$\begin{aligned}(b-a) \boxplus a \;&=\; (b-a) + \mathrm{gyr}[b-a\,;-a\,]\,a\\ &=\; (b-a) + \mathrm{gyr}[b\,;-a\,]\,a\\ &=\; b + (-a+a)\\ &=\; b\end{aligned}$$

Moreover, the square binary operation $\boxplus$ in a gyrocommutative gyrogroup is commutative as we will see in Theorem 5.10.

The square binary operation $\boxplus$ in (4.16) pairs elegantly with its gyrogroup operation as we see in the following

THEOREM 4.2 Let $(G,\oplus)$ be a gyrogroup and let $\boxplus$ be a second binary operation in G given by

$$a \boxplus b \;=\; a \oplus \mathrm{gyr}[a\,;-b\,]b \tag{4.19a}$$

Then

$$a \oplus b \;=\; a \boxplus \mathrm{gyr}[a\,;b\,]b \tag{4.19b}$$

and

$$\mathrm{Aut}(G,\oplus) \;=\; \mathrm{Aut}(G,\boxplus) \tag{4.20}$$

Proof Let a and b be any two elements of G. By (3.2) we have

$$\mathrm{gyr}[a\,;-gyr\,[a\,;b\,]b\,] \;=\; \mathrm{gyr}^{-1}[a\,;b\,]$$

where $\mathrm{gyr}^{-1}[a\,;b\,]$ is the inverse automorphism of $\mathrm{gyr}[a\,;b\,]$. Hence, by (4.19a),

$$\begin{aligned}a \boxplus \mathrm{gyr}[a\,;b\,]b \;&=\; a \oplus \mathrm{gyr}[a\,;-\mathrm{gyr}[a\,;b\,]b\,]\,\mathrm{gyr}[a\,;b\,]b\\ &=\; a \oplus \mathrm{gyr}^{-1}[a\,;b\,]\mathrm{gyr}[a\,;b\,]b\\ &=\; a \oplus b\end{aligned}$$

thus verifying (4.19b).

Let $\tau \in \mathrm{Aut}(G, \oplus)$. Then, by (3.4) in Theorem 3.3,

$$\begin{aligned} \tau(a \boxplus b) &= \tau(a \oplus \mathrm{gyr}[a; -b]b) \\ &= \tau a \oplus \tau\, \mathrm{gyr}[a; -b]b \\ &= \tau a \oplus \mathrm{gyr}[\tau a; -\tau b]\tau b \\ &= \tau a \boxplus \tau b \end{aligned}$$

so that $\tau \in \mathrm{Aut}(G, \boxplus)$. Conversely, let $\tau \in \mathrm{Aut}(G, \boxplus)$. Then, similarly,

$$\begin{aligned} \tau(a \oplus b) &= \tau(a \boxplus \mathrm{gyr}[a; b]b \\ &= \tau a \boxplus \tau\, \mathrm{gyr}[a; b]b \\ &= \tau a \boxplus \mathrm{gyr}[\tau a; \tau b]\tau b \\ &= \tau a \oplus \tau b \end{aligned}$$

so that $\tau \in \mathrm{Aut}(G, \oplus)$, as desired.

The two binary operations $\oplus$ and $\boxplus$ in a gyrogroup $(G, \oplus)$ prove useful in the study of gyrovector spaces in Section 7 where they give rise to two natural metrics and to their corresponding interesting geodesics; see e.g. Eqs. (7.9) - (7.10). The metric resulting from $\oplus$ in the Poincaré disk model of hyperbolic geometry, called the Poincaré metric, is well known.[22] It has gone unnoticed however, that there exists a second natural metric which is useful in the study of hyperbolic geometry.

REMARK 4.1 According to (4.16) the square binary operation in Einstein gyrogroup $(V_c, \oplus_E)$ is $\boxplus_E$ given by

$$a \boxplus_E b = a \oplus_E \mathrm{gyr}[a; -b]b \,. \tag{4.21}$$

For later reference we remark here that in the special case when V_c is realized by $\mathbb{R}_c = (-c, c)$ Thomas gyrations vanish (that is, all the gyrations reduce to the identity automorphism), and the binary operation $\boxplus_E$ reduces to the binary operation $\oplus_E$ discussed in Remark 2.1.

5. The Study of Gyrogroups by means of their Gyrosemidirect Product groups

We demonstrate in this section the use of group theoretic considerations to obtain results in gyrogroup theory by means of the gyrosemidirect product.

THEOREM 5.1 A gyrogroup $(G,+)$ contains only one identity element. It is both a right and a left identity element.

Proof Following Section 3, the gyrosemidirect product $G\times\mathrm{Aut}_0(G)$ of the gyrogroup G and a gyroautomorphism group $\mathrm{Aut}_0(G)$ forms a group of bijections of G. The left identity element, $(0,I)$, of any group must also be a right identity element,

$$\begin{aligned}(a,A) &= (a,A)(0,I)\\ &= (a+A\,0, \mathrm{gyr}[a;A\,0]A\,I)\\ &= (a+0, \mathrm{gyr}[a;0]A).\end{aligned}$$

Hence, for all $a\in G$ we have

$$a = a+0 \qquad (5.1)$$

and

$$\mathrm{gyr}[a;0] = I. \qquad (5.2)$$

Eq. (5.1) completes the proof.

While it is more elegant to verify gyrogroup theorems within the frame of gyrogroup theory, as we did in Theorem 3.1, the proof of Theorem 5.1 within the frame of group theory indicates that the verification of gyrogroup theorems by means of gyrosemidirect product groups is relatively simple, and rewarding. It is relatively simple since well known group theoretic considerations can be employed (e.g., in the proof of Theorem 5.1 we used the fact that in a group a left identity element must be a right identity element); and it is rewarding since the exposition of a desired result in a gyrogroup G, like (5.1), exposes a companion result in the corresponding automorphism group $\mathrm{Aut}(G)$, like (5.2). We therefore prefer in this section to verify gyrogroup theorems by means of gyrosemidirect product groups and group theoretic considerations.

THEOREM 5.2 If a and x are two elements of a gyrogroup $(G,+)$ such that $x+a=0$, then $a+x=0$. Further, there is no element in the gyrogroup, other than x, that satisfies either of these equations.

Proof Let $\mathrm{Aut}_0(G,+)$ be a gyroautomorphism group of the gyrogroup $(G,+)$. If $-a$ is a left inverse of a in G, then $(-A^{-1}a, A^{-1})$ is a left inverse of (a,A) in $G\times\mathrm{Aut}_0(G,+)$. Since $G\times\mathrm{Aut}_0(G,+)$ is a group, the element $(-A^{-1}a, A^{-1})$ must also be a right inverse of (a,A),

$$\begin{aligned}(0,I) &= (a,A)(-A^{-1}a, A^{-1})\\ &= (a - AA^{-1}a, AA^{-1})\\ &= (a-a\ ,I).\end{aligned}$$

Hence, we have the equation $a-a=0$ so that $-a$ is also a right inverse of a in G. The uniqueness of the inverse $-a$ of a in the gyrogroup G follows from the uniqueness of the inverse $(-A^{-1}a, A^{-1})$ of (a,A) in the gyrosemidirect product group $G\times \mathrm{Aut}_0(G,+)$. The proof is thus complete.

THEOREM 5.3 For any two elements a,b of a gyrogroup $(G,+)$

$$-(a+b) = \mathrm{gyr}[a;b](-b-a) \tag{5.3}$$

and

$$\mathrm{gyr}^{-1}[a;b] = \mathrm{gyr}[-b;-a]. \tag{5.4}$$

Proof The product of two elements of the gyrosemidirect product group $G\times \mathrm{Aut}_0(G,+)$,

$$(a_1,A_1)(a_2,A_2) = (a_1+A_1a_2,\ \mathrm{gyr}[a_1;A_1a_2]A_1A_2) \tag{5.5}$$

has a unique inverse, which can be calculated in two different ways. If we use the notation

$$\begin{aligned}A_2^{-1}\,a_2 &= a\\ A_2^{-1}\,A_1^{-1}a_1 &= b,\end{aligned}$$

then the inverse of the left-hand side of Eq. (5.5) is given by the expression

$$\begin{aligned}(a_2,A_2)^{-1}(a_1,A_1)^{-1} &= (-A_2^{-1}a_2, A_2^{-1})(-A_1^{-1}a_1, A_1^{-1})\\ &= (-A_2^{-1}a_2 - A_2^{-1}\,A_1^{-1}a_1,\ t[-A_2^{-1}a_2; -A_2^{-1}\,A_1^{-1}a_1]A_2^{-1}\,A_1^{-1}) \qquad (5.5a)\\ &= (-a-b,\ \mathrm{gyr}[-a;-b]A_2^{-1}\,A_1^{-1}).\end{aligned}$$

On the other hand, the inverse of the right-hand side of Eq. (5.5) is given by the expression

$$\begin{aligned}&(-A_2^{-1}A_1^{-1}\ \mathrm{gyr}^{-1}[a_1;A_1a_2]\,(a_1+A_1a_2),\ A_2^{-1}A_1^{-1}\ \mathrm{gyr}^{-1}[a_1;A_1a_2])\\ &= (-\mathrm{gyr}^{-1}[A_2^{-1}A_1^{-1}\ a_1;\ A_2^{-1}a_2]A_2^{-1}A_1^{-1}\ (a_1+A_1a_2),\ \mathrm{gyr}^{-1}[A_2^{-1}A_1^{-1}\ a_1; A_2^{-1}a_2]A_2^{-1}A_1^{-1}) \qquad (5.5b)\\ &= (-\mathrm{gyr}^{-1}[b;a](b+a),\ \mathrm{gyr}^{-1}[b;a]A_2^{-1}A_1^{-1}),\end{aligned}$$

which is obtained by a use of Theorem 3.3. These two expressions, (5.5a) and (5.5b) must be identically equal. The equality of their respective entries,

$$-a-b \;=\; -\mathrm{gyr}^{-1}[b\,;a\,](b+a) \tag{5.6}$$

$$\mathrm{gyr}[-a\,;-b\,] \;=\; \mathrm{gyr}^{-1}[b\,;a\,]\,, \tag{5.7}$$

gives the two identities of the Theorem, as desired.

The first identity of Theorem 5.3 is viewed as a gyrosum inversion rule as indicated by Eq. (5.17) below. We may remark that the proof can be simplified by selecting $A_1=A_2=I$, I being the identity automorphism. Instructively, however, we preferred in this article to show how the proof works harmoniously for any two automorphisms A_1 and A_2 giving us an opportunity to employ Theorem 3.3.

THEOREM 5.4 Let $(G\,,+)$ be a gyrogroup. Then for all $a\,,b \in G$

$$\mathrm{gyr}^{-1}[a\,;b\,] \;=\; \mathrm{gyr}[a\,;-\mathrm{gyr}[a\,;b\,]\,b\,] \tag{5.8}$$

$$\mathrm{gyr}^{-1}[a\,;b\,] \;=\; \mathrm{gyr}[-a\,;a+b\,] \tag{5.9}$$

$$\mathrm{gyr}^{-1}[a\,;b\,] \;=\; \mathrm{gyr}[b\,;a+b\,] \tag{5.10}$$

$$\mathrm{gyr}[a\,;b\,] \;=\; \mathrm{gyr}[b\,;-b-a\,] \tag{5.11}$$

$$\mathrm{gyr}[a\,;b\,] \;=\; \mathrm{gyr}[-a\,;-b-a\,] \tag{5.12}$$

$$\mathrm{gyr}[a\,;b\,] \;=\; \mathrm{gyr}[-(a+b);a\,] \tag{5.13}$$

Proof Identity (5.8) follows from (3.2). Identity (5.9) follows from (3.3). Identity (5.10) follows from an application to (5.9) of the left loop property followed by a left cancellation. Identity (5.11) follows from (5.4) and (5.9),

$$\mathrm{gyr}[a\,;b\,] \;=\; \mathrm{gyr}^{-1}[-b\,;-a\,] \;=\; \mathrm{gyr}[b\,;-b-a\,]\,.$$

Identity (5.12) follows from an application to the right hand side of (5.11) of the left loop property followed by a left cancellation. Identity (5.13) follows by inverting (5.9) by means of (5.4).

THEOREM 5.5 *(The Inverse Gyroautomorphism Law)* Let $(G\,,+)$ be a gyrogroup. Then for all $a\,,b \in G$

$$\mathrm{gyr}[a\,;b\,] \;=\; \mathrm{gyr}[-a\,;-b\,] \tag{5.14}$$

and

$$\mathrm{gyr}^{-1}[a\,;b\,] \;=\; \mathrm{gyr}[b\,;a\,] \tag{5.15}$$

Proof By (5.4) and the left loop property we have

$$\begin{aligned}\mathrm{gyr}^{-1}[a\,;b\,] &= \mathrm{gyr}[-b\,;-a\,]\\ &= \mathrm{gyr}[-b-a\,;-a\,]\\ &= \mathrm{gyr}^{-1}[a\,;-(-b-a\,)]\end{aligned}$$

implying

$$\mathrm{gyr}[a\,;b\,] \;=\; \mathrm{gyr}[a\,;-(-b-a\,)]$$

which, by means of (5.12), results in the equation

$$\mathrm{gyr}[a\,;-(-b-a\,)] \;=\; \mathrm{gyr}[-a\,;-b-a\,]$$

for all $a\,,b \in G$. If for any given $c \in G$ we select b to be the unique solution of the equation $-b-a=-c$, by Theorem 4.1, then the resulting equation can be written as

$$\mathrm{gyr}[a\,;c\,] \;=\; \mathrm{gyr}[-a\,;-c\,]$$

for all $a\,,c \in G$, thus verifying (5.14). Finally, (5.15) follows from (5.4) and (5.14), and the proof is complete.

THEOREM 5.6 Let $(G\,,+)$ be a gyrogroup. Then, for any $a\,,b \in G$ we have

$$a \boxplus b \;=\; -\{(-b\,) \boxplus (-a\,)\} \tag{5.16}$$

Proof To verify (5.16) we note that by Eqs. (4.16), (5.3) and (5.8) we have

$$\begin{aligned}
a \boxplus b \;&=\; a + \mathrm{gyr}[a\,;-b\,]b \\
&=\; \mathrm{gyr}[a\,;\mathrm{gyr}[a\,;-b\,]b\,]\{-(-\mathrm{gyr}[a\,;-b\,]b \;-\; a\,)\} \\
&=\; \mathrm{gyr}[a\,;-\mathrm{gyr}[a\,;-b\,](-b\,)]\{-(-\mathrm{gyr}[a\,;-b\,]b \;-\; a\,)\} \\
&=\; \mathrm{gyr}^{-1}[a\,;-b\,]\{-(-\mathrm{gyr}[a\,;-b\,]b \;-\; a\,)\} \\
&=\; -(-b \;-\; \mathrm{gyr}[b\,;-a\,]a\,) \\
&=\; -\{(-b\,) \boxplus (-a\,)\}
\end{aligned}$$

THEOREM 5.7 Let $(G\,,+)$ be a gyrogroup. Then, for any $a\,,b\,,c \in G$ we have

$-(a+b\,) = \mathrm{gyr}[a\,;b\,](-b-a\,)$	Gyrosum Inversion Rule
$(a+b\,)+c \;= a+(b+\mathrm{gyr}[b\,;a\,]c\,)$	Right Gyroassociative Law
$\mathrm{gyr}[a\,;b\,] = \mathrm{gyr}[a\,;b+a\,]$	Right Loop Property

Proof The gyrosum inversion rule has been verified in Theorem 5.3. The right gyroassociative law follows from the left one and (5.15),

$$\begin{aligned}
a + (b + \mathrm{gyr}[b\,;a\,]c\,) \;&=\; (a+b\,) \;+\; \mathrm{gyr}[a\,;b\,]\mathrm{gyr}[b\,;a\,]c \\
&=\; (a+b\,)+c\;.
\end{aligned}$$

The right loop property follows from (5.10) and (5.15). The proof of the theorem is thus complete.

The group counterpart of the gyrosum inversion rule in Theorem 5.7, written multiplicatively, is

$$(ab\,)^{-1} \;=\; b^{-1}a^{-1} \tag{5.17a}$$

and, written additively, is

$$-(a+b) = -b-a \tag{5.17b}$$

and may be called the product (or, sum in additive notation) inversion rule. Its gyrogroup counterpart, Eq. (5.3), written additively, is therefore called the gyrosum inversion rule.

Groups that underlie vector spaces must be commutative. For analogous reasons, gyrogroups that underlie the gyrovector spaces that will be developed in Section 6 must be gyrocommutative. The following theorem presents an identity that holds in gyrocommutative gyrogroups, which will prove useful in Section 7 in the study of geodesics in various models of hyperbolic geometry by means of their underlying gyrovector spaces.

THEOREM 5.8 Let $(G,+)$ be a gyrocommutative gyrogroup. Then for all $a,b,c \in G$

$$(a+b)-(a+c) = gyr[a;b](b-c). \tag{5.18}$$

Proof We use the notation $g_{a,b} = \mathrm{gyr}[a;b]$ whenever convenient. By the gyrocommutative law, the inverse gyroautomorphism law (5.15) and the left gyroassociative law we have

$$\begin{aligned}(a+b)-a &= g_{a,b}(b+a)-a \\ &= g_{a,b}((b+a)-g_{b,a}a) \\ &= g_{a,b}(b+(a-a)) \\ &= g_{a,b}\,b\,.\end{aligned}$$

Furthermore, by the loop property (right and left) and by left cancellation we have

$$\begin{aligned}\mathrm{gyr}[a+b;-a] &= \mathrm{gyr}[a+b;-a+(a+b)] \\ &= \mathrm{gyr}[a+b\,;\,b] \\ &= g_{a,b}\,.\end{aligned}$$

Hence, finally, by left gyroassociativity and the two identities above we have

$$\begin{aligned}(a+b)-(a+c) &= ((a+b)-a)-\mathrm{gyr}[a+b;-a]c \\ &= g_{a,b}\,b - g_{a,b}\,c \\ &= g_{a,b}(b-c)\end{aligned}$$

as desired.

DEFINITION 5.1 *(Automorphic Inverse Gyrogroups)* A gyrogroup is *automorphic inverse* if it possesses the *automorphic inverse property*

$$-(a+b) = -a-b\,. \tag{5.19}$$

THEOREM 5.9 A gyrogroup is gyrocommutative if and only if it is automorphic inverse.

Proof If $(G,+)$ is automorphic inverse then (5.3) specializes to the gyrocommutative law $(G6)$. Conversely, if $(G6)$ is valid then by (5.3)

$$\mathrm{gyr}[a;b]\{-(-b-a)\} = a+b = \mathrm{gyr}[a;b](b+a).$$

Since $\mathrm{gyr}[a;b]$ is an automorphism, we have $-(-b-a)=(b+a)$ for all $a,b\in G$ as desired.

As an application of Theorem 5.9 we show that the "square" binary operation $\boxplus$ in a gyrocommutative gyrogroup $(G,+)$ is commutative.

THEOREM 5.10

The "square" binary operation $\boxplus$, (4.16), in a gyrocommutative gyrogroup $(g,+)$ is commutative, that is,

$$a \boxplus b = b \boxplus a$$

for any $a,b\in G$.

Proof From the chain of equations in the proof of Theorem 5.6 we have

$$a \boxplus b = -(-b-\mathrm{gyr}[b;-a]a)$$

Being gyrocommutative, the gyrogroup $(G,+)$ is automorphic inverse, Theorem 5.9. Hence, by (4.16) we have

$$a \boxplus b = b+\mathrm{gyr}[b;-a]a = b \boxplus a$$

as desired.

THEOREM 5.11 Any three elements a,b,c of a gyrogroup $(G,+)$ satisfy the identity

$$\mathrm{gyr}[a;b+c]\,\mathrm{gyr}[b;c] = \mathrm{gyr}[a;b]\,\mathrm{gyr}[-(-b-a);c] \tag{5.20}$$

If the gyrogroup $(G,+)$ is gyrocommutative then

$$\mathrm{gyr}[a;b+c]\,\mathrm{gyr}[b;c] = \mathrm{gyr}[a;b]\,\mathrm{gyr}[b+a;c] \tag{5.21}$$

Proof By Eq. (3.1), by the gyrosum inversion rule, and by Theorem 3.3 with $A=\mathrm{gyr}[a;b]$ we have

$$\begin{aligned}\mathrm{gyr}[a;b+c]\,\mathrm{gyr}[b;c] &= \mathrm{gyr}[a+b;\mathrm{gyr}[a;b]c]\,\mathrm{gyr}[a;b] \\ &= \mathrm{gyr}[\mathrm{gyr}[a;b]\{-(-b-a)\};\mathrm{gyr}[a;b]c]\,\mathrm{gyr}[a;b] \\ &= \mathrm{gyr}[a;b]\,\mathrm{gyr}[-(-b-a);c]\end{aligned}$$

thus verifying (5.20), which reduces to (5.21) in a gyrocommutative gyrogroup as desired.

The gyrogroups in Examples 2.1-2.4 are automorphic inverse, while the gyrogroup in Example 2.5 is not. We may note that the elements of the non-gyrocommutative gyrogroup of Example 2.5 obey both (5.20) and

(5.21). In fact, we do not have a concrete example of a gyrogroup whose elements violate (5.21).

Following the gyrogroup theorems that we have established we can now present an alternative, equivalent definition of gyrogroups.

DEFINITION 5.2 *(Gyrogroups - an Alternative Definition)* A groupoid $(G,+)$ is a *gyrogroup* if its binary operation satisfies the following axioms and properties. In G there exists a unique element, 0, called the identity, satisfying

$(g1)$	$0+a = a+0 = a$	Identity

for all $a \in G$. For any $a \in G$ there exists in G a unique inverse, $-a$, satisfying

$(g2)$	$-a+a = a+(-a) = 0$	Inverse

Moreover, if for any $a,b \in G$ the map $\text{gyr}[a;b]$ of G into itself is given by the equation

$$\text{gyr}[a;b]z = -(a+b)+(a+(b+z))$$

for all $z \in G$, the the following hold for all $a,b,c \in G$.

$(g3)$	$\text{gyr}[a;b] \in \text{Aut}(G,+)$	Gyroautomorphism
$(g4a)$	$a+(b+c) = (a+b)+\text{gyr}[a;b]c$	Left gyroassociative Law
$(g4b)$	$(a+b)+c = a+(b+\text{gyr}[b;a]c)$	Right gyroassociative Law
$(g5a)$	$\text{gyr}[a;b] = \text{gyr}[a+b;b]$	Left Loop Property
$(g5b)$	$\text{gyr}[a;b] = \text{gyr}[a;b+a]$	Right Loop Property
$(g6)$	$-(a+b) = \text{gyr}[a;b](-b-a)$	Gyrosum Inversion Rule

A gyrogroup is *gyrocommutative* if it satisfies

$(g7)$	$a+b = \text{gyr}[a;b](b+a)$	Gyrocommutative Law

THEOREM 5.12 Definitions 5.2 and 2.2 of gyrogroups are equivalent.

Proof All the gyrogroup properties in Definition 5.1 are either gyrogroup axioms, defined in Definition 2.2, or results that follow from the axioms, as shown in the present article.

6. Scalar Multiplication - Paving the Way for Gyrovector Spaces

In group theory both commutative and noncommutative groups are considered. Similarly, also in gyrogroup theory both gyrocommutative and non-gyrocommutative gyrogroups should be considered. In contrast, groups that underlie vector spaces are always commutative. By analogy, gyrovector spaces are expected to have underlying gyrocommutative gyrogroups. The gyrogroups of interest in this section are, accordingly, gyrocommutative. They are, specifically,

1. Einstein gyrogroup, $(V_c, \oplus_E)$;
2. Möbius gyrogroup, $(V_c, \oplus_M)$; and
3. the whole-space gyrogroup, $(V, \oplus_W)$.

In particular, therefore, Eq. (5.18) is valid, and the square binary operation $\boxplus$, Eq. (4.16), is commutative, by Theorem 5.10.

Einstein, Möbius, and the whole-space gyrogroups admit scalar multiplication, expressible in terms of the norm of the vector spaces in which they reside. The scalar multiplication, in turn, gives rise to gyrovector spaces that possess amazing analogies with vector spaces.

Scalar multiplication $\odot_E$ in Einstein gyrogroup $(V_c, \oplus_E)$ is given by

$$\begin{aligned} r \odot_E \mathbf{v} &= c \frac{\left(1+\frac{\|\mathbf{v}\|}{c}\right)^r - \left(1-\frac{\|\mathbf{v}\|}{c}\right)^r}{\left(1+\frac{\|\mathbf{v}\|}{c}\right)^r + \left(1-\frac{\|\mathbf{v}\|}{c}\right)^r} \frac{\mathbf{v}}{\|\mathbf{v}\|} \\ &= c \tanh(r \tanh^{-1} \frac{\|\mathbf{v}\|}{c}) \frac{\mathbf{v}}{\|\mathbf{v}\|} \end{aligned} \tag{6.1}$$

where $r \in \mathbb{R}$, $\mathbf{v} \in V_c$, $\mathbf{v} \neq \mathbf{0}$; and by $r \odot_E \mathbf{0} = \mathbf{0}$.

As in Einstein gyrogroup, scalar multiplication $\odot_M$ in Möbius gyrogroup $(V_c, \oplus_M)$ is given by

$$\begin{aligned} r \odot_M \mathbf{v} &= c \frac{\left(1+\frac{\|\mathbf{v}\|}{c}\right)^r - \left(1-\frac{\|\mathbf{v}\|}{c}\right)^r}{\left(1+\frac{\|\mathbf{v}\|}{c}\right)^r + \left(1-\frac{\|\mathbf{v}\|}{c}\right)^r} \frac{\mathbf{v}}{\|\mathbf{v}\|} \\ &= c \tanh(r \tanh^{-1} \frac{\|\mathbf{v}\|}{c}) \frac{\mathbf{v}}{\|\mathbf{v}\|} \end{aligned} \tag{6.2}$$

where $r \in \mathbb{R}$, $\mathbf{v} \in V_c$, $\mathbf{v} \neq \mathbf{0}$; and by $r \odot_M \mathbf{0} = \mathbf{0}$.

Finally, scalar multiplication $\odot_W$ in the whole-space gyrogroup $(V_\infty, \oplus_W)$ is given by

$$\begin{aligned} r \odot_W \mathbf{v} &= \frac{c}{2} \left\{ \left(\sqrt{1+\frac{\|\mathbf{v}\|^2}{c^2}} + \frac{\|\mathbf{v}\|}{c} \right)^r - \left(\sqrt{1+\frac{\|\mathbf{v}\|^2}{c^2}} - \frac{\|\mathbf{v}\|}{c} \right)^r \right\} \frac{\mathbf{v}}{\|\mathbf{v}\|} \\ &= c \sinh(r \sinh^{-1} \frac{\|\mathbf{v}\|}{c}) \frac{\mathbf{v}}{\|\mathbf{v}\|} \end{aligned} \tag{6.3}$$

where $r \in I\!R$, $\mathbf{v} \in V_\infty$, $\mathbf{v} \neq \mathbf{0}$; and by $r \odot_W \mathbf{0} = \mathbf{0}$.

Denoting collectively the two operations in each of the three gyrogroups that we consider in this section by $\oplus$ (gyroaddition) and $\odot$ (scalar gyromultiplication), the resulting three *gyrovector spaces* $(V_c, \oplus_E, \odot_E)$, $(V_c, \oplus_M, \odot_M)$ and $(V_c, \oplus_W, \odot_W)$ possess the following properties. For any element $\mathbf{v}$ of each of these gyrovector spaces, for any integer n, and for any real numbers r, r_1 and r_2 we have

$$\lim_{c \to \infty} r \odot \mathbf{v} = r\mathbf{v} \tag{6.4}$$

$$n \odot \mathbf{v} = \mathbf{v} \oplus \mathbf{v} \oplus \quad \cdots \quad \oplus \mathbf{v} \qquad (n \ terms) \tag{6.5}$$

$$(r_1 + r_2) \odot \mathbf{v} = r_1 \odot \mathbf{v} \oplus r_2 \odot \mathbf{v} \tag{6.6}$$

$$(r_1 r_2) \odot \mathbf{v} = r_1 \odot (r_2 \odot \mathbf{v}) \tag{6.7}$$

$$\|r \odot \mathbf{v}\| = |r| \odot \|\mathbf{v}\| \tag{6.8}$$

The scalar $\|\mathbf{v}\|$ on the right hand side of Eq. (6.8) is viewed as a one-dimensional vector, noting that while $1 \odot \|\mathbf{v}\| = \|\mathbf{v}\|$, in general, $r \odot 1 \neq r$. In fact, if $\mathbf{v} \in \partial V_c$ is an element of the boundary of the open c-ball V_c then $r \odot \|\mathbf{v}\| = c$ for all $r \in I\!R$, where $\odot = \odot_E = \odot_M$.

The operation pair $(\oplus, \odot)$ in Eqs. (6.4)-(6.8), as well as in the rest of this section, is a substitution for each of the following operation pairs: $(\oplus_E, \odot_E)$, $(\boxplus_E, \odot_E)$, $(\oplus_M, \odot_M)$, $(\boxplus_M, \odot_M)$, $(\oplus_W, \odot_W)$, and $(\boxplus_W, \odot_W)$. Clearly, $\odot_E = \odot_M$ as we see from Eqs. (6.1) and (6.2).

Since gyrations are automorphisms, we clearly have $\mathrm{gyr}[\mathbf{a};\mathbf{b}](n \odot \mathbf{v}) = n \odot \mathrm{gyr}[\mathbf{a};\mathbf{b}]\mathbf{v}$ for any two elements $\mathbf{a}$ and $\mathbf{b}$ of the gyrogroup under consideration and any integer n. This remains valid for any real number r as well so that we have

$$\mathrm{gyr}[\mathbf{a};\mathbf{b}]\, r \odot \mathbf{v} = r \odot \mathrm{gyr}[\mathbf{a};\mathbf{b}]\mathbf{v} \tag{6.9}$$

If the abstract gyrogroup $(G, \oplus)$ underlying the abstract Lorentz-boost gyrogroup $(I\!R^+ \times (G, \oplus))$, Example 2.6, admits scalar multiplication then so does the Lorentz-boost gyrogroup $(I\!R^+ \times (G, \oplus))$. Since gyroaddition in $(I\!R^+ \times (G, \oplus))$ is conveniently written multiplicatively, scalar multiplication in $(I\!R^+ \times (G, \oplus))$ is written exponentially. It is given by

$$\begin{pmatrix} t \\ v \end{pmatrix}^r = \begin{pmatrix} \dfrac{\rho_{r \odot v}}{\rho_v^r} t^r \\ \\ r \odot v \end{pmatrix} \tag{6.10}$$

as we see from the composition law (2.11). It satisfies (6.4)-(6.7) written multiplicatively for $\oplus$ and exponentially for $\odot$, where the norm is given by (2.12). It also satisfies a slight modification of (6.8) into (6.11) in which $|r|$ is replaced by r, that is,

$$\left\| \begin{pmatrix} t \\ v \end{pmatrix}^r \right\| = \left\| \begin{pmatrix} t \\ v \end{pmatrix} \right\|^r \tag{6.11}$$

where the power on the right hand side is r rather than $|r|$. In this sense, the Lorentz boost gyrogroup is slightly different from the relativity velocity gyrogroups. In particular, the Lorentz boost gyrogroup with the "scalar multiplication" given by (6.10) is not a gyrovector space.

A gyrovector space is thus a gyrogroup with gyration invariant inner product and scalar multiplication satisfying (6.5)-(6.8). The associative-commutative laws of vector spaces are relaxed in gyrovector spaces by means of Thomas gyration. It is hoped that one day a *gyrodistributive* law connecting $\oplus$ and $\odot$ will be discovered. If exists, it is expected to be the standard distributive law relaxed by Thomas gyration in some unexpected way. In the present article we study gyrovector spaces only in terms of three concrete examples which are not yet fully understood. These are the E-gyrovector spaces, the M-gyrovector spaces and the W-gyrovector spaces. It is too early to approach gyrovector spaces axiomatically at the present state of the art since new discoveries followed by a better understanding of gyrovector spaces are expected. Thus, for instance, in an attempt to extend our concrete examples of gyrovector spaces by abstraction, Oliver Jones has shown that (i) any *Riemannian globally symmetric space of non-compact type*(54) can be given the structure of a gyrovector space, in some sense, whose geodesics are of the form $g(t) = a \oplus t \odot b$ in a gyrovector space $(G, \oplus, \odot)$ with a parameter $t \in \mathbb{R}$; and that (ii) conversely, a metric can be defined on a gyrovector space making it a Riemannian globally symmetric space of non-compact type.(55) The applicability of gyrovector spaces to hyperbolic geometry and to relativistic physics is indicated in the next section.

Let

$$L: \ \mathbf{a} + t\,(\mathbf{b} - \mathbf{a}) \tag{6.12a}$$

be the straight line containing two points, **a** and **b**, in a Euclidean space. The parameter $t \in \mathbb{R}$ in (6.12a) measures length along L in units $\|\mathbf{b} - \mathbf{a}\|$ in the sense that **a** corresponds to $t = 0$ on L, **b** corresponds to $t = 1$ on L, and for all $-\infty < t < \infty$ we have

$$\|\{\mathbf{a} + t\,(\mathbf{b} - \mathbf{a})\} - \mathbf{a}\| \;=\; |t|\,\|\mathbf{b} - \mathbf{a}\| \tag{6.12b}$$

that is, t is the ratio between (i) the distance from **a** to a generic point $\mathbf{a} + t\,(\mathbf{b} - \mathbf{a})$ on L parametrized by t, and (ii) the distance from **a** to **b**. The parameter t in (6.12) is thus an *arc length parameter* relative to the metric

$$d(\mathbf{a}, \mathbf{b}) \;=\; \|\mathbf{a} - \mathbf{b}\|\,. \tag{6.12c}$$

In analogy with the Euclidean lines L in (6.12), and since unlike the commutative binary operation + in a vector space its gyrocommutative counterpart $\oplus$ is not commutative, *gyrolines* take two forms. These are the left and the right gyrolines

$$L_{Left} \ : \quad \mathbf{a} \oplus t \odot (-\mathbf{a} \oplus \mathbf{b}) \tag{6.13a}$$

$$L_{Right} \ : \quad t \odot (\mathbf{b} \boxminus \mathbf{a}) \oplus \mathbf{a} \tag{6.14a}$$

containing the points **a** and **b** corresponding respectively to $t = 0$ and $t = 1$, as one can check by applying left cancellation to (6.13a) with $t = 1$ and by applying right cancellation to (6.14a) with $t = 1$.

The parameter $t \in \mathbb{R}$ in (6.13a) and (6.14a) is an arc length parameter relative to the respective metrics

$$d_{\bigcirc}(\mathbf{a},\mathbf{b}) \;=\; \|\mathbf{a}\ominus\mathbf{b}\| \tag{6.13c}$$

and

$$d_{\square}(\mathbf{a},\mathbf{b}) \;=\; \|\mathbf{a}\boxminus\mathbf{b}\| \tag{6.14c}$$

satisfying respectively

$$\|\{\mathbf{a}\oplus t\odot(-\mathbf{a}\oplus\mathbf{b})\}\ominus\mathbf{a}\| \;=\; |t|\odot\|\mathbf{b}\ominus\mathbf{a}\| \tag{6.13b}$$

and

$$\|\{t\odot(\mathbf{b}\boxminus\mathbf{a})\oplus\mathbf{a}\}\boxminus\mathbf{a}\| \;=\; |t|\odot\|\mathbf{b}\boxminus\mathbf{a}\| \tag{6.14b}$$

The arc length parameters t in (6.13a) and in (6.14a) thus share remarkable analogies with the Euclidean arc length parameter t in (6.12b). Eq. (6.13b) can readily be verified by (5.18), noting that $\mathrm{gyr}[\mathbf{u};\mathbf{v}]$ is an isometry for any two elements $\mathbf{u}$ and $\mathbf{v}$ in any gyrocommutative gyrogroup, and by a use of (6.8). Eq. (6.14b) is verified by a right cancellation, (4.18b), and by a use of (6.8). Related graphical presentations will be studied in the next section.

The selection of the points $\mathbf{a}$ and $\mathbf{b}$ in Eq. (6.12a) to be points that form a unit interval, $\|\mathbf{b}-\mathbf{a}\|=1$, in a Euclidean space simplifies the right hand side of Eq. (6.12b). We may note that a similar simplification in gyrovector spaces does not hold since, in general, $t\odot 1\neq t$.

The norm in a gyrovector space $V=(V,\oplus,\odot)$ is derived from a gyration invariant inner product. As such, the norm is gyration invariant. Since, in addition, gyrations respect the two binary operations $\oplus$ and $\boxplus$ in V, they are isometries of V relative to its two metrics $d_{\bigcirc}$ and $d_{\square}$.

Moreover, left gyrotranslations are also isometries of V, but only relative to the metric $d_{\bigcirc}$. Let $\mathbf{a},\mathbf{b}\in V$ be left gyrotranslated by some $\mathbf{x}\in V$. Then, by Theorem 5.8

$$(\mathbf{x}\oplus\mathbf{a})\ominus(\mathbf{x}\oplus\mathbf{b}) \;=\; \mathrm{gyr}[\mathbf{x};\mathbf{a}]\,(\mathbf{a}\ominus\mathbf{b}) \tag{6.15}$$

implying

$$d_{\bigcirc}(\mathbf{x}\oplus\mathbf{a},\mathbf{x}\oplus\mathbf{b}) \;=\; \|(\mathbf{x}\oplus\mathbf{a})\ominus(\mathbf{x}\oplus\mathbf{b})\| \;=\; \|\mathbf{a}\ominus\mathbf{b}\| \;=\; d_{\bigcirc}(\mathbf{a},\mathbf{b}) \tag{6.16}$$

The analogies shared by gyrolines with Euclidean lines suggest that the midpoint $m_{\mathbf{a},\mathbf{b}}$ of any two points $\mathbf{a}$ and $\mathbf{b}$ in a gyrovector space $(G,\oplus,\odot)$ be defined by the equation

$$m_{\mathbf{a},\mathbf{b}} \;=\; \mathbf{a}\oplus\frac{1}{2}\odot(-\mathbf{a}\oplus\mathbf{b}) \tag{6.17}$$

in the metric $d_{\bigcirc}$, (6.13c), and by the equation

$$m_{\mathbf{a},\mathbf{b}} \;=\; \frac{1}{2}\odot(\mathbf{b}\boxminus\mathbf{a})\oplus\mathbf{a} \tag{6.18}$$

in the metric $d_{\square}$, (6.14c), as we did in Ref. 35. In terms of midpoints we define triangle medians in obvious analogy with Euclidean geometry.

(I) We find that in the metric (6.13c)

  (i) triangle medians are concurrent, but

  (ii) the parallel postulate is not valid; see Fig. 12 of Ref. 35 and Fig. 7.7 in Section 7. On the other hand,

(II) we find that in the metric (6.14c)

  (i) triangle medians are not concurrent; see Fig. 10 of Ref. 35 and Fig. 7.8 in Section 7. But,

  (ii) the parallel postulate is valid if parallelism is appropriately defined.

In Euclidean geometry the two metrics (6.13c) and (6.14c) coincide into (6.12c), and "hence" both

  (i) triangle medians are concurrent, and

  (ii) the parallel postulate is valid.

These observations suggest the need to further explore the distribution of Euclidean geometric properties in the two metrics of *gyrogeometry.*

An interesting property resulting from the metric (6.14c) is that its geodesics are *gyrotransitive* in the sense that for any arbitrarily ordered set of k points $p_1, p_2, \cdots, p_k$ on a geodesic relative to the metric $d_{\square}$ successive gyrations are equivalent to a single gyration according to the equation

$$\mathrm{gyr}[p_1; p_2]\,\mathrm{gyr}[p_2; p_3] \quad \cdots \quad \mathrm{gyr}[p_{k-1}; p_k] = \mathrm{gyr}[p_1; p_k] \tag{6.19}$$

as explained in Eq. (7.3b) of Ref. 35.

7. Gyrovector Spaces, Euclidean and Hyperbolic Geometry, and Relativistic Physics

The idea of introducing scalar multiplication into loops is not new.[56]. In this section we are interested in gyrocommutative gyrogroups that admit scalar multiplication, in analogy with commutative groups that are turned into vector spaces by the introduction of scalar multiplication. Unexpectedly, the resulting gyrovector spaces share remarkable analogies with vector spaces exhibiting strong links with classical results in hyperbolic geometry as well as with new ones. In particular, it turns out that gyrovector spaces underlie hyperbolic geometry in full analogy with vector spaces, that underlie Euclidean geometry.

Let $\mathbf{v}_0, \mathbf{a} \in \mathbb{R}^3$ be two vectors in a Euclidean three-space, representing respectively a constant velocity and a constant acceleration. The time dependent velocity of an object uniformly accelerated relative to an inertial frame with initial relative velocity $\mathbf{v}_0$ and relative acceleration $\mathbf{a}$, $\mathbf{a} \neq \mathbf{0}$, is given in prerelativistic physics by the equation

$$\mathbf{v}_1(t) \;=\; \mathbf{v}_0 + \mathbf{a}t \tag{7.1a}$$

where $t \in \mathbb{R}$ is a real parameter representing time. Graphically, the right hand side of Eq. (7.1a) represents the family of all straight lines in $\mathbb{R}^3$ passing through any given point $\mathbf{v}_0$ in any given direction determined by the unit vector $\mathbf{a}/\|\mathbf{a}\|$. These straight lines form the set of all the geodesics of $\mathbb{R}^3$ with its natural metric

$$d_1(\mathbf{x}, \mathbf{y}) \;=\; \|\mathbf{x} - \mathbf{y}\| \,. \tag{7.1b}$$

In analogy with (7.1), but since $\oplus_E$ is not commutative, two distinct candidates for the definition of (the time dependent velocity of) relativistic uniform acceleration relative to an inertial frame are suggestive. These, shown in Figs. 7.1 and 7.2, are

$$\mathbf{v}_2(t) \;=\; \mathbf{v}_0 \oplus_E \mathbf{a} \odot t \qquad\qquad (\mathit{Fig.}\ 7.1) \tag{7.2a}$$

and

$$\mathbf{v}_3(t) \;=\; \mathbf{a} \odot t \oplus_E \mathbf{v}_0 \qquad\qquad (\mathit{Fig.}\ 7.2) \tag{7.3a}$$

for any initial relative velocity $\mathbf{v}_0 \in \mathbb{R}_c^3$ and relative acceleration $\mathbf{a} \in \mathbb{R}_c^3$ in $\mathbb{R}_c^3$, where $\mathbb{R}_c^3$ is the c-ball of $\mathbb{R}^3$ given by

$$\mathbb{R}_c^3 \;=\; \left\{ \mathbf{v} \in \mathbb{R}^3 : \|\mathbf{v}\| < c \right\}$$

c being the speed of light in empty space.

FIGURE 7.1
FIGURE 7.2

It turns out that the lines in (7.2a) are geodesics relative to the metric

$$d_2(\mathbf{x}, \mathbf{y}) \;=\; \|\mathbf{x} \ominus_E \mathbf{y}\| \tag{7.2b}$$

of $\mathbb{R}_c^3$, and the lines in (7.3a) are geodesics relative to the metric

$$d_3(\mathbf{x}, \mathbf{y}) \;=\; \|\mathbf{x} \boxminus_E \mathbf{y}\| \tag{7.3b}$$

of $I\!R_c^3$. The lines (7.2a) and (7.3a) are shown respectively in Figs. 7.1 and 7.2 for the unit disk $I\!R_{c=1}^2$ of the Euclidean plane $I\!R^2$,

$$I\!R_c^2 = \left\{ \mathbf{v} \in I\!R^2 : \|\mathbf{v}\| < c \right\}.$$

The metric function (7.1b) satisfies the triangle inequality in V_∞ (including $V_\infty = I\!R^3$ as a special case)

$$d_1(\mathbf{x},\mathbf{z}) \le d_1(\mathbf{x},\mathbf{y}) + d_1(\mathbf{y},\mathbf{z})$$

and members of the family of lines $\mathbf{v}_1(t)$ in (7.1a) are geodesics in the this metric (7.1b) since

$$d_1(\mathbf{v}_1(t_1),\mathbf{v}_1(t_2)) + d_1(\mathbf{v}_1(t_2),\mathbf{v}_1(t_3)) = d_1(\mathbf{v}_1(t_1),\mathbf{v}_1(t_3))$$

for any $t_1, t_2, t_3 \in I\!R$ satisfying $t_1 \le t_2 \le t_3$.

In full analogy, the metric functions (7.2b) and (7.3b) satisfy the triangle inequality in V_c (including $V_c = I\!R_c^3$ as a special case)

$$d_k(\mathbf{x},\mathbf{z}), \le d_k(\mathbf{x},\mathbf{y}) \oplus_E d_k(\mathbf{y},\mathbf{z})$$

for $k = 1{,}2$ (Note that $\oplus_E$ in the triangle inequality is a binary operation in $I\!R_c$. For the application of the binary operation $\oplus_E$ in $I\!R_c$ see Remark 2.1), and members of each of the family of curves $\mathbf{v}(t)$ in (7.2a)-(7.3a) are geodesics in their respective metrics $d_2(\mathbf{x},\mathbf{y})$, (7.2b), and $d_3(\mathbf{x},\mathbf{y})$, (7.3b), since for $k = 2{,}3$

$$d_k(\mathbf{v}_k(t_1),\mathbf{v}_k(t_2)) \oplus_E d_k(\mathbf{v}_k(t_2),\mathbf{v}_k(t_3)) = d_k(\mathbf{v}_k(t_1),\mathbf{v}_k(t_3))$$

for any $t_1, t_2, t_3 \in I\!R$ satisfying $t_1 \le t_2 \le t_3$ (for the application of the binary operation $\oplus_E$ in $I\!R_c$ see Remark 4.1). The results in (7.2)-(7.3) in Einstein gyrovector space $(V_c, \oplus_E, \odot_E)$ remain valid in Mӧbius gyrovector space $(V_c, \oplus_M, \odot_M)$ as shown in Ref. 35, and in the whole-space gyrovector space $(V, \oplus_W, \odot_W)$, as we will see in the sequel.

The time dependent velocity $\mathbf{v}_2(t)$ in (7.2a) seems to be preferable over the one, $\mathbf{v}_3(t)$, in (7.3a) as a candidate for the definition of relativistic uniform acceleration for the following two (among other) reasons.

Reason I: (i) The graph of the curve $\mathbf{v}_2(t)$ in (7.2a) is a Euclidean straight line in the hyperbolic geometry of the ball $I\!R_c^3$ determined by the metric (7.2b), Fig. 7.1; while (ii) the graph of the curve $\mathbf{v}_3(t)$ in (7.3a) is a Euclidean elliptic arc supported by a diameter in the hyperbolic geometry of the ball $I\!R_c^3$ determined by the metric (7.3b), Fig. 7.2.

Reason II: As opposed to (7.3a), the state of uniform acceleration as defined by (7.2a) is invariant for all inertial observers, as shown below in Eq. (7.5).

Let $\Sigma_\mathbf{u}$ be an inertial observer moving with velocity $\mathbf{u}$ relative to another inertial observer Σ_0. If $\Sigma_\mathbf{u}$ observes a uniformly accelerated frame $\Sigma_\mathbf{a}$ moving relative to him with velocity

$$\mathbf{v}(t) = \mathbf{v}_0 \oplus a \odot t , \tag{7.4}$$

where $\oplus = \oplus_E$, in accordance with the definition in Eq. (7.2a) of uniform acceleration, then the velocity $\mathbf{w}(t)$ of $\Sigma_\mathbf{a}$ relative to Σ_0 is

$$\begin{aligned}\mathbf{w}(t) &= \mathbf{u} \oplus \mathbf{v}(t) \\ &= \mathbf{u} \oplus (\mathbf{v}_0 \oplus \mathbf{a} \odot t) \\ &= (\mathbf{u} \oplus \mathbf{v}_0) \oplus \mathrm{gyr}[\mathbf{u};\mathbf{v}_0](\mathbf{a} \odot t) \\ &= (\mathbf{u} \oplus \mathbf{v}_0) \oplus (\mathrm{gyr}[\mathbf{u};\mathbf{v}_0]\,\mathbf{a}) \odot t\ .\end{aligned} \tag{7.5}$$

The uniformly accelerated velocities $\mathbf{v}(t)$ and $\mathbf{w}(t)$ in (7.4) and (7.5) have the same form. Hence, $\Sigma_{\mathbf{a}}$ is also viewed by Σ_0 as a uniformly accelerated frame. The frame $\Sigma_{\mathbf{a}}$ moves with acceleration $\mathrm{gyr}[\mathbf{u};\mathbf{v}_0]\,\mathbf{a}$ and initial velocity $\mathbf{u} \oplus \mathbf{v}_0$ relative to Σ_0.

Moreover, as noted by Urbantke,[57] it is accepted in the literature on special relativity theory that in relativity velocity space the hyperbolic geodesics describe uniformly accelerated velocities. Since gyrogroup theory reveals that these hyperbolic geodesics are given by the parametric curves $\mathbf{v}_2(t)$, Eq. (7.2a), it lends physical interpretation to the gyrovector space expression of $\mathbf{v}_2(t)$ which describes relativistic uniformly accelerated velocities in full analogy with $\mathbf{v}_1(t)$ of Eq. (7.1a) that describes prerelativistic uniformly accelerated velocities. Both $\mathbf{v}_2(t)$ and $\mathbf{v}_3(t)$ in Eqs. (7.2a) and (7.3a) have geometric significance since they are both geodesics relative to their respective two distinct metrics (7.2b) and (7.3b) which reciprocally interplay harmoniously, as shown in Fig. 7.9. Hence, it is hoped that together with the standard uniformly accelerated velocity $\mathbf{v}_2(t)$ also the non-standard uniformly accelerated velocity $\mathbf{v}_3(t)$ will be found physically significant.

We would like to emphasize that (7.2a) and (7.3a) are only two among other possibilities to define uniformly accelerated velocities. As an example, the expression

$$\mathbf{v}(t) = \mathbf{v}_0 \boxplus_M \mathbf{a} \odot t \qquad (\mathit{Fig.}\ 7.3) \tag{7.6}$$

may also be considered a candidate for a definition of a uniformly accelerated velocity $\mathbf{v}(t)$ with acceleration gyrovector $\mathbf{a}$ and initial velocity $\mathbf{v}_0$. Graphically, $\mathbf{v}(t)$ of Eq. (7.6) describes a Euclidean semihyperbola through $\mathbf{v}_0 \in \mathbb{R}_c^2$ whose two Euclidean asymptotes are perpendicular and whose two "disk-asymptotes" (that is, the two tangent lines of the semihyperbola at the points where it intersects the boundary of the disk $\mathbb{R}_c^2$) intersect at the point $2 \odot \mathbf{v}_0$, as shown in Fig. 7.3.

FIGURE 7.3

If $\mathbf{a}$ and $\mathbf{v}_0$ in the disk $\mathbb{R}_c^2$ of Fig. 7.3 are given by their components relative to Cartesian coordinates (x, y) with origin at the center of the disk, $\mathbf{a} = (a_1, a_2)$ and $\mathbf{v}_0 = (v_{01}, v_{02})$, then the equation of the hyperbola (7.6) of Fig. 7.3 relative to the Cartesian coordinates is

$$Q\,(x^2 - y^2) + Ax + By + 2Kxy + D = 0 \tag{7.6a}$$

where the coefficients Q, A, B, K and D are determined from $\mathbf{v}_0$ and $\mathbf{a}$ according to the equations

$$Q = a_1 v_{02} + a_2 v_{01} \tag{7.6b}$$

$$A = -(c^2 + \mathbf{v}_0^2)\, a_2 \tag{7.6c}$$

$$B = (c^2 + \mathbf{v}_0^2)\, a_1 \tag{76.d}$$

$$K = -(a_1 v_{01} - a_2 v_{02}) \tag{7.6e}$$

$$D = -(a_1 v_{02} - a_2 v_{01})\, c^2 \tag{7.6f}$$

Gyrovector spaces thus provide the setting for non-Euclidean geometries in which various candidates for the definition of uniformly accelerated velocity can be tested. Obviously, the final verdict about the best definition of *relativistic uniformly accelerated velocity* belongs to experiment.

Classically, the set of all uniformly accelerated velocities (7.1a) which share a fixed acceleration **a** forms a family of parallel straight lines that share their directions in space. It would therefore be interesting to explore the hyperbolic-geometric property shared by each of the family of hyperbolic curves in (7.2a), in (7.3a) and in (7.6) that share a fixed acceleration gyrovector **a**. Indeed, the set of all elliptic uniformly accelerated velocities, (7.3a), and the set of all hyperbolic uniformly accelerated velocities, (7.6), with a fixed acceleration gyrovector **a** share a common supporting diameter which is Euclidean-parallel to **a**, as shown in Figs. 7.2 and 7.3.

As in the classical case, linear uniformly accelerated velocities relative to uniform velocities, Fig. 7.1, remain linear uniformly accelerated velocities according to Eq. (7.5). This is, however, not the case of linear uniformly accelerated velocities relative to a linear uniformly accelerated velocity,

$$(\mathbf{u}_0 \oplus_E t \odot_E \mathbf{a}_1) \oplus_E (\mathbf{v}_0 \oplus_E t \odot_E \mathbf{a}_2) \qquad (\textit{Fig. } 7.4) \tag{7.7}$$

as shown in Fig. 7.4. It is clear from Fig. 7.4 that the resulting time dependent relative velocity undergoes the "unexpected" sign change that has been pointed out by Rindler and Mishra.[58]

FIGURE 7.4

Exploring various graphs in various two-dimensional gyrovector spaces, Kinyon discovered that the curves

$$a \oplus_M t \odot_M (-a \oplus_M b) \tag{7.8}$$

$a, b \in \mathbb{R}^2_{c=1}$, in the Möbius gyrogroup $(\mathbb{R}^2_{c=1}, \oplus_M)$, parametrized by $t \in \mathbb{R}$, are the well known geodesics through points a and b of the disk $\mathbb{R}^2_{c=1}$ in the Poincaré model of hyperbolic geometry. The gyrocurves (7.8) are Euclidean circles that intersect the boundary of the disk $\mathbb{R}^2_{c=1}$ orthogonally; see Fig. 11 in Ref. 35. By his discovery, Kinyon thus closed a circle of development in relativistic physics and hyperbolic geometry. By means of its Thomas precession effect, (i) relativistic physics leads us in this paper to the (ii) gyrovector space notion, that provides the appropriate setting to (iii) hyperbolic geometry in full analogy with vector spaces, which provide the setting for Euclidean geometry. Finally, hyperbolic geometry is the geometry underlying spacetime in (iv) relativistic physics,[4] thus closing the circle. We thus see that Thomas precession of relativistic physics is tailor made for hyperbolic geometry.

In the relativistic momentum equation $p = m\gamma_{\mathbf{v}}\mathbf{v}$ the relativistic counterpart of the classical velocity $\mathbf{v}$ is the proper velocity $\gamma_{\mathbf{v}}\mathbf{v}$ rather than the relativistic velocity $\mathbf{v}$ itself. This is also the case in Example 2.4 of the whole-space gyrogroup. It would therefore be interesting to explore uniform proper velocity accelerations as well. Similar to Eqs. (7.2a), (7.3a) and (7.6) we thus graphically explore in the gyrovector space $(V, \oplus_W, \odot_W)$ of proper velocities the two possible uniform accelerations

$$\mathbf{v}_4(t) = \mathbf{v}_0 \oplus_W \mathbf{a} \odot t \qquad (\textit{Hyperbolic geodesics in } V, \textit{Fig.}\ 7.5) \tag{7.9a}$$

and

$$\mathbf{v}_5(t) = \mathbf{a} \odot t \oplus_W \mathbf{v}_0 \qquad (\textit{Linear geodesics in } V, \textit{Fig.}\ 7.6) \tag{7.10a}$$

for $\mathbf{v}_0, \mathbf{a} \in V$ and $t \in \mathbb{R}$, which are shown in Figs. 7.5 and 7.6 for the special case when $V = \mathbb{R}^2$.

FIGURE 7.5

FIGURE 7.6

In analogy with the curve in Eqs. (7.1) and the gyrocurves in Eqs. (7.2) and (7.3) the gyrocurves (7.9a) and (7.10a), parametrized by $t \in \mathbb{R}$, are respectively geodesics in the whole-space gyrovector space $V = (V, \oplus_W, \odot_W)$ relative to the metrics

$$d_4(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} \ominus_W \mathbf{y}\| \qquad (\textit{Metric in } V \text{ for } \textit{the hyperbolic geodesics}) \tag{7.9b}$$

and

$$d_5(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} \boxminus_W \mathbf{y}\| \qquad (\textit{Metric in } V \text{ for } \textit{the linear geodesics}) \tag{7.10b}$$

The curve (7.9a) is a Euclidean semihyperbola passing through the points $\mathbf{v}_0$ and $\mathbf{v}_0 \oplus_W \mathbf{a}$ in V, having asymptotes that intersect each other at the origin of V, Fig. 7.5. It exhibits no notion of parallelism with respect to which the parallel postulate is valid.

Surprisingly, the curve (7.10a) is a Euclidean straight line passing through the points $\mathbf{v}_0$ and $\mathbf{a} \oplus_W \mathbf{v}_0$ in V which is Euclidean parallel to $\mathbf{a}$, Fig. 7.6, in full analogy with the Euclidean line (7.1a) which is parallel to $\mathbf{a}$.

The parallel postulate that has seemingly disappeared in the transition from Euclidean geometry to hyperbolic geometry thus reappears in the whole-space gyrovector space model of hyperbolic geometry. We should note that the geodesics (7.9a), which do not obey the parallel postulate, form triangles in which the medians are concurrent, Fig. 7.7. In contrast, the geodesics (7.10a), which do obey the parallel postulate, form triangles in which the medians are not concurrent, Fig. 7.8.

FIGURE 7.7

FIGURE 7.8

Thus, in agreement with the observations we made at the end of Section 6, in order to avoid Euclidean geometry, the geodesics relative to the two distinct metrics (7.9b) and (7.10b) of the whole-space gyrogroup take two distinct "strategies":

(i) geodesics (7.9a) of the gyrovector space $(V, \oplus_W, \odot_W)$ relative to the metric (7.9b) violate the parallel postulate while obeying the concurrency of triangle medians; and in contrast,

(ii) geodesics (7.10a) of the gyrovector space $(V, \oplus_W, \odot_W)$ relative to the metric (7.10b) violate the concurrency of triangle medians while obeying the the parallel postulate.

In addition, unlike (7.9a), the geodesics (7.10a) are gyrotransitive in the sense of (6.19).

Each of the gyrovector spaces considered in this section admits two natural metrics that we denote collectively by

$$d_{\bigcirc}(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} \ominus \mathbf{y}\| \tag{7.11a}$$

and

$$d_{\square}(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} \boxminus \mathbf{y}\| \tag{7.11b}$$

The two metrics in each of the gyrovector spaces become coincident when the gyrovector space reduces to the vector space $(V, +)$. That is, in the limit of large c, $c \to \infty$, we have

$$d_{\bigcirc}(\mathbf{x}, \mathbf{y}) = d_{\square}(\mathbf{x}, \mathbf{y}) = d(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\| .$$

The role that the two metrics $d_{\bigcirc}$ and $d_{\square}$ jointly play in a gyrovector space is well exhibited in the mixed-metric parallelogram shown in Fig. 7.9. The two points $\mathbf{a}, \mathbf{b} \in (\mathbb{R}^2, \oplus_W)$ in Fig. 7.9 determine a geodesic line in the metric $d_4(\mathbf{x}, \mathbf{y})$, Eq. (7.9b). This geodesic turns out to be a Euclidean semihyperbola. Left gyrotranslation of both $\mathbf{a}$ and $\mathbf{b}$ by some $\mathbf{x} \in (\mathbb{R}^2, \oplus_W)$ gives the points $\mathbf{c}, \mathbf{d} \in (\mathbb{R}^2, \oplus_W)$ in Fig. 7.9. The two lines containing the two pairs $(\mathbf{a}, \mathbf{c})$ and $(\mathbf{b}, \mathbf{d})$ are Euclidean parallel. In contrast, the two lines containing the two pairs $(\mathbf{a}, \mathbf{b})$ and $(\mathbf{c}, \mathbf{d})$ are not Euclidean parallel. Yet, the quadruplet $(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d})$ defines in some sense a "parallelogram" called a *mixed-metric parallelogram,* shown in Fig. 7.9. Each of the two opposite sides of the mixed-metric parallelogram $(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d})$ in Fig. 7.9 have equal lengths, but ... relative to the two distinct natural metrics of the gyrovector space $(V, \oplus_W, \odot_W)$: $d_{\bigcirc}(\mathbf{a}, \mathbf{b}) = d_{\bigcirc}(\mathbf{c}, \mathbf{d})$ and $d_{\square}(\mathbf{a}, \mathbf{c}) = d_{\square}(\mathbf{b}, \mathbf{d})$, where $d_{\bigcirc} = d_4$ and $d_{\square} = d_5$, Eqs. (7.9b) - (7.10b).

FIGURE 7.9

Rotations (including gyrations) take hyperbolic (linear) geodesics into hyperbolic (linear) geodesics. A left gyrotranslation of a hyperbolic geodesic (7.9a) is again a hyperbolic geodesic. To show this straightforwardly we note that with $\oplus_W = \oplus$ and $\odot_W = \odot$ we have in the whole-space gyrovector space $V = (V, \oplus, \odot)$

$$\begin{aligned} \mathbf{x} \oplus \{\mathbf{v}_0 \oplus a \odot t\} &= (\mathbf{x} \oplus \mathbf{v}_0) \oplus \mathrm{gyr}[\mathbf{x}; \mathbf{v}_0] (\mathbf{a} \odot t) \\ &= (\mathbf{x} \oplus \mathbf{v}_0) \oplus (\mathrm{gyr}[\mathbf{x}; \mathbf{v}_0]\, \mathbf{a}) \odot t \end{aligned}$$

Thus, for instance, the hyperbolic geodesic through $\mathbf{c}$ and $\mathbf{d}$ in Fig. 7.9 is a left gyrotranslation by $\mathbf{x}$ of the hyperbolic geodesic through $\mathbf{a}$ and $\mathbf{b}$.

By the left gyroassociative law we have

$$\{\mathbf{a} \odot t \oplus \mathbf{v}_0\} \oplus \mathrm{gyr}[\mathbf{a} \odot t; \mathbf{v}_0]\, \mathbf{x} = \mathbf{a} \odot t \oplus (\mathbf{v}_0 \oplus \mathbf{x})$$

Hence, appropriately gyrated right gyrotranslations take linear geodesics into linear geodesics.

The four sides of the mixed-metric parallelogram in Fig. 7.9 are gyroline segments in the whole-space gyrogroup $(V, \oplus, \odot)$ $(\oplus = \oplus_W)$ that can be parametrized by $0 \le t \le 1$ as

$$\begin{aligned} L_{ab}(t) &= \mathbf{a} \oplus (-\mathbf{a} \oplus \mathbf{b}) \odot t \\ L_{cd}(t) &= \mathbf{c} \oplus (-\mathbf{c} \oplus \mathbf{d}) \odot t \\ L_{ac}(t) &= (\mathbf{c} \boxminus \mathbf{a}) \odot t \oplus \mathbf{a} \\ L_{bd}(t) &= (\mathbf{d} \boxminus \mathbf{b}) \odot t \oplus \mathbf{b} \end{aligned} \tag{7.12}$$

The two sides L_{ab} and L_{cd} of the mixed-metric parallelogram are geodesic segments relative to the metric $d_{\bigcirc} = d_4$, and the two sides L_{ac} and L_{bd} of the mixed-metric parallelogram are geodesics relative to the metric $d_{\square} = d_5$.

These geodesic segments can be extended to their corresponding geodesics by allowing $-\infty < t < \infty$. Points on opposite sides of the mixed-metric parallelogram corresponding to equal values of t are separated by a constant distance in the relevant metric. Thus, for points on the opposite geodesics L_{ac} and L_{bd} containing respectively the pair of points $(\mathbf{a}, \mathbf{c})$ and $(\mathbf{b}, \mathbf{d})$ in Fig. 7.9 we have

$$d_{\bigcirc}(L_{ac}(t), L_{bd}(t)) = d_{\bigcirc}(\mathbf{a}, \mathbf{b}) = d_{\bigcirc}(\mathbf{c}, \mathbf{d}) \tag{7.13}$$

and for points on the opposite geodesics L_{ab} and L_{cd} containing respectively the pair of points $(\mathbf{a}, \mathbf{b})$ and $(\mathbf{c}, \mathbf{d})$ in Fig. 7.9 we have

$$d_{\square}(L_{ab}(t), L_{cd}(t)) = d_{\square}(\mathbf{a}, \mathbf{c}) = d_{\square}(\mathbf{b}, \mathbf{d}) \tag{7.14}$$

for all $t \in \mathbb{R}$ where the two metrics $d_{\bigcirc} = d_4$ and $d_{\square} = d_5$ of V are given in Eqs. (7.9b) and (7.10b).

The mixed-metric parallelogram thus exhibits the harmonious interplay between the two metrics $d_{\bigcirc}$ and $d_{\square}$ of a gyrovector space. In the literature on hyperbolic geometry only the metric $d_{\bigcirc}$ is studied. In contrast, gyrogeometry is the study of hyperbolic geometry with its two natural metrics, one of which is derivable from the other by a Thomas gyration as we see from Eqs. (4.19) and (7.11).

The mixed-metric parallelogram in Fig. 7.9 shows that the two opposite sides which are geodesics relative to the metric $d_{\square}$ (and hence are Euclidean straight lines) are indeed Euclidean parallel. It also suggests that we define parallelism between geodesics relative to the metric $d_{\square}$ (which are Euclidean semihyperbolas) as follows:

DEFINITION 7.1 *(parallelism between hyperbolic geodesics in W-gyrovector spaces)*

(i) Two geodesics in a W-gyrovector space relative to the metric $d_{\square}$ (which are Euclidean straight lines and hence are called *linear geodesics)* are parallel if they are Euclidean parallel; and

(ii) two geodesics in a W-gyrovector space relative to the metric $d_{\bigcirc}$ (which are Euclidean semihyperbolas and hence are called *hyperbolic geodesics)* are *gyroparallel* if one of them is a left gyrotranslation of the other one (Clearly, gyro-parallelism, as opposed to parallelism, is not transitive since in general the composition of two successive left gyrotranslations is not a left gyrotranslation but, rather, a left gyrotranslation followed, or preceded, by a Thomas rotation).

Furthermore, the mixed-metric parallelogram allows us to introduce a coordinate system into the W-gyroplane $\mathbb{R}^2 = (\mathbb{R}^2, \oplus, \odot)$ where $\oplus = \oplus_W$ and $\odot = \odot_W$. Let $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^2$ be any three non-cogyrolinear points (that is, they do not lie on a geodesic), Fig. 7.9. They define two basic gyrovectors, $-\mathbf{a} \oplus \mathbf{b}$ and $-\mathbf{a} \boxplus \mathbf{c}$ in $\mathbb{R}^2$. These two basic gyrovectors give rise to the two coordinate gyrolines $c_{\bigcirc}$ and $c_{\square}$ given by

$$c_{\bigcirc} : \quad \mathbf{a} \oplus (-\mathbf{a} \oplus \mathbf{b}) \odot s\,, \qquad -\infty < s < \infty$$

and

$$c_{\square} : \quad (-\mathbf{a} \boxplus \mathbf{c}) \odot t \oplus \mathbf{a}\,, \qquad -\infty < t < \infty$$

where s and t are arc length parameters in their respective metrics $d_{\bigcirc}$ and $d_{\square}$, as explained in Eqs. (6.13)-(6.14).

Every point P of the gyroplane $\mathbb{R}^2$ can now be written as a "gyrolinear combination" of the two basic gyrovectors,

$$P(s,t) \;=\; (-\mathbf{a} \boxplus \mathbf{c}) \odot t \;\oplus\; \{\mathbf{a} \oplus (-\mathbf{a} \oplus \mathbf{b}) \odot s\} \tag{7.15}$$

for some $(s,t) \in \mathbb{R}^2$, describing a transport of the point $\mathbf{a} = P(0,0)$ to the point $P(s,t)$ in either of two different, but equivalent ways:

(i) a transport of $\mathbf{a}$ in the direction of $\mathbf{b}$ a distance of s units of $\|\mathbf{b} \ominus \mathbf{a}\|$ (that is, a distance $s \odot \|\mathbf{b} \ominus \mathbf{a}\|$) in the metric $d_{\bigcirc}$), followed by its transport in the direction parallel to $c_{\square}$ a distance of t units of $\|\mathbf{c} \boxminus \mathbf{a}\|$ (that is, a distance $t \odot \|\mathbf{c} \boxminus \mathbf{a}\|$) in the metric $d_{\square}$); or

(ii) a a transport of $\mathbf{a}$ in the direction of $\mathbf{c}$ a distance of t units of $\|\mathbf{c} \boxminus \mathbf{a}\|$ (that is, a distance $t \odot \|\mathbf{c} \boxminus \mathbf{a}\|$) in the metric $d_{\square}$), followed by its transport in the direction parallel to $c_{\bigcirc}$ a distance of s units of $\|\mathbf{b} \boxminus \mathbf{a}\|$ (that is, a distance $s \odot \|\mathbf{b} \ominus \mathbf{a}\|$) in the metric $d_{\bigcirc}$).

Obviously, Eq. (7.15) can be simplified if one selects $\mathbf{a} = (0,0)$, obtaining

$$P(s,t) \;=\; \mathbf{c} \odot t \;\oplus\; \mathbf{b} \odot s \tag{7.16}$$

for $(s,t) \in \mathbb{R}^2$. The extension of (7.16) to n-dimensional W-gyrovector spaces is obvious. It is however not unique due to the lack of associativity. The mixed-metric parallelogram allows us to parallel-transport geodesics without changing their magnitudes and orientations.

Einstein gyrovector space $(V_c, \oplus_E, \odot_E)$ with $V_c = \mathbb{R}^2_{c=1}$ underlies the Klein disk model of hyperbolic geometry (see Fig. 11-12 in Millman and Parker,[59] p. 300; and Fig. 11 in Ref. 35) and, similarly, Möbius gyrovector space $(V_c, \oplus_M, \odot_M)$ with $V_c = \mathbb{R}^2_{c=1}$ underlies the Poincaré disk model of hyperbolic geometry (see Fig. 11-16 in Millman and Parker,[59] p. 302; and Fig. 11 in Ref. 35 as well as Figs. 7.10 and 7.11 of the present section).

Over the last century and a half several models for hyperbolic geometry have been developed. A new, elegant model is exposed in the present article; it is the Relativity-Proper-Velocity Gyrogroup or, the Whole-Space Gyrogroup (W-Gyrogroup, in short) of Example 2.4 in Section 2. Its geometrical elegance, well exhibited in Figs. 7.5-7.9, stems from the fact that it has physical significance being the gyrovector space of relativity proper velocities. One of the most elegant model of hyperbolic geometry is the Poincaré disk model offered by the French mathematician Henri Poincaré (1854-1912).[60] Its elegance stems from the fact that the Poincaré disk model within Euclidean geometry exhibit Euclidean angles; that is, the Poincaré measure of an angle is given by the Euclidean measure of the angle formed by the Euclidean tangent rays.[61]

Accordingly, let us consider the geodesic $L_{ab}(t)$ in (7.12) with $\oplus = \oplus_M$,

$$L_{ab}(t) \;=\; \mathbf{a} \oplus (-\mathbf{a} \oplus \mathbf{b}) \odot t \tag{7.17}$$

containing the points **a** and **b** in the Möbius disk gyrovector space $M_2 = (\mathbb{R}^2_{c=1}, \oplus, \odot)$ where $\oplus = \oplus_M$ and $\odot = \odot_M$, shown in Fig. 7.10. The slope of the tangent line at $\mathbf{a} \in L_{ab}(t)$ equals the slope of the Euclidean vector $-\mathbf{a} \oplus \mathbf{b}$,[62] and more generally, the slope of the tangent line at any point $p \in L_{ab}(t)$ equals the slope of the Euclidean vector $\mathrm{gyr}[\mathbf{p}; -\mathbf{a}](-\mathbf{a} \oplus \mathbf{b})$.[63] These equalities between slopes can readily be verified, and are clearly seen in Fig. 7.10.

FIGURE 7.10

Hence, the cosine of the Euclidean angle α between the two geodesics $L_{ab}(t)$ and $L_{ac}(t)$ in (7.12) with $\oplus = \oplus_M$, at their point of intersection a, Fig. 7.11, is given by the equation

$$\cos\alpha \;=\; \frac{-\mathbf{a} \oplus \mathbf{b}}{\|-\mathbf{a} \oplus \mathbf{b}\|} \cdot \frac{-\mathbf{a} \oplus \mathbf{c}}{\|-\mathbf{a} \oplus \mathbf{c}\|} \tag{7.18}$$

where $\cdot$ and $\|\cdot\|$ are the inner product and the norm that Möbius gyrovector space inherits from its underlying Euclidean space within which it is described. The angle measure (7.18) exhibits a remarkable analogy shared with angles in vector spaces.

The Möbius gyrovector space presented in Fig. 7.10 within the Euclidean plane $\mathbb{R}^2$ is, in fact, the Poincaré disk model of hyperbolic geometry. The geodesics of a Möbius gyrovector space relative to the Poincaré metric $d(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} \ominus \mathbf{y}\|$, given by (7.17), are the usual Poincaré hyperbolic geodesics given by circular arcs that intersect the boundary of the disk orthogonally. Thus, for instance, the Möbius geodesic through two given points **a** and **b** of M_2 is shown in Fig. 7.10.

Since the Poincaré measure of an angle is given by the Euclidean measure of the angle, analogies shared by angles in Euclidean and in hyperbolic geometries can clearly be visualized in a Möbius gyrovector space taken within a Euclidean vector space, as we show in Fig. 7.10 for the 2-dimensional Euclidean plane $\mathbb{R}^2$. Thus, the Euclidean tangent line at any point **p** of the geodesic (7.17) in M_2, $\mathbf{a} \oplus (-\mathbf{a} \oplus \mathbf{b}) \odot t$, is Euclidean-parallel to the Euclidean vector $\mathrm{gyr}[\mathbf{p}; -\mathbf{a}](-\mathbf{a} \oplus \mathbf{b})$, Fig. 7.10, in full analogy with Euclidean geometry where the tangent line at any point **p** of the Euclidean straight line $\mathbf{a} + (\mathbf{b} - \mathbf{a})t$, $t \in \mathbb{R}$, is parallel to $\mathbf{b} - \mathbf{a}$.

It is thus due to the presence of Thomas gyration that Möbius geodesics are *curved* in Euclidean geometry as opposed to their Euclidean counterparts, which are *straight.*

The *hyperbolic rigid motions* of a Möbius gyrovector space are left gyrotranslations and rotations about its origin. The angle measure $\cos\alpha$ in a Möbius gyrovector space is invariant under its hyperbolic rigid motions. The invariance under rotations about the origin is obvious, and the invariance under left gyrotranslations follows from Theorem 3.1 (vi) in Ref. 35. In the special case of the Möbius gyrovector space M_2 the invariance also follows from the conformality of Möbius transformations of the complex unit disk.

FIGURE 7.11

Fig. 7.11 shows two congruent triangles and their interior angles in the Möbius gyrovector space $M_2 = (\mathbb{R}^2_{c=1}, \oplus, \odot)$ where $\oplus = \oplus_M$ and $\odot = \odot_M$. The sides of the triangles are formed by geodesics that join the vertices of a triangle, and the two triangles are congruent in the sense that their corresponding vertices are related by a common left gyrotranslation. Thus, the vertices $(\mathbf{A}, \mathbf{B}, \mathbf{C})$ are obtained in Fig. 7.11 by a left gyrotranslation of the vertices $(\mathbf{a}, \mathbf{b}, \mathbf{c})$ by some $\mathbf{x} \in M_2$, and the equality of their corresponding angles is clearly seen.

The angle measure (7.18) is valid in any gyrovector space. As we remarked above, it can be clearly visualized in a Möbius gyrovector space taken within a Euclidean space, shown in Figs. 7.10 and 7.11 for a 2-dimensional Euclidean space $\mathbb{R}^2$, since the Poincaré measure of an angle equals the Euclidean measure of the angle.

Angles in gyrovector spaces, called *gyro-angles,* are measured by Eq. (7.18) in full analogy with the measure of angles in Euclidean vector spaces. Since they are invariant under hyperbolic rigid motions, they have geometric significance in the sense of Klein's *Erlangen Program* according to which geometry is viewed as the study of the invariants of an acting group. The acting group, the group of hyperbolic rigid motions, acts on a relativity velocity space and is isomorphic with the Lorentz transformation group acting on a corresponding *spacetime.* As such, gyro-angles have physical significance as well.

If the three gyro-angles of the hyperbolic triangle Δabc are denoted by α, β and γ then the hyperbolic area of Δabc is given by its angular defect $\pi - (\alpha + \beta + \gamma)$. The angular defect of the triangle Δabc, in turn, is the rotation angle of the composite Thomas rotation

$$\mathrm{gyr}[a; -b]\mathrm{gyr}[b; -c]\mathrm{gyr}[c; -a] \tag{7.19}$$

of the disk about its origin, as seen from Fig. 7.10. It can be written as a single Thomas rotation

$$\mathrm{gyr}[a; -b]\mathrm{gyr}[b; -c]\mathrm{gyr}[c; -a] = \mathrm{gyr}[a \ominus b; -a \oplus c] \tag{7.20}$$

which is an identity similar to the one in Theorem 3.2(iv) of Ref. 35. Each of the two sides of (7.20) is invariant under permutations of $\{a, b, c\}$ that preserve the orientation of Δabc, and is inverse-invariant under permutations of $\{a, b, c\}$ that reverse the orientation of Δabc. Thus, for instance, interchanging a and b results in reversing the orientation of the hyperbolic triangle Δabc and, accordingly,

$$\mathrm{gyr}[b \ominus a; -b \oplus c] = \mathrm{gyr}^{-1}[a \ominus b; -a \oplus c] \tag{7.21}$$

The extension of (7.19) from triangles to polygons is straightforward. By employing the fact that the defect of a hyperbolic triangle equals the sum of the defects of subdivided triangles various extensions of the right hand side of (7.20) are possible. Thus for instance, if a point d lies on the geodesic segment connecting a and b then the hyperbolic triangle Δabc is subdivided into the two hyperbolic triangles Δadc and Δdbc and, accordingly, the defect of the hyperbolic triangle Δabc is the sum of the defects of the two hyperbolic triangles Δadc and Δdbc resulting in the Thomas gyration identity

$$\mathrm{gyr}[a \ominus c; -a \oplus b] = \mathrm{gyr}[a \ominus c; -a \oplus d]\,\mathrm{gyr}[d \ominus c; -d \oplus b] \tag{7.22}$$

which is valid for any three points a, b, c in a gyrovector space where d lies on the $d_\ominus$-geodesic containing a and b, that is, when d is given by the equation $d = a \oplus (-a \oplus b) \odot t$ for any $t \in \mathbb{R}$.

In Eq. (7.20) we have thus established the following

THEOREM 7.1 The defect of a hyperbolic triangle Δabc in 3-dimensional hyperbolic geometry is the rotation angle of the Thomas gyration $\mathrm{gyr}[a \ominus b; -a \oplus c]$

By means of Theorem 7.1 and the right loop property one can find points c' such that the two hyperbolic triangles Δabc and $\Delta abc'$ have equal defects. One such example follows.

$$\begin{aligned}\mathrm{gyr}[a \ominus b; -a \oplus c] &= \mathrm{gyr}[a \ominus b; (-a \oplus c) \oplus (a - b)] \\ &= \mathrm{gyr}[a \ominus b; -a \oplus (c \oplus \mathrm{gyr}[c; -a](a \ominus b))] \\ &= \mathrm{gyr}[a \ominus b; -a \oplus c']\end{aligned}$$

Thus Δabc and $\Delta abc'$ have equal defects, where $c' = c \oplus \mathrm{gyr}[c; -a](a \ominus b)$.

8. Abstract Spacetime and its Abstract Lorentz Transformation Group

Let $(G, \oplus)$ be an abstract gyrogroup, and let $(t, v)^t$ be an element of the gyrogroup $(I\!R^{+} \times (G, \oplus)$ described in Example 2.6. When G happens to be a subset of a real vector space we introduce the *spacetime notation* according to which an element $(t, v)^t$ of the gyrogroup $(I\!R^{+} \times (G, \oplus)$ is written as $(t, vt)^t$,

$$\begin{pmatrix} t \\ v \end{pmatrix} \to \begin{pmatrix} t \\ vt \end{pmatrix} = \begin{pmatrix} t \\ x \end{pmatrix} \tag{8.1}$$

where $x = vt$. The pair $(t, x)^t$ is called a spacetime event. An application of a Lorentz-boost $B(u) = (\gamma_u, u)^t$, parametrized by $u \in G$, to a spacetime event $(t, x)^t$, in the notation of Example 2.6, is given by the product (2.11),

$$B(u)\begin{pmatrix} t \\ v \end{pmatrix} = \begin{pmatrix} \rho_u \\ u \end{pmatrix}\begin{pmatrix} t \\ v \end{pmatrix} = \begin{pmatrix} \dfrac{\rho_{u \oplus v}}{\rho_v}\, t \\ u \oplus v \end{pmatrix} \tag{8.2}$$

It can be shown that the abstract boost $B(u)$ keeps the norm (2.12) invariant. More generally, one can readily find that the inner product

$$IN\left(\begin{pmatrix} t_1 \\ v_1 \end{pmatrix}, \begin{pmatrix} t_2 \\ v_2 \end{pmatrix}\right) = \frac{\rho(v_1 \ominus v_2)}{\rho(v_1)\,\rho(v_2)}\, t_1 t_2 \tag{8.3}$$

satisfies

$$IN\left(\begin{pmatrix} t \\ v \end{pmatrix}, \begin{pmatrix} t \\ v \end{pmatrix}\right) = \left\|\begin{pmatrix} t \\ v \end{pmatrix}\right\|^2 \tag{8.4}$$

and it is boost invariant, that is,

$$IN\left(\begin{pmatrix} t_1 \\ v_1 \end{pmatrix}, \begin{pmatrix} t_2 \\ v_2 \end{pmatrix}\right) = IN\left(B(u)\begin{pmatrix} t_1 \\ v_1 \end{pmatrix}, B(u)\begin{pmatrix} t_2 \\ v_2 \end{pmatrix}\right) \tag{8.5}$$

for any $u, v_1, v_2 \in (G, \oplus)$ and $t_1, t_2 \in I\!R^{+}$.

Let $\mathrm{Aut}_0(G, \oplus)$ be any gyroautomorphism group of the gyrogroup $(G, \oplus)$, that is, any subgroup of $\mathrm{Aut}(G, \oplus)$ containing all the gyroautomorphisms $\mathrm{gyr}[u; v]$ of $(G, \oplus)$. Let U, V be generic elements of $\mathrm{Aut}_0(G, \oplus)$, and let $R(V)$ be a map of $I\!R^{+} \times (G, \oplus)$ into itself given by

$$R(U) : \begin{pmatrix} t \\ v \end{pmatrix} \mapsto \begin{pmatrix} t \\ Uv \end{pmatrix} \tag{8.6}$$

Finally, the abstract Lorentz transformation $L\{v; V\}$ parametrized by $v \in G$ and $V \in \mathrm{Aut}_0(G, \oplus)$ is the abstract Lorentz boost $B(v)$ preceded by the automorphism $R(V)$,

$$L\{v; V\} = B(v)R(V) : I\!R^{+} \times (G, \oplus) \to I\!R^{+} \times (G, \oplus) \tag{8.7}$$

The application of the abstract Lorentz transformation to an abstract timespace event is therefore, by Eqs. (8.7), (8.6) and (8.2),

$$
\begin{aligned}
L\{u;U\}\begin{pmatrix} t \\ v \end{pmatrix} &= B(u)R(U)\begin{pmatrix} t \\ v \end{pmatrix} \\
&= B(u)\begin{pmatrix} t \\ Uv \end{pmatrix} \\
&= \begin{pmatrix} \dfrac{\rho_{u \oplus v}}{\rho_v}\, t \\ u \oplus Uv \end{pmatrix}
\end{aligned}
\tag{8.8}
$$

The abstract Lorentz group $L\{v;V\}$, $v \in G$ and $V \in \mathrm{Aut}_0(G, \oplus)$, has a gyrosemidirect product structure having the composition law

$$
L\{u;U\}L\{v;V\} = L\{u \oplus Uv\, ;\, \mathrm{gyr}[u\,;Uv]UV\}
\tag{8.9}
$$

EXAMPLE 8.1 *(The Lorentz group of special theory of relativity)* In the spacetime notation in (8.1), Eq. (8.2) takes the form, with $x = vt$,

$$
B(u)\begin{pmatrix} t \\ x \end{pmatrix} = \begin{pmatrix} \rho_u \\ \rho_u u \end{pmatrix}\begin{pmatrix} t \\ x \end{pmatrix} = \begin{pmatrix} \dfrac{1}{\rho_v}\rho_{u \oplus v}\, t \\ \dfrac{1}{\rho_v}\rho_{u \oplus v}\,(u \oplus v)\, t \end{pmatrix}
\tag{8.10}
$$

If the abstract gyrogroup $(G, \oplus)$ is realized by Einstein gyrogroup $(V_c, \oplus_E)$ then ρ_v may be selected to be the Lorentz factor (2.4b), $\rho_v = \gamma_{\mathbf{v}}$. In this case the norm (2.12) of $(t, \mathbf{x})^t$ is

$$
\left\| \begin{pmatrix} t \\ \mathbf{x} \end{pmatrix} \right\| = \sqrt{t^2 - \mathbf{x}^2/c^2}
\tag{8.11}
$$

and Eq. (8.10) specializes to, with $\oplus = \oplus_E$,

$$
\begin{aligned}
B(\mathbf{u})\begin{pmatrix} t \\ \mathbf{x} \end{pmatrix} &= \begin{pmatrix} \gamma_{\mathbf{u}} \dfrac{\gamma_{\mathbf{u}\oplus\mathbf{v}}}{\gamma_{\mathbf{u}}\gamma_{\mathbf{v}}}\, t \\ \gamma_{\mathbf{u}} \dfrac{\gamma_{\mathbf{u}\oplus\mathbf{v}}}{\gamma_{\mathbf{u}}\gamma_{\mathbf{v}}}\,(\mathbf{u}\oplus\mathbf{v})\, t \end{pmatrix} = \begin{pmatrix} \gamma_{\mathbf{u}} t\,(1 + \dfrac{1}{c^2}\,\mathbf{u}\cdot\mathbf{v}) \\ (\mathbf{u}\oplus\mathbf{v})\,\gamma_{\mathbf{u}}\, t\,(1 + \dfrac{1}{c^2}\,\mathbf{u}\cdot\mathbf{v}) \end{pmatrix} \\
&= \begin{pmatrix} \gamma_{\mathbf{u}}(t + \dfrac{1}{c^2}\,\mathbf{u}\cdot\mathbf{x}) \\ \dfrac{1}{1 + \mathbf{u}\cdot\mathbf{v}/c^2}\left\{\mathbf{u} + \dfrac{1}{\gamma_{\mathbf{u}}}\mathbf{v} + \dfrac{1}{c^2}\dfrac{\gamma_{\mathbf{u}}}{1+\gamma_{\mathbf{u}}}(\mathbf{u}\cdot\mathbf{v})\mathbf{u}\right\}\gamma_{\mathbf{u}} t\,(1 + \dfrac{1}{c^2}\,\mathbf{u}\cdot\mathbf{v}) \end{pmatrix} \\
&= \begin{pmatrix} \gamma_{\mathbf{u}}(t + \dfrac{1}{c^2}\,\mathbf{u}\cdot\mathbf{x}) \\ \gamma_{\mathbf{u}}\mathbf{u}t + \mathbf{x} + \dfrac{1}{c^2}\dfrac{\gamma_{\mathbf{u}}^2}{1+\gamma_{\mathbf{u}}}(\mathbf{u}\cdot\mathbf{x})\mathbf{u} \end{pmatrix}
\end{aligned}
\tag{8.12}
$$

noting that for $\oplus = \oplus_E$ we have

$$\gamma_{\mathbf{u}\oplus\mathbf{v}} \; = \; \gamma_{\mathbf{u}}\gamma_{\mathbf{v}}\Big(1 + \frac{\mathbf{u}\cdot\mathbf{v}}{c^2}\Big) \tag{8.13}$$

as we see from the real form of Eq. (2.3c). Eq. (8.12) is identical with the standard definition of the Lorentz-boost in STR.(64)

The relativity velocity gyrogroup $(G, \oplus_E) = (\mathbb{R}^3_c, \oplus_E)$ of STR is a subset of the Euclidean 3-space $\mathbb{R}^3$. We therefore select $\mathrm{Aut}_0(\mathbb{R}^3_c, \oplus_E) = SO(3)$ where $SO(3)$ is the rotation group of $\mathbb{R}^3$ about its origin. The Lorentz transformation of STR then follows from Eqs. (8.8) and (8.12),

$$L\{\mathbf{u};U\}\begin{pmatrix} t \\ \mathbf{x} \end{pmatrix} \; = \; \begin{pmatrix} \gamma_{\mathbf{u}}(t + \dfrac{1}{c^2}\,\mathbf{u}\cdot U\mathbf{x}) \\ \gamma_{\mathbf{u}}\mathbf{u}t \, + U\mathbf{x} + \dfrac{1}{c^2}\,\dfrac{\gamma_{\mathbf{u}}^2}{1+\gamma_{\mathbf{u}}}\,(\mathbf{u}\cdot U\mathbf{x})\mathbf{u} \end{pmatrix} \tag{8.14}$$

and its composition law follows from Eq. (8.9),

$$L\{\mathbf{u};U\}\,L\{\mathbf{v};V\} \; = \; L\{\mathbf{u}\oplus_E U\mathbf{v};\, \mathrm{gyr}[\mathbf{u};U\mathbf{v}]UV\} \tag{8.15}$$

for any $\mathbf{u},\mathbf{v}\in(\mathbb{R}^3_c, \oplus_E)\subset\mathbb{R}^3$ and any $U, V \in SO(3)$. The Lorentz transformation composition law (8.15) is thus given by parameter composition; and the parameter composition law is the gyrosemidirect product in the gyrosemidirect product group $(\mathbb{R}^3_c, \oplus_E)\times SO(3)$. It shares analogies with the Galilei transformation composition law,(12, 19)

$$B\{\mathbf{u};U\}\,B\{\mathbf{v};V\} \; = \; B\{\mathbf{u}+U\mathbf{v};\, UV\} \tag{8.16}$$

thus demonstrating the power and elegance of gyrogroup theory. We may note that the task of understanding the composition law of Lorentz transformations before the emergence of gyrogroup theory was overwhelmingly involved.(65)

EXAMPLE 8.2 *(The Lorentz group of special theory of relativity with vectorial time)* Proper time in STR, $\sqrt{t^2 - \mathbf{x}^2/c^2}$, is spacetime coordinate independent and, hence, it is the analog of time in Newtonian physics. The fact that it involves $\sqrt{t^2}$ rather than t triggers the exploration of a six-dimensional STR with proper time $\sqrt{\mathbf{t}^2 - \mathbf{x}^2/c^2}$, where both $\mathbf{x}$ and $\mathbf{t}$ are three dimensional vectors.(66) It would therefore be interesting to extend Eq. (8.14) of the previous Example to the case where both $\mathbf{x}$ and t are vectors. As a first step towards the extension we rewrite Eq. (8.14) in an equivalent form which is suitable for the extension, and select $c = 1$ for simplicity,

$$L\{\mathbf{v};V\}\begin{pmatrix} t \\ \mathbf{x} \end{pmatrix} \; = \; \begin{pmatrix} t \; + \{\gamma_{\mathbf{v}}(\mathbf{v}\cdot V\mathbf{x}) + \dfrac{\gamma_{\mathbf{v}}^2}{1+\gamma_{\mathbf{v}}}\,\mathbf{v}^2\, t\,\} \\ V\mathbf{x} + \{\gamma_{\mathbf{v}}t \; + \dfrac{\gamma_{\mathbf{v}}^2}{1+\gamma_{\mathbf{v}}}\,(\mathbf{v}\cdot V\mathbf{x})\}\,\mathbf{v} \end{pmatrix} \tag{8.14r}$$

where $t\in\mathbb{R}$, $\mathbf{x}\in\mathbb{R}^3$, $\mathbf{v}\in\mathbb{R}^3_{c=1}$, and $V\in SO(3)$. One can readily check that (8.14r) and (8.14) are indeed equivalent, noting that $\mathbf{v}^2 = (\gamma_{\mathbf{v}}^2 - 1)/\gamma_{\mathbf{v}}^2$.

The desired extension of (8.14r) takes the elegant form that has gone unnoticed in the literature on six-dimensional relativity,(66)

$$L\{\mathbf{u},U;\mathbf{v},V\}\begin{pmatrix}\mathbf{t}\\ \mathbf{x}\end{pmatrix} = \begin{pmatrix} U\mathbf{t} + \{\gamma_{\mathbf{uv}}(\mathbf{v}\cdot V\mathbf{x}) + \dfrac{\gamma_{\mathbf{uv}}^2}{1+\gamma_{\mathbf{uv}}}\mathbf{v}^2(\mathbf{u}\cdot U\mathbf{t})\}\,\mathbf{u} \\ V\mathbf{x} + \{\gamma_{\mathbf{uv}}(\mathbf{u}\cdot U\mathbf{t}) + \dfrac{\gamma_{\mathbf{uv}}^2}{1+\gamma_{\mathbf{uv}}}\mathbf{u}^2(\mathbf{v}\cdot V\mathbf{x})\}\,\mathbf{v}\end{pmatrix} \tag{8.14e}$$

where $\mathbf{u},\mathbf{v}\in\{\mathbf{u}\in V_{\mathbf{t}},\mathbf{v}\in V_{\mathbf{x}}: \mathbf{u}^2\mathbf{v}^2<c^2=1\}$, $U\in \mathrm{Aut}(V_{\mathbf{t}})$ and $V\in\mathrm{Aut}(V_{\mathbf{x}})$, and where $\mathbf{t}\in V_{\mathbf{t}}$ and $\mathbf{x}\in V_{\mathbf{x}}$ are elements of their respective inner product spaces $V_{\mathbf{t}}$ and $V_{\mathbf{x}}$, and finally where

$$\gamma_{\mathbf{uv}} = \frac{1}{\sqrt{1-\mathbf{u}^2\mathbf{v}^2}}$$

The *extended* Lorentz transformations (8.14e) form a transformation group that keeps the relativistic distance $\sqrt{\mathbf{t}^2-\mathbf{x}^2}$ invariant.

The corresponding extension of (8.15) to compositions of the extended Lorentz transformation $L\{\mathbf{u},U;\mathbf{u},V\}$ in Eq. (8.14e) involves a Thomas gyration of both space and time, and the resulting extension of gyrogroup theory is likely to be useful in the exploration of symplectic groups $SO(p,q)$ similarly to the exploration of $SO(1,3)$ which is implicit in Example 8.1. Indeed, effects of both temporal and spatial Thomas gyrations have been studied in Refs. 18 and 19 where time is complex (and hence can be smoothly rotated) rather than real (which allows only two discrete rotations, $t\rightarrow t$ and $t\rightarrow -t$, so that there can be no temporal Thomas gyration to accompany the spatial Thomas gyration).

EXAMPLE 8.3 *(The Lorentz Inner Product)* In the spacetime notation in (8.1), Eq. (8.3) takes the form, with $x_1=\mathbf{u}t_1$ and $x_2=\mathbf{v}t_2$,

$$IN\left(\begin{pmatrix}t_1\\ \mathbf{x}_1\end{pmatrix},\begin{pmatrix}t_2\\ \mathbf{x}_2\end{pmatrix}\right) = \frac{\rho_{\mathbf{u}\ominus\mathbf{v}}}{\rho_{\mathbf{u}}\rho_{\mathbf{v}}}t_1t_2 \tag{8.17}$$

where $\mathbf{u}=\mathbf{x}_1 t_1$ and $\mathbf{v}=\mathbf{x}_2 t_2$. If the abstract gyrogroup $(G,\oplus)$ is realized by Einstein gyrogroup $(V_c,\oplus_E)$ then ρ_v may be selected to be the Lorentz factor $\rho_v=\gamma_v$, (2.4b). Then, by means of (8.13), Eq. (8.17) gives

$$\begin{aligned} IN\left(\begin{pmatrix}t_1\\ \mathbf{x}_1\end{pmatrix},\begin{pmatrix}t_2\\ \mathbf{x}_2\end{pmatrix}\right) &= \frac{\gamma_{\mathbf{u}\ominus\mathbf{v}}}{\gamma_{\mathbf{u}}\gamma_{\mathbf{v}}}t_1t_2 \\ &= (1-\frac{\mathbf{u}\cdot\mathbf{v}}{c^2})t_1t_2 \\ &= t_1t_2-\mathbf{x}_1\cdot\mathbf{x}_2/c^2 \end{aligned} \tag{8.18}$$

which is the Lorentz inner product of the two spacetime events $(t_1,\mathbf{x}_1)^t$ and $(t_2,\mathbf{x}_2)^t$.

EXAMPLE 8.4 *(The Lorentz group generated by a Möbius gyrogroup)* In this Example we realize the abstract gyrogroup $(G,\oplus)$ in Eq. (8.10) by $(I\!R_c^3,\oplus_M)\subset I\!R^3$, and the abstract Lorentz factor by $\rho_v = \gamma_{\mathbf{v}}$. The resulting Lorentz transformation group turns out to be the *nonstandard Lorentz group* studied in Ref. 10, and which is experimentally equivalent to the standard Lorentz transformation group of Example 8.1 as pointed out by Urbantke.[67]

Following the exposition of analogies shared by Lorentz and Galilean groups one can now solve problems in Lorentz groups that previously could only be solved in Galilean groups. Thus, for instance, the problem of determining the Galilean transformations that link two given spacetime events can easily be solved. In contrast, its relativistic counterpart could only be solved partially,[65] before it was completely solved by employing analogies provided by gyrogroup theory.[15] Elegant and powerful analogies shared by Galilean and Lorentz transformation groups are presented in the "Introduction and Demonstration" section of Ref. 19.

Acknowledgments It is hoped that this work will broaden participation in *abstract Thomas precession* research by theoretical physicists, geometers, and algebraists, and will find its way into the lore learned by students of physics and mathematics, thereby highlighting Gravity Probe B, a NASA project aimed at the measurement of the Thomas precession of gyroscopes in Earth orbit.[68] The work presented in this article was supported by NASA Joint Venture (JOVE) program, Grant NAG8-1007, aimed at increasing participation in space research by faculty and students at colleges and universities; and by ND EPSCoR through NSF Grant #OSR-9452892. The author wishes to thank Oliver Jones of North Dakota State University and Michael Kinyon of Indiana University, South Bend, for useful conversations and suggestions, and Heinrich Wefelscheid of Duisburg University for information about the pioneering work of Varićak on hyperbolic geometry and its applications to Einstein's theory of relativity.

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FIGURE CAPTIONS

Fig. 1.1 Thomas Precession: Shown is the Thomas rotation (or, precession) gyr$[\mathbf{u};\mathbf{v}]$ of an inertial frame Σ'' relative to another inertial frame Σ with relative velocities in the disk $\mathbb{R}_c^2$ of the Euclidean plane $\mathbb{R}^2$. It is generated by the two relativistically admissible velocities $\mathbf{u},\mathbf{v}\in\mathbb{R}_c^2$, and is related to Einstein's velocity addition $\oplus$ by the relation shown in the figure as well as, unexpectedly, by Eq. (1.2). In the geometry of relativity velocity spaces Euclidean parallelism between moving inertial frames is not transitive due to the presence of Thomas precession. Thomas rotation vanishes when the angle θ between $\mathbf{u}$ and $\mathbf{v}$ is either $\theta=0$ or $\theta=\pi$. The relationship between the angle θ and its resulting Thomas rotation angle ε for any $\mathbf{u},\mathbf{v}\in\mathbb{R}_c^2$ is studied and presented graphically in Fig. 2 of Ref. 1 and in Fig. 3 of Ref. 14.

Fig. 7.1 A *linear uniformly accelerated velocity* in Einstein gyrovector space of the disk $(\mathbb{R}_{c=1}^2, \oplus_E, \odot)$ of the Euclidean plane $\mathbb{R}^2$ is represented by the gyrocurve $\mathbf{v}_2(t)=\mathbf{v}_0\oplus_E\mathbf{a}\odot t$, $\mathbf{v}_0,\mathbf{a}\in\mathbb{R}_{c=1}^2$, Eq. (7.2a), parametrized by $t\in\mathbb{R}$, passing through $\mathbf{v}_0$ for $t=0$ and through $\mathbf{v}_0\oplus_E\mathbf{a}$ for $t=1$. This gyrocurve is a Euclidean straight line. It is a geodesic line in the metric $d_2(\mathbf{x},\mathbf{y})=\|\mathbf{x}\ominus_E\mathbf{y}\|$ of Einstein gyrovector space of the gyrovector space of the disk, that turns out to be the Klein disk model of hyperbolic geometry. Extension to an Einstein c-ball gyrovector space $(V_c, \oplus_E, \odot)$ of any real inner product space V_∞ is straightforward. Medians of triangles generated by gyrocurves like the one shown here are concurrent, as shown in Fig. 12 of Ref. 35.

Fig. 7.2 An *elliptic uniformly accelerated velocity* in Einstein gyrovector space of the disk $(\mathbb{R}_{c=1}^2, \oplus_E, \odot)$ of the Euclidean plane $\mathbb{R}^2$ is represented by the gyrocurve $\mathbf{v}_3(t)=\mathbf{a}\odot t\oplus_E\mathbf{v}_0$, $\mathbf{v}_0,\mathbf{a}\in\mathbb{R}_{c=1}^2$, Eq. (7.3a), parametrized by $t\in\mathbb{R}$, passing through $\mathbf{v}_0$ for $t=0$ and through $\mathbf{a}\oplus_E\mathbf{v}_0$ for $t=1$. This gyrocurve is a Euclidean semiellipse supported by a diameter of the disk $\mathbb{R}_{c=1}^2$ which is Euclidean parallel to the acceleration gyrovector $\mathbf{a}$. It is a geodesic line in the metric $d_3(\mathbf{x},\mathbf{y})=\|\mathbf{x}\boxminus_E\mathbf{y}\|$ of Einstein disk. Extension to an Einstein gyrovector space of the c-ball $(V_c, \oplus_E, \odot)$ of any real inner product space V_∞ is straightforward. Medians of triangles generated by gyrocurves like the one shown here are not concurrent, as shown in Fig. 10 of Ref. 35.

Fig. 7.3 A *hyperbolic uniformly accelerated velocity* in Möbius gyrovector space of the disk $(\mathbb{R}_{c=1}^2, \oplus_M, \odot)$ of the Euclidean plane $\mathbb{R}^2$ is represented by the gyrocurve $\mathbf{v}_0\boxplus_M\mathbf{a}\odot t$, $\mathbf{v}_0,\mathbf{a}\in\mathbb{R}_{c=1}^2$, Eq. (7.6), parametrized by $t\in\mathbb{R}$, passing through $\mathbf{v}_0$ for $t=0$. This gyrocurve is a Euclidean semihyperbola supported by a diameter of the disk $\mathbb{R}_{c=1}^2$ which is Euclidean parallel to the acceleration vector $\mathbf{a}$. The two "disk-asymptotes" of the semihyperbola intersect at the point $2\odot\mathbf{v}_0$. Extension to a Möbius gyrovector space of the ball c-ball $(V_c, \oplus_M, \odot)$ of any real inner product space V_∞ is straightforward.

Fig. 7.4 Composite linear uniformly accelerated velocity, $\mathbf{v}(t)=(\mathbf{u}_0\oplus_E\mathbf{a}_1\odot t)\oplus_E(\mathbf{v}_0\oplus_E\mathbf{a}_2\odot t)$, $-\infty<t<\infty$. The composition of two linear uniformly accelerated velocities is no longer a linear uniformly accelerated velocity; the "unexpected sign change" reported by Rindler and Mishra in Ref. 58 is clearly seen.

Fig. 7.5 A *hyperbolic uniformly accelerated proper velocity* in the whole-space gyrovector space $(\mathbb{R}^2, \oplus_W, \odot_W)$ of the Euclidean plane $\mathbb{R}^2$ is represented by the gyrocurve $\mathbf{v}_4(t) = \mathbf{v}_0 \oplus_W \mathbf{a} \odot t$, $\mathbf{v}_0, \mathbf{a} \in \mathbb{R}^2$, Eq. (7.9a), parametrized by $t \in \mathbb{R}$, passing through $\mathbf{v}_0$ for $t = 0$ and through $\mathbf{v}_0 \oplus_W \mathbf{a}$ for $t = 1$. This gyrocurve is a Euclidean semihyperbola whose two Euclidean asymptotes intersect at the origin of the space, and whose equation in a Euclidean xy-plane $Ax^2 + By^2 + Kxy = c^2$, $A, B \in \mathbb{R}$, $K^2 = 4(A+1)(B+1)$, and $c \in \mathbb{R}$ is the parameter of the gyrogroup operation $\oplus = \oplus_W$, Eq. 2.5a). No parallelism notion is observed between the gyrocurve and its generating acceleration vector $\mathbf{a}$. The gyrocurve shown here is a geodesic line in the metric $d_4(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} \ominus_W \mathbf{y}\|$ of the whole-space gyrovector space. Medians of gyrotriangles generated by geodesics like the one shown here are concurrent, as shown in Fig. 7.7 below. Extension to a whole-space gyrovector space $(V, \oplus_W, \odot_W)$ of any real inner product space V is straightforward.

Fig. 7.6 A *linear uniformly accelerated proper velocity* in the whole-space gyrovector space $(\mathbb{R}^2, \oplus_W, \odot_W)$ of the Euclidean plane $\mathbb{R}^2$ is represented by the gyrocurve $\mathbf{v}_5(t) = \mathbf{a} \odot t \oplus_W \mathbf{v}_0$, $\mathbf{v}_0, \mathbf{a} \in \mathbb{R}^2$, Eq. (7.10a), parametrized by $t \in \mathbb{R}$, passing through $\mathbf{v}_0$ for $t = 0$ and through $\mathbf{a} \oplus_W \mathbf{v}_0$ for $t = 1$. This gyrocurve is a Euclidean straight line which is Euclidean parallel to its generating acceleration vector $\mathbf{a}$. The gyrocurve shown here is a geodesic line in the metric $d_5(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} \boxminus_W \mathbf{y}\|$ of the whole-space gyrovector space. While Euclidean parallelism is clearly observed, medians of gyrotriangles generated by geodesics like the one shown here are not concurrent, as shown in Fig. 7.8 below. Extension to a whole-space gyrogroup $(V, \oplus_W, \odot_W)$ of any real inner product space V is straightforward.

Fig. 7.7 The medians of any triangle in the whole-space gyrovector space $(V, \oplus_W, \odot_W)$ made out of geodesics relative to the metric $d_{\bigcirc} = d_4(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} \ominus_W \mathbf{y}\|$, Eq. (7.9b), are concurrent. The geodesics are Euclidean semihyperbolas the asymptotes of which intersect at the origin. Note that the origin of the space is privileged in the sense that it is the point of intersection of the Euclidean asymptotes of any geodesic relative to the metric d_4, as clearly seen in Fig. 7.5.

Fig. 7.8 The medians of any triangle in the whole-space gyrovector space $(V, \oplus_W, \odot_W)$ made out of geodesics relative to the metric $d_{\Box} = d_5(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} \boxminus_W \mathbf{y}\|$, Eq. (7.10b), are not concurrent.

Fig. 7.9 The mixed-metric parallelogram in the gyrovector space $(\mathbb{R}^2, \oplus_W)$. Each of its two opposite sides have equal lengths, but in different metrics: $d_{\bigcirc}(\mathbf{a}, \mathbf{b}) = d_{\bigcirc}(\mathbf{c}, \mathbf{d})$ and $d_{\Box}(\mathbf{a}, \mathbf{c}) = d_{\Box}(\mathbf{b}, \mathbf{d})$.

Fig. 7.10 Geodesics in the Möbius gyrovector space M_2 visually share remarkable analogies with geodesics in Euclidean vector spaces since the Poincaré angle measure in Möbius gyrovector spaces and the Euclidean angle measure in vector spaces are equal, as pointed out by Millman and Parker in Ref. 59, p. 304. Being present in gyrovector spaces but absent in vector spaces, the effect of Thomas gyration is clearly seen. The geodesic L_{ab}, Eq. (7.12), passing through two given points $\mathbf{a}, \mathbf{b} \in M_2$ is shown. For an arbitrarily fixed point $\mathbf{p} \in L_{ab}$ and any $\mathbf{x} \in L_{ab}$ the Euclidean vector $-\mathbf{p} \oplus \mathbf{x}$ is Euclidean parallel to the gyrated vector $\mathrm{gyr}[\mathbf{p}; -\mathbf{a}](-\mathbf{a} \oplus \mathbf{b})$ as shown for $-\mathbf{p} \oplus \mathbf{a}$, for $-\mathbf{p} \oplus \mathbf{b}$, and generally, for $-\mathbf{p} \oplus \mathbf{x}$. In fact, the map $\mathbf{x} \to -\mathbf{p} \oplus \mathbf{x}$ maps bijectively the geodesic L_{ab} to the diameter of the disk which is parallel to the tangent line of L_{ab} at the point $\mathbf{p}$.

Fig. 7.11 The vertices of two congruent triangles in a Möbius gyrovector space are related to one another by a hyperbolic rigid motion. Hyperbolic rigid motions, in turn, keep gyro-angles invariant. Shown are two congruent triangles in the Möbius gyrovector space M_2. Their respective gyro-angles are, accordingly, equal.