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From Pythagoras To Einstein: The Hyperbolic Pythagorean Theorem

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ABSTRACT

A new form of the *Hyperbolic Pythagorean Theorem,* which has a striking intuitive appeal and offers a strong contrast to its standard form, is presented. It expresses the square of the hyperbolic length of the hypotenuse of a hyperbolic right angled triangle as the "Einstein sum" of the squares of the hyperbolic lengths of the other two sides, Fig. 1, thus completing the long path from Pythagoras to Einstein.

Following the pioneering work of Varičak it is well known that relativistic velocities are governed by hyperbolic geometry in the same way that prerelativistic velocities are governed by Euclidean geometry. Unlike prerelativistic velocity composition, given by the ordinary vector addition, the composition of relativistic velocities, given by the Einstein addition, is neither commutative nor associative due to the presence of Thomas precession. Following the discovery of the mathematical regularity that Thomas precession stores, it is now possible to extend Thomas precession by abstraction, (i) allowing hyperbolic geometry to be studied by means of analogies that it shares with Euclidean geometry; and, similarly (ii) allowing velocities and accelerations in relativistic mechanics to be studied by means of analogies that they share with velocities and accelerations in classical mechanics. The abstract Thomas precession, called the Thomas gyration, gives rise to gyrovector space theory in which the prefix *gyro* is used extensively in terms like *gyrogroups* and *gyrovector spaces, gyroassociative* and *gyrocommutative* laws, *gyroautomorphisms, gyrotranslations,* etc. We demonstrate the superiority of our gyrovector space formalism in capturing analogies by deriving the *Hyperbolic Pythagorean Theorem* in a form fully analogous to its Euclidean counterpart, thus contrasting it with the standard form in which the Hyperbolic Pythagorean Theorem is known in the literature. The hyperbolic metric, that supports the Hyperbolic Pythagorean Theorem, has a dual metric. We show that the dual metric does not support a Pythagorean theorem but, instead, it supports the $\pi-$Theorem according to which the sum of the three dual angles of a hyperbolic triangle is π.

1. Introduction

Some time in the sixth century B.C. Pythagoras of Samos discovered the theorem that now bears his name. The conception of the Pythagorean Theorem is one of the most profound accomplishments in the history of mathematics, marking the first known intellectual leap from empirical speculation into deductive reasoning. This celebrated theorem is one of the most important theorems in the whole realm of geometry and is known in history as the 47th proposition, that being its number in the first book of Euclid’s Elements. Not unexpectedly, therefore, Stillwell’s *Mathematics and Its History* book begins with the theorem of Pythagoras.(1) The Pythagorean theorem attracts remarkable attention, as evidenced from the number of proofs that have been given to it.(2) It allows the concept of orthogonality to readily be accepted in mathematics, playing an important role in the theory of vector spaces which, in turn, form the setting for Euclidean geometry - the geometry that underlies classical mechanics.

Our recent extension of vector spaces into their hyperbolic counterparts, called *gyrovector spaces*,(3) accommodates the setting for hyperbolic geometry - the geometry that underlies Einstein’s relativistic mechanics. The Hyperbolic Pythagorean Theorem appears in the present article as an identity in a gyrovector space that expresses the square of the hyperbolic length of the hypotenuse of a hyperbolic right angled triangle as the Einstein sum of the squares of the hyperbolic lengths of the other two sides,

$$\|A\|^2 \oplus \|B\|^2 = \|C\|^2 \tag{1.1}$$

shown in Fig. 1 and in Theorem 4.3, where the binary operation $\oplus$ is the Einstein velocity addition, Eq. (2.12). As such, it extends the validity of the Pythagorean theorem in its original spirit beyond Euclidean geometry, and highlights the long path from Pythagoras to Einstein.

FIGURE 1

Clearly, a modified hyperbolic Pythagorean theorem fails when one applies the ordinary addition, +, in (1.1) instead of the Einstein addition, $\oplus$. The failure of (1.1) with + instead of $\oplus$ has been emphasized by Wallace and West. Since the validity of (1.1) has gone unnoticed in the literature they concluded that "the Pythagorean theorem [in its original spirit, expressing a sum of squares as a square] is strictly Euclidean."(4) It is therefore interesting to realize that the Pythagorean theorem is valid in non-Euclidean geometry as well if it is appropriately linked to the Einstein theory of relativity.

Physicists and mathematicians tend to think of symmetry as being virtually synonymous with the theory of groups and their actions.(5) However, being nonassociative, the Einstein velocity addition demonstrates that also some non-group groupoids can measure symmetry. Unlike velocity addition in classical mechanics, which is a group operation, the Einstein velocity addition is not a group operation. Is the breakdown of the associativity of the velocity composition law in the transition from classical to relativistic velocity addition associated with loss of symmetry? It has been discovered in 1988 that the seemingly lost symmetry is, in fact, concealed in the relativistic effect known as Thomas precession, or, Thomas gyration.(6, 7) Taking the role played by the Thomas precession into consideration, the Einstein velocity addition appears to be a gyrocommutative gyrogroup operation,(3) in full analogy with its classical counterpart, which is a commutative group operation.

The grand scientific achievement of this century in mathematical beauty and experimental verifications has been special theory of relativity, with its Einstein addition and Thomas gyration. The theory of gyrogroups and gyrovector spaces that has been presented in Ref. 3 captures the symmetry that has seemingly been lost in the transition from the ordinary vector addition to the Einstein velocity addition. It is particularly interesting to realize in this paper that the Einstein velocity addition law captures the lasting beauty of the Euclidean Pythagorean Theorem that has seemingly been lost in its transition to hyperbolic geometry; it can now be seen by inhabitants of hyperbolic worlds as well.

Furthermore, the Hyperbolic Pythagorean Theorem in its present, new form constitutes an important step towards our envisaged axiomatic approach to gyrovector spaces, guided by analogies shared with vector spaces and Euclidean geometry, to which Thomas precession gives rise. The basic role that Thomas precession plays in our gyrovector space theory and in hyperbolic geometry highlights Gravity Probe B, a NASA project aimed at the measurement of the Thomas precession of gyroscopes in Earth orbit,[(8)] to test general relativity.

The fascinating journey from Pythagoras to Einstein presents itself in this article by means of our gyrovector space theory that we have developed in Ref. 3. It allows, by means of the Einstein velocity addition, the Hyperbolic Pythagorean Theorem to be presented in a form fully analogous with the form originally derived by Pythagoras, as shown in Fig. 1. The long path from Pythagoras to Einstein has been described (i) by Friedrichs in his book *From Pythagoras to Einstein*,[(9)] tracing the Pythagorean Theorem through its various metamorphoses leading to $E = mc^2$; and (ii) by Lanczos in his book *Space Through the Ages The Evolution of Geometrical Ideas from Pythagoras to Hilbert and Einstein.*[(10)] Our presentation of the Hyperbolic Pythagorean Theorem as an identity that expresses the square of the hyperbolic length of the hypotenuse of a hyperbolic right angled triangle as the Einstein sum of the squares of the hyperbolic lengths of the other two sides, exhibits a novel feature of the path from Pythagoras to Einstein.

By deciphering the algebraic structure concealed in the Thomas precession of the special theory of relativity it became possible to understand the Einstein velocity addition in terms of analogies that it shares with the vector addition of Euclidean geometry. More generally, the Thomas precession is abstracted to the Thomas gyration, giving rise to a grouplike structure called a gyrogroup. Exploring the resulting gyrogroup theory along lines parallel to group theory,[(3)] we introduce a scalar multiplication into some gyrocommutative gyrogroups in the same way that scalar multiplication is introduced into some commutative groups to construct vector spaces. The resulting gyrovector spaces then provide the setting for hyperbolic geometry in the same way that vector spaces provide the setting for Euclidean geometry. Interestingly, there are more gyrovector spaces than vector spaces since two vector spaces with equal dimensions are isomorphic while any two non-isomorphic symmetric spaces give rise to corresponding non-isomorphic gyrovector spaces.[(11)]

The name "hyperbolic geometry" for the Non-Euclidean geometry of Bolyai and Lobachevsky was introduced by Klein in 1871.[(12)] Five years after Einstein's 1905 paper that founded special relativity theory,[(13)] a Croatian mathematician, Vladimir Varičak,[(14)] pointed out that relativistic velocity spaces are governed by hyperbolic geometry. Following Varičak's pioneering work and recently discovered analogies shared by hyperbolic and Euclidean geometries, hyperbolic geometry can now effectively be used in the study of velocity spaces in relativistic mechanics in the same way that Euclidean geometry is employed for the study of velocity spaces in classical mechanics. Specifically, (i) following the discovery of the mathematical regularity stored in the relativistic effect known as Thomas precession in 1988;[(6)] (ii) following the abstraction of the Thomas precession into the Thomas gyration in 1991;[7] and (iii) following the introduction of compatible scalar multiplication in 1992,[15] it is now possible to study hyperbolic geometry by means of novel striking analogies that it shares with Euclidean geometry, resulting in corresponding analogies shared by relativistic mechanics and classical mechanics.

In order to develop the setting for Euclidean geometry one considers a commutative group of elements, called vectors, for which inner product and scalar multiplication are defined. In full analogy, to develop the setting for hyperbolic geometry we consider a gyrocommutative gyrogroup of elements called gyrovectors, for which inner product and scalar multiplication are defined. The prefix "gyro" that we extensively use stems from the underlying Thomas gyration.

Historically, the concept of gyrogroup evolved from the 1988 discovery of the gyroassociative law that the Einstein velocity addition obeys.[6] Our first axiomatic approach was proposed in 1989.[16] The various forerunners of gyrogroups and gyrocommutative gyrogroups were K-loops, proposed in 1989,[16] WAGs (Weakly Associative Groups) and WACGs (Weakly Associative Commutative Groups), proposed in 1990.[17] WACGs were later termed *gyrogroups* in 1991,[7] a term which was modified in 1997 into *gyrocommutative gyrogroups* to accommodate non-gyrocommutative gyrogroups as well.[3] Gyrocommutative gyrogroups that admit scalar multiplication became gyrovector spaces in Ref. 3. It is the exotic grouplike structure to which the Thomas gyration gives rise which accommodates the gyrovector space that allows the Hyperbolic Pythagorean Theorem to be presented in this article.

We prefer the term "gyrogroup", that we have coined in 1991,[7] over the term "K-loop" that we have coined in 1989,[16] since it is an integral part of our gyroterminology, in which we emphasize by means of Thomas gyration analogies shared by Euclidean geometry and hyperbolic geometry, as well as corresponding analogies shared by classical mechanics and relativistic mechanics. Thus, for instance, the Einstein velocity addition is a gyrocommutative gyrogroup operation, in full analogy with the ordinary vector addition, which is a commutative group operation.[3] Similarly, the (homogeneous, proper, orthochronous) Lorentz group is the gyrosemidirect product of a gyrogroup of left gyrotranslations and a group of rotations in full analogy with the (homogeneous) Galilei group which is the semidirect product of a group of translations and a group of rotations.[18, 19] The latter, in turn, is isomorphic with the group of Euclidean rigid motions and the former, by analogy, is considered as the group isomorphic with the group of relativistic "rigid motions".[20]

Of particular interest are analogies shared by gyrovector spaces and vector spaces and analogies shared by their respective geometries, the hyperbolic and the Euclidean ones. Thus, for instance, the unique geodesic line that contains two given points $\mathbf{a}$ (for $t=0$) and $\mathbf{b}$ (for $t=1$) of a vector space is given by the parametric equation

$$\mathbf{a} + (-\mathbf{a} + \mathbf{b})t\,, \qquad\qquad t \in \mathbb{R} \tag{1.2}$$

where + is a commutative group operation and where the product between a vector and a scalar is the common scalar multiplication in a vector space. In full analogy, it has been shown in Ref. 3 that the unique geodesic line that contains two given points $\mathbf{a}$ (for $t=0$) and $\mathbf{b}$ (for $t=1$) of a gyrovector space is given by

$$\mathbf{a} \oplus (-\mathbf{a} \oplus \mathbf{b}) \odot t\,, \qquad\qquad t \in \mathbb{R} \tag{1.3}$$

where $\oplus$ is a gyrocommutative gyrogroup operation and $\odot$ is the scalar multiplication in a gyrovector space that we define in Defs. 2.1 – 2.3 of Section 2.

Furthermore, curves that describe uniform acceleration in velocity spaces are geodesics.[21] Hence, as explained in Ref. 3, the time dependent velocity $\mathbf{v}_g(t)$ of an initial, relativistically admissible velocity $\mathbf{v}_0$ and a constant, relativistically admissible acceleration $\mathbf{a}$ is given in relativistic mechanics by the equation

$$\mathbf{v}_g(t) = \mathbf{v}_0 \oplus \mathbf{a} \odot t, \qquad t \in \mathbb{R} \qquad (Fig.\ 2) \tag{1.4}$$

in full analogy with its classical mechanics counterpart, $\mathbf{v}(t) = \mathbf{v}_0 + \mathbf{a}t$. The curves (1.4) are the well known geodesics of hyperbolic geometry relative to the hyperbolic metric (5.8) that will be presented in Section 5. One of these curves in the Poincaré disk model of hyperbolic geometry is shown in Fig. 2.

FIGURE 2

Since we wish to emphasize analogies shared by classical and relativistic mechanics that correspond to analogies shared by Euclidean and hyperbolic geometry, we explore in Section 5 the dual metric, relative to which the curves

$$\mathbf{v}_d(t) = \mathbf{a} \odot t \oplus \mathbf{v}_0, \qquad t \in \mathbb{R} \qquad (Fig.\ 3) \tag{1.5}$$

are geodesics, called dual geodesics. Since the Einstein velocity addition is gyrocommutative, Def. 2.2, rather than commutative, the curves (1.4) are different from the curves (1.5) whenever $\mathbf{v}_0$ and $\mathbf{a}$ are nonparallel. The curves (1.5) are known in hyperbolic geometry as *hypercircles* or *equidistant curves*.[22] One of these curves in the Poincaré disk model of hyperbolic geometry is shown in Fig. 3.

FIGURE 3

Being equal, $\mathbf{v}_0 + \mathbf{a}t$ and $\mathbf{a}t + \mathbf{v}_0$ are equally significant in classical mechanics, where they describe the velocity of a uniformly accelerated object. If analogies are to be validated, the physical and the geometric significance of the curves in (1.4) and (1.5) should be equal as well. Promising results indicating that, indeed, (1.5) is likely to be found as significant as (1.4) are presented in Section 5. It is shown in that section that, like (1.4), the curves (1.5) are also geodesics, but relative to a *dual metric* that emerges naturally in Section 5. We then discover that the Euclidean Pythagorean Theorem has no hyperbolic counterpart in the dual metric. As to compensate for this loss, we show in Section 5 that the sum of the three dual angles of any dual triangle in hyperbolic geometry is π. This observation strengthens our conjecture that no Euclidean property has been lost in the transition from Euclidean geometry to hyperbolic geometry. The seemingly lost properties, like parallelism, in fact reappear with the dual metric and its dual geodesics and dual angles.

It is interesting to realize from Figs. 2 and 3 that in hyperbolic geometry geodesics and equidistant curves are closely related by the noncommutativity of the gyrogroup operation $\oplus$ in Eqs. (1.4) and (1.5). We will see in this paper that (i) on one hand, only the curves (1.4) give rise to the Hyperbolic Pythagorean Theorem, in Section 4. But, (ii) on the other hand, only the curves (1.5) give rise to the hyperbolic $\pi-$Theorem, in Section 5.

2. The Poincaré ball model of hyperbolic geometry and its associated Möbius gyrovector space

Hyperbolic geometry is studied in the literature by means of several standard models. The various models of hyperbolic geometry are equivalent to each other in the sense that if inhabitants of one model of hyperbolic geometry were to communicate with those of another model by telephone, they would not be able to tell their worlds apart. In fact, it was Poincaré who analyzed the role of geometry in physics and concluded that one model of a geometry cannot be more true than another model of the same geometry, it can only be more convenient.(23) Thus, in particular, in the study of relativistic velocities we are free to select any model of hyperbolic geometry. Three mutually isomorphic gyrovector spaces which underlie three attractive models of hyperbolic geometry have been studied in Ref. 3. These are (i) the Einstein gyrovector space, (ii) the Möbius gyrovector space, and (iii) the Weierstrass gyrovector space, which underlie respectively (i) the Klein disk model of hyperbolic geometry, (ii) the Poincaré disk model of hyperbolic geometry; and (iii) the Weierstrass model of hyperbolic geometry. The latter is an attractive model of hyperbolic geometry since, unlike other models, its underlying space is not restricted to the ball or to a half-space of an inner product space.

It seems that the natural choice for the study of special relativity would be the Einstein gyrovector space since its binary operation is the Einstein velocity addition. However, due to their mutual equivalence, the selection of any particular model of hyperbolic geometry for the study of relativistic physics is legitimate. Being conformal, it is the Poincaré disk model which is particularly suitable for graphical presentation of the Hyperbolic Pythagorean Theorem. Within Euclidean geometry, this model of hyperbolic geometry exhibits Euclidean angles since the Poincaré measure of an angle is given by the Euclidean measure of the angle formed by Euclidean tangent rays.(24) Therefore, we select in this article the Poincaré disk model and its generalization into the ball of the abstract real inner product space for the presentation of the Hyperbolic Pythagorean Theorem.

Gyrogroups are defined in Ref. 3 in terms of their underlying axioms. Gyrovector spaces, in turn, are studied in Ref. 3 by means of several concrete examples. We feel that concrete examples of gyrovector spaces should be further explored before an axiomatic approach can be taken. Our present Hyperbolic Pythagorean Theorem in Möbius gyrovector spaces in a form that exhibits analogies shared with its Euclidean counterpart justifies an optimistic outlook for eventual construction of the axiomatic foundation of gyrovector spaces. The gyrogroup definition and the definition of the Möbius gyrovector space, as presented in Ref. 3, follow.

DEFINITION 2.1 *(Gyrogroups)* A groupoid $(G, \oplus)$ (that is, a non-empty set with a binary operation) is a *gyrogroup* if its binary operation satisfies the following axioms and properties. In G there exists a unique element, 0, called the identity, satisfying

$(G1)$ $0 \oplus a = a \oplus 0 = a$ Identity

for all $a \in G$. For any $a \in G$ there exists in G a unique inverse, $-a$, satisfying

$(G2)$ $-a \oplus a = a \oplus (-a) = 0$ Inverse

Moreover, for any $a, b \in G$ the map $\mathrm{gyr}[a;b]$ of G into itself, given by the equation

$$\mathrm{gyr}[a;b]z = -(a \oplus b) \oplus (a \oplus (b \oplus z)) \tag{2.1}$$

for all $z \in G$, is an automorphism of G (that is, a bijection of G that respects its binary operation $\oplus$), and the following hold for all $a, b, c \in G$.

$(G3a)$	$a \oplus (b \oplus c) = (a \oplus b) \oplus \mathrm{gyr}[a;b]c$	Left gyroassociative Law
$(G3b)$	$(a \oplus b) \oplus c = a \oplus (b \oplus \mathrm{gyr}[b;a]c)$	Right gyroassociative Law
$(G4a)$	$\mathrm{gyr}[a;b] = \mathrm{gyr}[a \oplus b, b]$	Left Loop Property
$(G4b)$	$\mathrm{gyr}[a;b] = \mathrm{gyr}[a, b \oplus a]$	Right Loop Property

DEFINITION 2.2 *(Gyrocommutative* Gyrogroups) A gyrogroup $(G, \oplus)$ is *gyrocommutative* if for all $a, b \in G$

$(G6)$ $a \oplus b = \mathrm{gyr}[a;b](b \oplus a)$ Gyrocommutative Law

Examples of gyrogroups, both gyrocommutative and non-gyrocommutative, abound in several areas of mathematics and physics.[3, 25] To gain experience with the beautiful gyrogroup structure, readers may consider the simplest non-group nongyrocommutative gyrogroup $(T_4, \odot)$ where T_4 is the set of all 4×4 real (or complex) upper triangular matrices with diagonal elements 1,

$$M(\mathbf{x}) = \begin{pmatrix} 1 & x_1 & x_2 & x_3 \\ 0 & 1 & x_4 & x_5 \\ 0 & 0 & 1 & x_6 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{2.2a}$$

$\mathbf{x} = (x_1, x_2, x_3, x_4, x_5, x_6) \in \mathbb{R}^6$ (or $\mathbb{C}^6$). The gyrogroup operation $\odot$ in T_4 is given in terms of matrix multiplication by the equation

$$M(\mathbf{x}) \odot M(\mathbf{y}) = M^2(\mathbf{x})M(\mathbf{y})M^{-1}(\mathbf{x}) \tag{2.2b}$$

where M^2 is the square of a matrix M and M^{-1} is its inverse. The gyrogroup inverse of a matrix $M(\mathbf{x}) \in (T_4, \odot)$ is, clearly, given by matrix inversion, $(M(\mathbf{x}))^{-1} = M^{-1}(\mathbf{x})$. Hence, for instance, according to Eq. (2.1) the gyroautomorphisms $\mathrm{gyr}[M(\mathbf{a}), M(\mathbf{b})]$ of the gyrogroup $(T_4, \odot)$ are given in terms of their effects on

$M(\mathbf{z})$ by the equation

$$\mathrm{gyr}[M(\mathbf{a}), M(\mathbf{b})]M(\mathbf{z}) = (M(\mathbf{a}) \odot M(\mathbf{b}))^{-1} \odot (M(\mathbf{a}) \odot (M(\mathbf{b}) \odot M(\mathbf{z}))) \tag{2.2c}$$

for all $M(\mathbf{a})$, $M(\mathbf{b})$, $M(\mathbf{z}) \in (T_4, \odot)$, where we use multiplicative, rather than additive, notation. This and other related non-gyrocommutative gyrogroups are studied in Ref. 25.

Like groups, some gyrocommutative gyrogroups give rise to gyrovector spaces. Of particular interest in this article is the Möbius gyrocommutative gyrogroup that has been presented in Ref. 3. In the following Definition we extend the definition of a Möbius gyrogroup into that of a Möbius gyrovector space.

A most elegant example of a gyrogroup arises in the study of the Möbius transformation group of the complex unit disk D, $D = \{ z \in \mathbb{C} : |z| < 1 \}$, of the complex plane $\mathbb{C}$. Möbius transformations of the disk,

$$z \mapsto e^{i\theta}\frac{z_0 + z}{1 + \bar{z}_0 z} = e^{i\theta}(z_0 \oplus z) \tag{2.3}$$

consist of rotations and translation like maps. The usefulness of (2.3) in string theory is evidenced from Ref. 26. The complex unit disk and its Möbius transformations (2.3) form the Poincaré disk model of hyperbolic geometry.[27] The binary operation $\oplus$ that Eq. (2.3) defines in the complex unit disk is a gyrogroup operation.[28] Rather than confining our study to the Poincaré disk model of hyperbolic geometry, we will present in the following definition the (generalized) Möbius transformation of the ball, allowing us to expand our study to the Poincaré ball model of hyperbolic geometry in any dimension, finite of infinite.

The Möbius addition $\oplus$ in the disk D is studied in the literature as a transformation ϕ_a of the disk rather than a binary operation in the disk. It is known to be useful in the modern geometric point of view in complex function theory which began with Ahlfors' classic paper that demonstrated that the Schwartz lemma can be viewed as an inequality of differential geometric quantities, curvatures, on the disk.[29] The usefulness of the Möbius addition in geometric function theory is evidenced from Ref. 30 where it is noted that the Möbius transformation ϕ_a of the disk,

$$\phi_a(z) = \frac{z + a}{1 + \bar{a}z}$$

$a, z \in D$, does not preserve Euclidean distance, but it does preserve Poincaré distance. We consider $\phi_a(z)$ as a left *gyrotranslation* of a by z, and express it in terms of the Möbius addition as $\phi_a(z) = z \oplus a$. The qualification of $\phi_a(z)$ to be viewed as a binary operation $\oplus$ between a and z stems from the fact that the resulting binary operation $\oplus$ is a gyrogroup operation.

Eq. (2.3) represents a special Cartan's decomposition. Further generalization of the Möbius transformation situation in (2.3) and its resulting gyrogroup to the so called *Riemannian globally symmetric spaces of non-compact type* is possible, as we will indicate at the end of this article. We are, however, interested in the generalization of (2.1) to the ball of any real inner product space that we present in the following definition.

DEFINITION 2.3 *(The Möbius Gyrovector space)* Let $(V_\infty, +, \cdot)$ be a real inner product space, and let V_s be its open s-ball, s being an arbitrary fixed positive constant,

$$V_s = \left\{ \mathbf{v} \in V_\infty : \ \|\mathbf{v}\| < s \right\}$$

The Möbius gyrovector space induced by V_∞ is the triple $(V_s, \oplus, \odot)$ equipped with inner product and norm that it inherits from V_∞, where (i) the binary operation $\oplus$ in V_s is given by the equation

$$\mathbf{u} \oplus \mathbf{v} = \frac{1}{1 + \frac{2}{s^2}\mathbf{u}\cdot\mathbf{v} + \frac{1}{s^4}\|\mathbf{u}\|^2\|\mathbf{v}\|^2} \left\{ (1 + \frac{2}{s^2}\mathbf{u}\cdot\mathbf{v} + \frac{1}{s^2}\|\mathbf{v}\|^2)\mathbf{u} + (1 - \frac{1}{s^2}\|\mathbf{u}\|^2)\mathbf{v} \right\} \tag{2.4}$$

representing a generalized Möbius transformation $\mathbf{x} \mapsto \mathbf{u} \oplus \mathbf{x}$ of the ball; and (ii) the scalar multiplication $\odot$ in V_s is given by the equation

$$\begin{aligned} r \odot \mathbf{v} &= s \frac{\left(1 + \frac{\|\mathbf{v}\|}{s}\right)^r - \left(1 - \frac{\|\mathbf{v}\|}{s}\right)^r}{\left(1 + \frac{\|\mathbf{v}\|}{s}\right)^r + \left(1 - \frac{\|\mathbf{v}\|}{s}\right)^r} \frac{\mathbf{v}}{\|\mathbf{v}\|} \\ &= s \tanh(r \tanh^{-1}\frac{\|\mathbf{v}\|}{s}) \frac{\mathbf{v}}{\|\mathbf{v}\|} \end{aligned} \tag{2.5}$$

where $r \in I\!R$, $\mathbf{v} \in V_s$, $\mathbf{v} \neq \mathbf{0}$; and by $r \odot \mathbf{0} = \mathbf{0}$.

The Möbius addition (2.4) in the ball V_s is reducible to the well known Möbius transformation $z_1 \oplus z_2 = (z_1 + z_2)/(1 + \bar{z}_1 z_2)$ of the complex disk, Eq. (2.3), as we will show in Eq. (2.18) below. In the limit of large s, $s \to \infty$, the ball V_s expands to the whole of its space V_∞, and the Möbius addition reduces to the ordinary addition of vectors in V_∞.

The pair $(V_s, \oplus)$ forms a gyrocommutative gyrogroup,[31] called a Möbius gyrogroup. The triple $(V_s, \oplus, \odot)$ is accordingly called a Möbius gyrovector space. The relationship between the Möbius addition and the Einstein velocity addition is presented in Ref. 3 where the two binary operations in the ball are respectively denoted by $\oplus_M$ and $\oplus_E$.

The scalar multiplication in a gyrovector space, consistent with certain elementary properties that one expects a notion of scalar multiplication to satisfy, is presented in Ref. 3. In addition, one may note for later reference that for any element $\mathbf{v}$ in a Möbius gyrovector space and $r \in I\!R^+$ a positive real number we have

$$\frac{r \odot \mathbf{v}}{\|r \odot \mathbf{v}\|} = \frac{\mathbf{v}}{\|\mathbf{v}\|} \tag{2.6}$$

The Lorentz factor $\gamma_\mathbf{v}$ in a Möbius gyrovector space $(V_s, \oplus, \odot)$ is given by the equation

$$\gamma_\mathbf{v} = \frac{1}{\sqrt{1 - \frac{\|\mathbf{v}\|^2}{s^2}}}, \tag{2.7}$$

satisfying

$$\gamma_{\mathbf{u}\oplus\mathbf{v}} = \gamma_{\mathbf{u}}\gamma_{\mathbf{v}}\sqrt{1+\frac{2}{s^2}\mathbf{u}\cdot\mathbf{v}+\frac{1}{s^4}\|\mathbf{u}\|^2\|\mathbf{v}\|^2}\,. \qquad (2.8)$$

The gyroautomorphism operation gyr of the Möbius gyrovector space V_s is given by the equation

$$\mathrm{gyr}[\mathbf{u};\mathbf{v}]\,\mathbf{z} = -(\mathbf{u}\oplus\mathbf{v}) \oplus \{\mathbf{u}\oplus(\mathbf{v}\oplus\mathbf{z})\} \qquad (2.9)$$

for all $\mathbf{u},\mathbf{v},\mathbf{z}\in V_s$, as wee see from Eq. (2.1). It can be shown that the gyroautomorphism $\mathrm{gyr}[\mathbf{u};\mathbf{v}]$ in (2.9), generated by any $\mathbf{u},\mathbf{v}\in V_s$, is an orthogonal transformation of V_∞.

A Möbius gyrovector space V_s is a metric space equipped with the Poincaré distance function

$$d(\mathbf{x},\mathbf{y}) = \|\mathbf{x}\ominus\mathbf{y}\| \qquad (2.10)$$

where we use the obvious notation $\mathbf{x}\ominus\mathbf{y}$ to denote $\mathbf{x}\oplus(-\mathbf{y})$. The Poincaré distance function maps $V_s\times V_s$ onto the open interval (or, the s-ball) $\mathbb{R}_s=(-s,s)$ of the real line $\mathbb{R}$, satisfying the *gyrotriangle inequality*

$$d(\mathbf{x},\mathbf{z}) \le d(\mathbf{x},\mathbf{y}) \oplus d(\mathbf{y},\mathbf{z}) \qquad (2.11)$$

for all $\mathbf{x},\mathbf{y},\mathbf{z}\in V_s$, where equality holds if and only if $\mathbf{y}$ lies on the geodesic segment which joins $\mathbf{x}$ and $\mathbf{z}$. The binary operation $\oplus$ in Eqs. (2.10) and (2.11) is the Möbius addition. However, the one in Eq. (2.11) can also be regarded as the Einstein velocity addition since, when applied to parallel vectors (particularly, vectors of dimension 1) the Möbius addition is identical with the Einstein velocity addition. The binary operation $\oplus$ in the gyrotriangle inequality (2.11) can thus be regarded as the Einstein addition in $\mathbb{R}_s$. The Einstein velocity addition in $\mathbb{R}_s$ follows from Def. 2.3, as explained in Eq. (2.12) below. In the special case when $V_s=\mathbb{R}_s^2$ is the s-disk of the Euclidean plane $V_\infty=\mathbb{R}^2$, one recovers from (2.11) the well known geodesics of plane hyperbolic geometry.[32] These are the circular arcs which intersect the boundary of the disk orthogonally,[24] as shown in Fig. 2.

The set of norms of all elements of the ball V_s of V_∞ is the ball $\mathbb{R}_s=(-s,s)$ of the real line $\mathbb{R}$. Inhabitants of the Poincaré ball model of hyperbolic geometry naturally consider $\mathbb{R}_s$ as the set of the whole of their real numbers that describe speeds. Accordingly, their hyperbolic metric spaces involve the gyrotriangle inequality (2.11) rather than the standard triangle inequality in metric spaces, like the one in Eq. (2.14) below.

Realizing the abstract real inner product space V_∞ by the real line $\mathbb{R}$, Def. 2.3 reduces to the definition of a binary operation $\oplus$ in $\mathbb{R}_s$, turning it into a commutative group $(\mathbb{R}_s,\oplus)$ with $\oplus$ given by the Einstein velocity addition of parallel velocities

$$\mathbf{a}\oplus\mathbf{b} = \frac{\mathbf{a}+\mathbf{b}}{1+\mathbf{ab}/s^2} \qquad (2.12)$$

$\mathbf{a},\mathbf{b}\in\mathbb{R}_s$. Moreover, Def. 2.3 also provides a scalar multiplication $\odot$ in the commutative group $(\mathbb{R}_s,\oplus)$, turning it into a vector space $(\mathbb{R}_s,\oplus,\odot)$ over $\mathbb{R}$ with $\odot$ given by the equation

$$r\odot\mathbf{a} = s\tanh\left(r\tanh^{-1}\frac{\mathbf{a}}{s}\right) \qquad (2.13)$$

that can be recovered from Eq. (2.5) when $V_\infty=\mathbb{R}$.

We may note that while $1\odot\mathbf{a}=\mathbf{a}$, in general, $r\odot 1\ne r$. Hence we view $1\in\mathbb{R}_s$ (when $s>1$), as well as every element of $\mathbb{R}_s$, as a vector rather than a scalar.

Incidentally, while the pair $(I\!R_s, \oplus)$ is frequently used in the literature as an example of an exotic group,[33] the triple $(I\!R_s, \oplus, \odot)$ is not used to exhibit an example of an exotic vector space. In fact, it is due to the absence of Thomas gyration in one dimensional space that $(I\!R_s, \oplus)$ and $(I\!R_s, \oplus, \odot)$ have their respective commutative group and vector space structure. It is the presence of the Thomas gyration in dimensions higher than 1 that distorts these group and vector space structures. It converts (i) the commutative group $(I\!R_s, \oplus)$ into a gyrocommutative gyrogroup $(V_s, \oplus)$ and (ii) the vector space $(I\!R_s, \oplus, \odot)$ into a gyrovector space $(V_s, \oplus, \odot)$.

The axioms underlying gyrocommutative gyrogroups are well understood in Ref. 3 in terms of analogies that they share with those of commutative groups. Unfortunately, we cannot presently offer an axiomatic approach to gyrovector spaces since a *gyrodistributive* law in $(V_s, \oplus, \odot)$ that reduces to the distributive law $r \odot (\mathbf{a} \oplus \mathbf{b}) = r \odot \mathbf{a} \oplus r \odot \mathbf{b}$ in $(I\!R_s, \oplus, \odot)$, $r \in I\!R$, is, as yet, unknown. The elusive gyrodistributive law defies, to date, the exploration of concrete examples of gyrovector spaces. We therefore hope that the hyperbolic law of cosines and its resulting hyperbolic polarization identity and the Hyperbolic Pythagorean Theorem that we present in this article constitute an important step towards our envisaged axiomatic approach to gyrovector spaces, guided by analogies shared with vector spaces.

The right hand side of the gyrotriangle inequality (2.11) can be written as

$$s \tanh(\tanh^{-1}\frac{d(\mathbf{x},\mathbf{y})}{s} + \tanh^{-1}\frac{d(\mathbf{y},\mathbf{z})}{s})$$

Hence the gyrotriangle inequality (2.11) can be written as a triangle inequality,

$$\tanh^{-1}\frac{d(\mathbf{x},\mathbf{z})}{s} \le \tanh^{-1}\frac{d(\mathbf{x},\mathbf{y})}{s} + \tanh^{-1}\frac{d(\mathbf{y},\mathbf{z})}{s} \tag{2.14}$$

Eq. (2.14) involves the standard addition of real numbers in $I\!R$ as opposed to Eq. (2.11), which involves the Einstein addition of real numbers in $I\!R_s$. Hence, it is customary in the literature to define the hyperbolic metric as

$$2\tanh^{-1}(d(\mathbf{x},\mathbf{y})/s) = \ln\frac{s+d(\mathbf{x},\mathbf{y})}{s-d(\mathbf{x},\mathbf{y})} \tag{2.15}$$

with $s = 1$.[34] The factor 2 in the metric (2.15) is chosen in order to make the resulting Gaussian curvature -1 when $s = 1$,[35] as opposed to the Gaussian curvature of the Poincaré metric (2.10) which is $-4/s^2$ according to Eq. (5.18b) in Section 5. We, however, prefer to leave s as a free positive parameter,[36] and employ the Poincaré metric $d(\mathbf{x},\mathbf{y})$ with its gyrotriangle inequality (2.11) in order to emphasize analogies shared by the hyperbolic and the Euclidean geometry. As a result, we will be rewarded in this article by the discovery that the Hyperbolic Pythagorean Theorem can be presented in a form fully analogous to its Euclidean counterpart, expressing an Einstein sum of squares as a square, Theorem 4.3 and Fig. 6.

The gyrotriangle inequality (2.11) follows from Eq. (2.8), Cauchy-Schwarz inequality, and Theorem 5.8 of Ref. 3. Details are given in Ref. 31. Möbius addition $\oplus$, Eq. (2.4), is known in the literature on Möbius groups in disguise. It has been studied by Ahlfors in connection with the map $x \to T_{\mathbf{a}}(\mathbf{x})$ that he advocated to use as a standard conformal map of the ball B^n of the Euclidean n-space $I\!R^n$ on itself. It turns out that this map expresses the Möbius addition $\oplus$ by means of the equation $T_{\mathbf{a}}(\mathbf{x}) = -\mathbf{a} \oplus \mathbf{x}$.[35, 37]

Moreover, Ahlfors did present the identity

$$T_{\mathbf{y}}(\mathbf{x}) \;=\; -\Delta(\mathbf{x},\mathbf{y})\,T_{\mathbf{x}}(\mathbf{y}) \tag{2.16}$$

for all $\mathbf{x},\mathbf{y}\in B^n$ that $T_{\mathbf{a}}$ obeys as an interesting but an isolated relation, where $\Delta(\mathbf{x},\mathbf{y})$ is an orthogonal transformation of B^n. It has gone unnoticed that (i) the Ahlfors rotation $\Delta(\mathbf{x},\mathbf{y})$ is analogous to the Thomas precession (or, rotation) of the special theory of relativity; that (ii) Identity (2.16) can be interpreted as a relaxed commutative law for the Möbius addition, our gyrocommutative law (G6) in Def. 2.2; and that (iii) the same Ahlfors rotation Δ that gives rise to a relaxed commutative law in Eq. (2.16), gives rise to a relaxed associative law for the Möbius addition as well, our gyroassociative law (G3) in Def. 2.2. In contrast, we place the Ahlfors rotation (or, equivalently, the Thomas precession) and its abstraction, the Thomas gyration, in the foundations of non-Euclidean geometry.[3]

The Möbius transformation of the ball in higher dimensions, $n\geq 3$, studied in Refs. 31, 35 and 37, is not well known in the standard literature on hyperbolic geometry. In contrast, its special case $n=2$, corresponding to the Poincaré disk is well known. To see this let us, therefore, realize the abstract real inner product space V_∞ in Def. 2.3 by the Euclidean plane $\mathbb{R}^2$, reducing the ball V_s into the Poincaré disk $\mathbb{R}_s^2$ whose points can be represented by the points of the complex s-disk D_s of the complex plane $\mathbb{C}$,

$$D_s \;=\; \left\{ z\in \mathbb{C} : |z| < s \right\} \tag{2.17}$$

Möbius addition $\oplus$, Eq. (2.4) reduces to the well known Möbius transformation of the disk,[31]

$$z_1 \oplus z_2 \;=\; \frac{z_1+z_2}{1+\bar{z}_1 z_2/s^2} \tag{2.18}$$

when V_∞ is realized by $\mathbb{R}^2$, and when complex number representation for vectors in $\mathbb{R}^2$ is employed. Accordingly, the distance function (2.10) reduces to the well known Poincaré distance function

$$d(z_1,z_2) \;=\; |z_1 \ominus z_2| \;=\; \left| \frac{z_1-z_2}{1-\bar{z}_1 z_2/s^2} \right| \tag{2.19}$$

in the Poincaré disk. The Einstein velocity addition for parallel velocities, Eq. (2.12), is recovered from Eq. (2.18) when z_1 and z_2 are real and when s represents the vacuum speed of light.

The binary operation (2.18) in the complex unit disk $D_{s=1}$ is known in the literature in disguise. It is viewed in the literature as a Möbius transformation of $D_{s=1}$ rather than as the Möbius addition. Similarly, also the scalar multiplication (2.5) for the special case when the ball V_s reduces to the complex unit disk $D_{s=1}$ is known in the literature, in disguise. It is viewed in the literature as a means of generating geodesics for the Teichmüller metric,[38] rather than as the Möbius scalar multiplication in a Möbius gyrovector space.

3. Hyperbolic geometry of the ball in a real inner product space

In physics vectors appear as a geometric concept. A basic notion in geometry is that of the point. Hence, if we wish to find a geometry closely tied to a gyrovector space we will have to establish a relationship between gyrovectors and points. This relationship is naturally analogous to the one between vectors and points in vector spaces; see, for instance, Artzy.[39]

Let $V_s = (V_s, \oplus, \odot)$ be the Möbius gyrovector space of the ball V_s of a real inner product space V_∞ with its natural metric $d(\mathbf{x},\mathbf{y}) = \|-\mathbf{x} \oplus \mathbf{y}\|$, known as the Poincaré metric. We call the elements of V_s points and associate a nonzero geometric gyrovector, $-\mathbf{a} \oplus \mathbf{b}$, with any ordered pair $(\mathbf{a},\mathbf{b})$ of distinct points $\mathbf{a},\mathbf{b} \in V_s$. The geometric gyrovector associated with the pair $(\mathbf{a},\mathbf{b})$ has length $d(\mathbf{a},\mathbf{b}) = \|-\mathbf{a} \oplus \mathbf{b}\|$, and it is viewed as a geodesic segment directed from $\mathbf{a}$ to $\mathbf{b}$. The analogies shared with the Euclidean geometric vector $\mathbf{b} - \mathbf{a}$, viewed as a straight arrow of length $\|\mathbf{b} - \mathbf{a}\|$ directed from $\mathbf{a}$ to $\mathbf{b}$ are obvious.

The motions of the ball V_s, which determine its hyperbolic geometry, are (i) the left gyrotranslations $L_\mathbf{x} : V_s \to V_s$, given by

$$L_\mathbf{x}\mathbf{v} = \mathbf{x} \oplus \mathbf{v}$$

$\mathbf{x},\mathbf{v} \in V_s$, and (ii) the rotations of V_s, that is, those isometries of V_s that possess a fixed point.

The length of geometric vectors in V_s is invariant under the motions of V_s. The invariance under rotations is obvious, and the invariance under left gyrotranslations follows from Theorem 5.8 in Ref. 3 noting that gyrations are rotations. Specifically, let $(\mathbf{x} \oplus \mathbf{a}, \mathbf{x} \oplus \mathbf{b})$ a be a left gyrotranslated pair of the pair $(\mathbf{a},\mathbf{b})$ by $\mathbf{x}$ in V_s. It then follows from Theorem 5.8 of Ref. 3 that

$$\begin{aligned} -L_\mathbf{x}\mathbf{a} \oplus L_\mathbf{x}\mathbf{b} &= -(\mathbf{x} \oplus \mathbf{a}) \oplus (\mathbf{x} \oplus \mathbf{b}) \\ &= \mathrm{gyr}[\mathbf{x};\mathbf{a}](-\mathbf{a} \oplus \mathbf{b}) \end{aligned} \tag{3.1}$$

so that, noting that gyrations are isometries, both the geometric gyrovectors associated with the pair $(\mathbf{a},\mathbf{b})$ and with its left gyrotranslated pair $(L_\mathbf{x}\mathbf{a}, L_\mathbf{x}\mathbf{b}) = (\mathbf{x} \oplus \mathbf{a}, \mathbf{x} \oplus \mathbf{b})$ have equal lengths

$$\|-L_\mathbf{x}\mathbf{a} \oplus L_\mathbf{x}\mathbf{b}\| = \|-\mathbf{a} \oplus \mathbf{b}\| \tag{3.2}$$

Hence, following Klein's Erlangen Program that Klein announced at the University of Erlangen in 1872,[24] the length of a geometric gyrovector has geometric significance in the geometry that is determined by the group of motions of V_s.

Eqs. (3.1) and (3.2) show that, unlike Euclidean geometry, in hyperbolic geometry a geometric vector from $\mathbf{a}$ to $\mathbf{b}$ is, in general, not equivalent to its left gyrotranslated gyrovector; it is only its length which remains invariant under a left gyrotranslation.

The origin is, however, a special point in hyperbolic geometry in the sense that every geometric vector from $\mathbf{a}$ to $\mathbf{b}$ is equivalent to a geometric vector from the origin, $\mathbf{0}$, to $-\mathbf{a} \oplus \mathbf{b}$. This follows from Eq. (3.1) with $\mathbf{x} = -\mathbf{a}$, noting that $\mathrm{gyr}[-\mathbf{a};\mathbf{a}] = id$ is the identity transformation. Specifically, thus, the two geometric gyrovectors determined by the two ordered pairs $(\mathbf{a},\mathbf{b})$ and $(\mathbf{0}, -\mathbf{a} \oplus \mathbf{b})$ are equivalent.

A vector in physics is determined by its length and relative orientation. By analogy, we wish that also a geometric gyrovector be determined by its length and relative orientation. Being guided by analogies, to accomplish this task we define the cosine of the angle α between the two geometric gyrovectors associated with the pairs $(\mathbf{a},\mathbf{b})$ and $(\mathbf{a},\mathbf{c})$, of which the first entries are coincident, Fig. 4, by the inner product

$$\cos\alpha = \frac{-\mathbf{a}\oplus\mathbf{b}}{\|-\mathbf{a}\oplus\mathbf{b}\|}\cdot\frac{-\mathbf{a}\oplus\mathbf{c}}{\|-\mathbf{a}\oplus\mathbf{c}\|} \tag{3.3}$$

Eq. (3.3) determines the angle between two rays emanated from a common point. The angle is either $\pm\alpha$ or $\pi\pm\alpha$, depending on the direction of the rays, $0\le\alpha\le\frac{\pi}{2}$, in full analogy with angles between directed rays in Euclidean geometry.

FIGURE 4

The angle α in (3.3) is in fact the angle between two geodesic rays, $L_{\mathbf{a},\mathbf{b}}$ and $L_{\mathbf{a},\mathbf{c}}$, emanated from a common point, $\mathbf{a}$, and containing respectively two given points, $\mathbf{b}$ and $\mathbf{c}$, Fig. 4. To show that α is independent of the choice of the points $\mathbf{b}$ and $\mathbf{c}$ on the two directed rays $L_{\mathbf{a},\mathbf{b}}$ and $L_{\mathbf{a},\mathbf{c}}$ that are emanated from $\mathbf{a}$, we note that the two rays are given in our analytic hyperbolic geometry by the parametric equations

$$L_{\mathbf{a},\mathbf{b}} = \mathbf{a}\oplus(-\mathbf{a}\oplus\mathbf{b})\odot t$$

and

$$L_{\mathbf{a},\mathbf{c}} = \mathbf{a}\oplus(-\mathbf{a}\oplus\mathbf{c})\odot t$$

where t is a real parameter running over $\mathbb{R}^+=(0,\infty)$.[(3)] Let therefore $\mathbf{b}^*$ and $\mathbf{c}^*$ be any two points other than $\mathbf{a}$ on $L_{\mathbf{a},\mathbf{b}}$ and on $L_{\mathbf{a},\mathbf{c}}$ respectively as shown in Fig. 4. Then, there exist t_1 and t_2 in $\mathbb{R}^+$ such that

$$\begin{aligned}\mathbf{b}^* &= \mathbf{a}\oplus(-\mathbf{a}\oplus\mathbf{b})\odot t_1\\ \mathbf{c}^* &= \mathbf{a}\oplus(-\mathbf{a}\oplus\mathbf{c})\odot t_2\end{aligned} \tag{3.4}$$

To show that α is independent of the choice of $\mathbf{b}^*$ and $\mathbf{c}^*$ on the geodesic rays $L_{\mathbf{a},\mathbf{b}}$ and $L_{\mathbf{a},\mathbf{c}}$ that define α we will show that the cosine of the angle α between the two geometric gyrovectors associated with the pairs $(\mathbf{a},\mathbf{b}^*)$ and $(\mathbf{a},\mathbf{c}^*)$ is independent of the choice of $t_1,t_2\in\mathbb{R}^+$ in (3.4).

According to eq. (3.3), the angle between the two geometric vectors associated with the pairs $(\mathbf{a},\mathbf{b}^*)$ and $(\mathbf{a},\mathbf{c}^*)$ is given by

$$\begin{aligned}\frac{-\mathbf{a}\oplus\mathbf{b}^*}{\|-\mathbf{a}\oplus\mathbf{b}^*\|}\cdot\frac{-\mathbf{a}\oplus\mathbf{c}^*}{\|-\mathbf{a}\oplus\mathbf{c}^*\|} &= \frac{(-\mathbf{a}\oplus\mathbf{b})\odot t_1}{\|(-\mathbf{a}\oplus\mathbf{b})\odot t_1\|}\cdot\frac{(-\mathbf{a}\oplus\mathbf{c})\odot t_2}{\|(-\mathbf{a}\oplus\mathbf{c})\odot t_2\|}\\ &= \frac{-\mathbf{a}\oplus\mathbf{b}}{\|-\mathbf{a}\oplus\mathbf{b}\|}\cdot\frac{-\mathbf{a}\oplus\mathbf{c}}{\|-\mathbf{a}\oplus\mathbf{c}\|}\\ &= \cos\alpha\end{aligned} \tag{3.5}$$

The first equality in (3.5) follows from an application of the left cancellation law, e.g.,

$$-\mathbf{a}\oplus\mathbf{b}^* \;=\; -\mathbf{a}\oplus\left\{\mathbf{a}\oplus(-\mathbf{a}\oplus\mathbf{b})\odot t_1\right\} \;=\; (-\mathbf{a}\oplus\mathbf{b})\odot t_1$$

The left cancellation law is presented in Ref. 3 as well as in Eq. (5.1) of Section 5.1. The second equality in (3.5) follows from Eq. (2.6).

The angle α is invariant under the motions of V_s as we see from Eqs. (3.1) − (3.3), noting that the inner product is preserved by rotations. Moreover, hyperbolic angles keep their numerical value invariant in the transition between the gyrovector spaces of Einstein, Möbius and Weierstrass. Hence, finally, geometric gyrovectors have geometric significance in V_s since their lengths and relative orientations are preserved under the motions of V_s. Unlike Euclidean geometric vectors, however, geometric gyrovectors are not invariant under (left or right) gyrotranslations, as we see from Eq. (3.1) according to which a left gyrotranslation of a geometric gyrovector results in a Thomas gyration (that is, a rotation) of the gyrovector.

Two geometric gyrovectors are orthogonal if the cosine of the angle between them is zero. By defining orthogonality in V_s we have completed setting the stage for the Hyperbolic Pythagorean Theorem in a Möbius gyrovector space, that is, in the Poincaré ball model of hyperbolic geometry.

4. The Hyperbolic Pythagorean Theorem in the Poincaré ball model of Hyperbolic Geometry

THEOREM 4.1 Let $(V_s, \oplus, \odot)$ be a Möbius gyrovector space. Then

$$\frac{\|\mathbf{a}\oplus\mathbf{b}\|^2}{s} = \frac{\|\mathbf{a}\|^2}{s} \oplus \frac{\|\mathbf{b}\|^2}{s} \oplus \frac{1}{s}\,\frac{2\,\mathbf{a}\cdot\mathbf{b}}{\left(1+\frac{\|\mathbf{a}\|^2}{s^2}\right)\left(1+\frac{\|\mathbf{b}\|^2}{s^2}\right)+2\frac{\mathbf{a}\cdot\mathbf{b}}{s^2}} \tag{4.1}$$

for all $\mathbf{a},\mathbf{b}\in V_s$.

In particular, if $\mathbf{a}$ and $\mathbf{b}$ are orthogonal then

$$\frac{\|\mathbf{a}\oplus\mathbf{b}\|^2}{s} = \frac{\|\mathbf{a}\|^2}{s} \oplus \frac{\|\mathbf{b}\|^2}{s} \tag{4.2}$$

Proof The proof is by straightforward algebra, noting that the $\oplus$ between elements of V_s, given by Eq. (2.4), is neither commutative nor associative, while the $\oplus$ between elements of $\mathbb{R}_s$, given by Eq. (2.4) as well, is both commutative and associative as shown in Eq. (2.12). Specifically, one can readily show that each of the two sides of (4.1) equals $\|\mathbf{a}+\mathbf{b}\|^2/(1+2\mathbf{a}\cdot\mathbf{b}/s^2+\|\mathbf{a}\|^2\|\mathbf{b}\|^2/s^4)$, QED.

Noting that

$$2\odot\mathbf{a} = \mathbf{a}\oplus\mathbf{a} = \frac{2\,\mathbf{a}}{1+\|\mathbf{a}\|^2/s^2}$$

Eq. (4.1) can be written as

$$\frac{\|\mathbf{a}\oplus\mathbf{b}\|^2}{s} = \frac{\|\mathbf{a}\|^2}{s} \oplus \frac{\|\mathbf{b}\|^2}{s} \oplus \frac{1}{s}\,\frac{\frac{1}{2}(2\odot\mathbf{a})\cdot(2\odot\mathbf{b})}{1+\frac{1}{2s^2}(2\odot\mathbf{a})\cdot(2\odot\mathbf{b})} \tag{4.3}$$

which, in turn, can be manipulated into the *hyperbolic polarization identity*,[40]

$$\frac{\|\mathbf{a}\oplus\mathbf{b}\|^2}{s} \ominus \frac{\|\mathbf{a}\ominus\mathbf{b}\|^2}{s} = \frac{(2\odot\mathbf{a})\cdot(2\odot\mathbf{b})}{s} \tag{4.4}$$

in Möbius gyrovector spaces $(V_s, \oplus, \odot)$ in full analogy with the polarization identity in real inner product spaces $(V_\infty, +, \cdot)$,

$$\|\mathbf{a}+\mathbf{b}\|^2 - \|\mathbf{a}-\mathbf{b}\|^2 = 4\,\mathbf{a}\cdot\mathbf{b}$$

We will now relate the identities in Theorem 4.1 to hyperbolic triangles thereby obtaining the hyperbolic law of cosines and the Hyperbolic Pythagorean Theorem. Let $\Delta\mathbf{abc}$ be the triangle in a Möbius gyrovector space V_s whose vertices are $\mathbf{a},\mathbf{b},\mathbf{c}\in V_s$. The special case of $V_s=\mathbb{R}_s^2$ is presented graphically in Fig. 5.

FIGURE 5

The sides of the triangle $\Delta\mathbf{abc}$ are formed by the three geometric gyrovectors $A=-\mathbf{c}\oplus\mathbf{b}$, $B=-\mathbf{c}\oplus\mathbf{a}$ and $C=-\mathbf{a}\oplus\mathbf{b}$. By Eq. (3.1), we have

$$(-\mathbf{c}\oplus\mathbf{b})\ \ominus\ (-\mathbf{c}\oplus\mathbf{a})\ =\ \mathrm{gyr}[-\mathbf{c};\mathbf{b}](\mathbf{b}\ominus\mathbf{a})$$

which, by the gyrocommutative law, can be written as

$$A\ominus B\ =\ \mathrm{gyr}[-\mathbf{c};\mathbf{b}]\,\mathrm{gyr}[\mathbf{b};-\mathbf{a}]\,C$$

Hence, noting that $\mathrm{gyr}[-\mathbf{c};\mathbf{b}]$ and $\mathrm{gyr}[\mathbf{b};-\mathbf{a}]$ are isometries, we have

$$\|C\|^2\ =\ \|A\ominus B\|^2$$

Noting that $A\ominus B = A\oplus(-B)$, we have, by Eq. (4.1)

$$\begin{aligned}\frac{1}{s}\|C\|^2\ &=\ \frac{1}{s}\|A\ominus B\|^2\\ &=\ \frac{1}{s}\|A\|^2\oplus\frac{1}{s}\|B\|^2\ominus\frac{1}{s}\,\frac{2A\cdot B}{\left(1+\frac{\|A\|^2}{s^2}\right)\left(1+\frac{\|B\|^2}{s^2}\right)-\frac{2}{s^2}A\cdot B}\end{aligned}\tag{4.5}$$

thus obtaining the hyperbolic law of cosines for a hyperbolic triangle $\Delta\mathbf{abc}$ with vertices $\mathbf{a},\mathbf{b},\mathbf{c}$ and sides A, B, C in the Poincaré ball model.

By Eq. (3.3), the inner product $A\cdot B$ in (4.5) satisfies the equation $A\cdot B = \|A\|\,\|B\|\cos\gamma$, where A, B and γ are shown in Fig. 5. The hyperbolic law of cosines can therefore be presented as

THEOREM 4.2 *(The Hyperbolic Law of Cosines)* Let A, B and C be the three sides of a triangle in a Möbius gyrovector space $(V_s, \oplus, \odot)$, and let γ be the hyperbolic angle between A and B. Then

$$\frac{1}{s}\|C\|^2\ =\ \frac{1}{s}\|A\|^2\oplus\frac{1}{s}\|B\|^2\ominus\frac{1}{s}\,\frac{2\,\|A\|\,\|B\|\cos\gamma}{\left(1+\frac{\|A\|^2}{s^2}\right)\left(1+\frac{\|B\|^2}{s^2}\right)-\frac{2}{s^2}\|A\|\,\|B\|\cos\gamma}$$

Finally, the Hyperbolic Pythagorean Theorem is recovered from the law of cosines when the two sides A and B of the hyperbolic triangle are orthogonal, Fig. 6.

THEOREM 4.3 *(The Hyperbolic Pythagorean Theorem)* Let $\Delta\mathbf{abc}$ be a hyperbolic triangle whose vertices are the three points $\mathbf{a},\mathbf{b}$ and $\mathbf{c}$ in a Möbius gyrovector space $(V_s, \oplus, \odot)$, and whose sides are (if directed counterclockwise) accordingly

$$\begin{aligned}A\ &=\ -\mathbf{b}\oplus\mathbf{c}\\ B\ &=\ -\mathbf{c}\oplus\mathbf{a}\\ C\ &=\ -\mathbf{a}\oplus\mathbf{b}\end{aligned}$$

If the two sides A and B are orthogonal then

$$\frac{1}{s}\|A\|^2\oplus\frac{1}{s}\|B\|^2\ =\ \frac{1}{s}\|C\|^2\tag{4.6}$$

FIGURE 6

Fig. 6 shows a practical way to draw hyperbolic right angled triangles in the Poincaré ball model. Since geodesics through the origin of the ball are Euclidean straight lines, it is easy to identify the special right angled triangles $\Delta \mathbf{a}^* \mathbf{b}^* \mathbf{c}^*$ with a right angle at the origin. More general hyperbolic right angled triangles $\Delta \mathbf{abc}$ can then be generated by left gyrotranslations and rotations.

Because of the large potential research rewards at stake for the exposition of more analogies to which Thomas gyration gives rise, natural selection made by various explorers is likely to favor a terminology which emphasizes analogies shared by Euclidean and non-Euclidean geometries and, correspondingly, analogies shared by classical mechanics and relativistic mechanics. Our gyroterminology, in which we extensively use the prefix gyro, is indeed sensitive to the need to accommodate new terms to describe further discoveries in gyrovector space theory that are likely to emerge from time to time. Thus, for instance, the term *gyrodistributive* law is waiting for the discovery of some unknown law that relates the two operations $\oplus$ and $\odot$ in gyrovector spaces $(V_s, \oplus, \odot)$ in such a way that it reduces to the common distributive law in vector spaces when Thomas gyration vanishes.

The Hyperbolic Pythagorean Theorem is well known in the literature on hyperbolic geometry,[12, 24] where it appears in a form that exhibits no obvious analogies shared with its Euclidean counterpart. Our gyrovector space version of the Hyperbolic Pythagorean Theorem, however, exposes analogies shared by the concept of Euclidean *Pythagorean orthogonality*,[41] and its hyperbolic counterpart that we may naturally call a hyperbolic Pythagorean orthogonality. It is therefore hoped that the present exposition of the Hyperbolic Pythagorean Theorem, as viewed in gyrovector space theory, will encourage further exploration of our analytic hyperbolic geometry approach, resulting in the discovery of more analogies shared by hyperbolic and Euclidean geometries.

The Hyperbolic Pythagorean Theorem provides a way to select elegant distance functions in the various models of hyperbolic geometry. Thus, for instance, the elegant form of the Pythagorean identity (4.6) results from the selection of the Poincaré distance function $d(\mathbf{x}, \mathbf{y})$ of Eq. (2.10) rather than, for instance, the hyperbolic distance function $2\tanh^{-1}(d(\mathbf{x}, \mathbf{y})/s)$ (with $s = 1$) that some authors prefer.[34, 35]

5. Is there a dual hyperbolic Pythagorean theorem?

We have emphasized in Ref. 3 that gyrovector spaces are *bimetric,* possessing the two distance functions (5.8) and (5.9) that are presented below. A gyrogroup $(G, \oplus)$ possesses a dual binary operation $\boxplus$ expressible in terms of the gyrogroup operation $\oplus$ and its Thomas gyration. The introduction of the dual binary operation into a gyrogroup G is natural since the binary operation $\oplus$ in G gives rise to a left cancellation law but not to a right cancellation law. It is with the help of the dual binary operation $\boxplus$ in G that a right cancellation law emerges. The left cancellation law and its two associated right cancellation laws are

$$-\mathbf{a} \oplus (\mathbf{a} \oplus \mathbf{b}) = \mathbf{b} \tag{5.1}$$

$$(\mathbf{a} \oplus \mathbf{b}) \boxminus \mathbf{b}) = \mathbf{a} \tag{5.2}$$

$$(\mathbf{a} \boxplus \mathbf{b}) \ominus \mathbf{b}) = \mathbf{a} \tag{5.3}$$

The duality between the two binary operations $\oplus$ and $\boxplus$ in G is expressed by the three relations

$$\mathbf{a} \boxplus \mathbf{b} = \mathbf{a} \oplus \mathrm{gyr}[\mathbf{a}; -\mathbf{b}]\mathbf{b} \tag{5.4}$$

$$\mathbf{a} \oplus \mathbf{b} = \mathbf{a} \boxplus \mathrm{gyr}[\mathbf{a}; \mathbf{b}]\mathbf{b} \tag{5.5}$$

$$\mathrm{Aut}(G, \oplus) = \mathrm{Aut}(G, \boxplus) \tag{5.6}$$

that they satisfy, as shown in Theorem 4.2 of Ref. 3, where Eq. (5.4) constitutes the definition of $\boxplus$. Furthermore, in a gyrocommutative gyrogroup the dual binary operation $\boxplus$ is commutative (but not associative), by Theorem 5.10 of Ref. 3. It can be shown by methods of Ref. 31 that the dual gyrogroup operation $\boxplus = \boxplus_M$, Eq. (5.4), in a Möbius gyrogroup $(V_s, \oplus_M)$, where $\oplus_M$ is the Möbius addition $\oplus$ in Eq. (2.4), can be written as

$$\mathbf{a} \boxplus_M \mathbf{b} = \frac{(1 - \|\mathbf{b}\|^2/s^2)\,\mathbf{a} + (1 - \|\mathbf{a}\|^2/s^2)\,\mathbf{b}}{1 - \|\mathbf{a}\|^2\,\|\mathbf{b}\|^2/s^4} \tag{5.7}$$

While the dual Möbius addition $\boxplus_M$ in Eq. (5.7) is far from being associative, it is commutative and looks simpler than Möbius addition $\oplus = \oplus_M$ in Eq. (2.4). It is interesting to realize that despite of being so different, the two binary operations $\oplus = \oplus_M$ and $\boxplus_M$, Eqs. (2.4) and (5.7), in the ball V_s are dual to one another in the sense of the duality symmetries (5.1) – (5.6).

In a gyrovector space $(V_s, \oplus, \odot)$ the gyrogroup operation $\oplus$ gives rise to the natural metric (2.10),

$$d_{\bigcirc}(\mathbf{a}, \mathbf{b}) = \|\mathbf{a} \ominus \mathbf{b}\| \tag{5.8}$$

Similarly, its dual binary operation gives rise to the dual metric

$$d_{\square}(\mathbf{a}, \mathbf{b}) = \|\mathbf{a} \boxminus \mathbf{b}\| \tag{5.9}$$

The introduction of the dual gyrogroup operation $\boxplus$ in a gyrogroup $(G, \oplus)$ is initially justified by the need to have a right cancellation law, (5.2), in addition to the left cancellation law (5.1) that the gyrogroup operation $\oplus$ offers. More justifications follow. The introduction of the resulting dual metric (5.9) is justified as well, as we will see in the sequel.

The curves $\mathbf{a}\oplus\mathbf{b}\odot t$, $\mathbf{a},\mathbf{b}\in V_s$, $t\in I\!R$, in the various gyrovector space models of hyperbolic geometry (e.g., the Einstein, the Möbius, and the Weierstrass gyrovector spaces that underlie respectively the Klein-Beltrami, the Poincaré, and the Weierstrass models of hyperbolic geometry) describe analytically the standard geodesics of hyperbolic geometry; see Fig. 2 in Section 1 of the present article and Figs. 2, 3, 6 and 7 of Ref. 3. This observation raises a natural question: Why are the curves

$$\mathbf{a}\oplus\mathbf{b}\odot t \tag{5.10}$$

which form the standard geodesics of hyperbolic geometry, seemingly more significant than the curves

$$\mathbf{b}\odot t\oplus\mathbf{a} \tag{5.11}$$

which are not geodesics? The two families of curves in (5.10) and (5.11) are different since their gyrocommutative gyrogroup operation $\oplus$ is noncommutative. In the special case when $\oplus$ is the Möbius addition of Definition 2.3 in the Poincaré disk $V_s = I\!R^2_{s=1}$, the two curves are shown in Figs. 2 and 3. The curves (5.10) in the disk, Fig. 2, are circular arcs that intersect the boundary of the disk orthogonally, while the curves (5.11) in the disk, Fig. 3, are circular arcs that intersect the boundary of the disk diametrically.

A most elegant answer, according to which both (5.10) and (5.11) are geometrically significant, is provided by the dual metric. While the former curves, (5.10), are geodesics relative to the natural hyperbolic metric (5.8) of a gyrovector space, the latter curves, (5.11), are geodesics relative to the dual metric (5.9). Accordingly, the curves (5.11) are called *dual geodesics,* and triangles made out of these are called *dual triangles.* The dual geodesics are known in hyperbolic geometry as hypercircles or equidistant curves.[22] The term "equidistant curve" is explained in terms of gyrogroup formalism in Ref. 42.

In the Poincaré disk the geodesics (5.10) are circular arcs that intersect the boundary of the disk orthogonally, Fig. 2, and the dual geodesics (5.11) are circular arcs that intersect the boundary of the disk diametrically (at antipodal points, that is, at diametrically opposite points), Fig. 3. As such, every dual geodesic has a *supporting diameter.* The hyperbolic orientation of the dual geodesic is, suggestively, defined to be the Euclidean orientation of its supporting diameter. It can be shown that, as a result, the dual angle between two dual geodesics is the one given by Eq. (5.12) below, which shares obvious analogies with the analytic description of Euclidean angles. Several geodesics in the Poincaré disk are shown in Figs. 2,4-6. Several dual geodesics are shown in Fig. 3, and in Figs. 1a, 2a, 3, 5a, 5b, 10 and 12 of Ref. 32.

The usefulness of geodesics in differential geometry and in mathematical physics is well known. Due to the similarity between the two families of curves in (5.10) and (5.11), and since they are both geodesics relative to their respective metrics, one should expect that duality in geodesics and in angles that they generate will be found useful as well. This expectation is indeed justified, as we will see in Eqs. (5.13) below.

It has been demonstrated in Ref. 3 that the two metrics, (5.8) and (5.9) of a gyrovector space interplay harmoniously. In addition, it was shown there that while

(i) triangle medians in hyperbolic triangles are concurrent (satisfying a corresponding Euclidean geometry property),

(ii) dual triangle medians in hyperbolic dual triangles are not concurrent (violating a corresponding Euclidean geometry property).

In contrast, however, it was shown in Ref. 3 that while

(i) the parallel postulate is not valid in geodesics in hyperbolic geometry (violating a corresponding Euclidean geometry property),

(ii) the parallel postulate is valid in dual geodesics (satisfying a corresponding Euclidean geometry property).

It is thus clear that the hyperbolic geometry as we presently know in the literature is only half of the story; the other half is concealed in the structure to which the dual metric gives rise. We say "story" rather than "theory" since at this early stage of the development our demonstration is anecdotal in nature. Thus, for instance, (I) the parallel postulate that has seemingly disappeared in the transition from Euclidean to hyperbolic geometry, reappears with the dual metric of hyperbolic geometry. Accordingly, the sum of the three angles of any hyperbolic triangle is less than π, but the sum of the three dual angles of any dual triangle equals π, as we will see in Eq. (5.13b) below. (II) Conversely, the triangle median concurrency is a property that did not disappear in the transition from Euclidean to Hyperbolic geometry. "Hence", it is being violated relative to the dual metric. The two dual hyperbolic geometries to which the hyperbolic dual metrics give rise are thus complementary, mutually making up what is lacking.

Having two metrics in hyperbolic geometry, it is natural to explore whether a hyperbolic Pythagorean theorem is valid relative to the dual metric as well. The complementarity of the two dual hyperbolic geometries that we have just observed suggests that the Hyperbolic Pythagorean Theorem is not valid in the dual metric "since" it is valid relative to the standard hyperbolic metric, as shown in Eq. (4.6). But, in compensation of losing the hyperbolic Pythagorean theorem in the dual metric, another important property of Euclidean triangles which is not valid in the standard hyperbolic metric (5.8) will hopefully be found valid relative to the dual metric. This is indeed the case. The compensation for the loss of the Hyperbolic Pythagorean Theorem in the dual metric is fully paid for by the dual metric establishing the $\pi-$Theorem according to which the sum of the three dual angles (to be defined below) of any dual triangle is π. Thus, the complementarity of the dual hyperbolic geometries emerges again. While

(i) right triangles obey the hyperbolic Pythagorean identity in the Poincaré model of hyperbolic geometry (satisfying a corresponding Euclidean geometry property),

(ii) dual right triangles do not obey it (violating a corresponding Euclidean geometry property).

In contrast, however, while

(i) the sum of the three angles of a triangle in hyperbolic geometry is less than π (violating a corresponding Euclidean geometry property),

(ii) the sum of the three dual angles of a dual triangle in hyperbolic geometry equals π (satisfying a corresponding Euclidean geometry property).

To establish our claim about the hyperbolic π we have to define in a natural way dual angles, that is, angles relative to the dual metric (5.9), and show that the sum of the dual angles of any dual triangle equals π.

Let us, accordingly, consider two arbitrary dual geodesics, that is, geodesics relative to the dual metric (5.9), that contain respectively the pair of points $(\mathbf{a}, \mathbf{b})$ and $(\mathbf{c}, \mathbf{d})$,

$$L_{\mathbf{a},\mathbf{b}} = (\mathbf{b} \boxminus \mathbf{a}) \odot t \oplus \mathbf{a}$$

$$L_{\mathbf{c},\mathbf{d}} = (\mathbf{d} \boxminus \mathbf{c}) \odot t \oplus \mathbf{c}$$

$t \in I\!R$, in a Möbius gyrovector space $(V_s, \oplus, \odot)$. The cosine of the angle α between the two dual geodesics is defined by the equation

$$\cos\alpha = \frac{-\mathbf{a} \boxplus \mathbf{b}}{\|-\mathbf{a} \boxplus \mathbf{b}\|} \cdot \frac{-\mathbf{c} \boxplus \mathbf{d}}{\|-\mathbf{c} \boxplus \mathbf{d}\|} \tag{5.12}$$

in full analogy with the definition of Euclidean hyperbolic angles, and in partial analogy with the hyperbolic angle definition in Eq. (3.3).

Supporting no parallelism, the two geometric vectors that define a hyperbolic angle α must be emanated from a common point, as we see from Eq. (3.3) and as illustrated in Fig. 4. In contrast, dual rays do support parallelism, and accordingly, the two dual geometric vectors that define a dual angle α in (5.12) need not be emanated from a common point, as is the case in Euclidean geometry. This is clearly seen in Fig. 2a of Ref. 32.

One can verify by means of arguments illustrated by Fig. 2a of Ref. 32 that the dual angle between dual geodesics is well defined (that is, it is independent of the choice of the two ordered points $(\mathbf{a}, \mathbf{b})$ and $(\mathbf{c}, \mathbf{d})$ that one selects on each of the two dual geodesics that $L_{\mathbf{a},\mathbf{b}}$ and $L_{\mathbf{c},\mathbf{d}}$ generate the hyperbolic dual angle α), and that the sum of the three dual angles of any dual triangle is π. In symbols, if $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are any three points in a gyrovector space $(V_s, \oplus, \odot)$ which do not lie on a dual geodesic, and if the dual angles of the dual triangle, whose vertices are these points, are α, β and γ, then

$$\begin{aligned}
\alpha &= \cos^{-1}\Big(\frac{-\mathbf{a} \boxplus \mathbf{b}}{\|-\mathbf{a} \boxplus \mathbf{b}\|} \cdot \frac{-\mathbf{a} \boxplus \mathbf{c}}{\|-\mathbf{a} \boxplus \mathbf{c}\|}\Big) \\
\beta &= \cos^{-1}\Big(\frac{-\mathbf{b} \boxplus \mathbf{a}}{\|-\mathbf{b} \boxplus \mathbf{a}\|} \cdot \frac{-\mathbf{b} \boxplus \mathbf{c}}{\|-\mathbf{b} \boxplus \mathbf{c}\|}\Big) \\
\gamma &= \cos^{-1}\Big(\frac{-\mathbf{c} \boxplus \mathbf{a}}{\|-\mathbf{c} \boxplus \mathbf{a}\|} \cdot \frac{-\mathbf{c} \boxplus \mathbf{b}}{\|-\mathbf{c} \boxplus \mathbf{b}\|}\Big)
\end{aligned} \tag{5.13a}$$

and

$$\alpha + \beta + \gamma = \pi \tag{5.13b}$$

The geometric meaning of identity (5.13b) in the Poincaré disk model of hyperbolic geometry is clearly seen in Fig. 2a of Ref. 32. Both angles and dual angles in hyperbolic geometry are model independent, that is, they keep their numerical value invariant in the transition between the gyrovector spaces of Einstein, Möbius and Weierstrass. Dual angles are preserved by rotations. Unlike angles, however, dual angles are not invariant under left gyrotranslations. Formally, we thus have

The $\pi-$*THEOREM 5.1* Let $\mathbf{a},\mathbf{b},\mathbf{c}\in V_s$ be any three distinct points in a gyrovector space $(V_s, \oplus, \cdot)$, and let α, β and γ be the three dual angles of the dual triangle formed by these three point, Eqs. (5.13a). Then, $\alpha+\beta+\gamma=\pi$.

The *defect* of a hyperbolic triangle with angles α, β and γ is $\pi-(\alpha+\beta+\gamma)$, and it equals the hyperbolic area of the triangle.(24) The $\pi-$Theorem suggests a natural extension of the notion of defect from the three angles of a triangle to individual angles.

DEFINITION 5.1 *(The defect of a hyperbolic angle)* Let (i) $\mathbf{a},\mathbf{b},\mathbf{c}\in V_s$ be any three distinct points in a gyrovector space $(V_s, \oplus, \odot)$; let (ii) α be the angle between the two rays $L_{\mathbf{a},\mathbf{b}}$ and $L_{\mathbf{a},\mathbf{c}}$ that are emanated from $\mathbf{a}$ and contain respectively $\mathbf{b}$ and $\mathbf{c}$; and let (iii) dual(α) be its dual angle. Then, the defect of α is given by

$$\mathrm{defect}(\alpha) \;=\; \mathrm{dual}(\alpha) - \alpha$$

that is, by

$$\mathrm{defect}(\alpha) \;=\; \cos^{-1}\Big(\frac{-\mathbf{a}\boxplus\mathbf{b}}{\|-\mathbf{a}\boxplus\mathbf{b}\|}\cdot\frac{-\mathbf{a}\boxplus\mathbf{c}}{\|-\mathbf{a}\boxplus\mathbf{c}\|}\Big) \;-\; \cos^{-1}\Big(\frac{-\mathbf{a}\oplus\mathbf{b}}{\|-\mathbf{a}\oplus\mathbf{b}\|}\cdot\frac{-\mathbf{a}\oplus\mathbf{c}}{\|-\mathbf{a}\oplus\mathbf{c}\|}\Big)$$

Following Def. 5.1 and the $\pi-$Theorem we can now state

THEOREM 5.2 The defect of a hyperbolic triangle equals the sum of the defects of its angles.

The $\pi-$Theorem demonstrates that a well known property of Euclidean triangles that has seemingly been lost in the transition to hyperbolic geometry, reappears in the novel structure of hyperbolic geometry to which the dual metric gives rise.

The significance of the definition in Eq. (3.3) of angles in a gyrovector space $(V_s, \oplus, \odot)$ relative to its natural metric is exhibited by the resulting Hyperbolic Pythagorean Theorem for hyperbolic right angled triangles in Theorem 4.3. Similarly, the significance of the definition in Eq. (5.12) of dual angles in a gyrovector space $(V_s, \oplus, \odot)$ relative to its dual metric is exposed by the resulting $\pi-$Theorem. Unlike angles and geodesics, however, dual angles and dual geodesics in a gyrovector space $(V_s, \oplus, \odot)$ are not preserved by left gyrotranslations. Thus, in particular, the defect of a hyperbolic angle is not invariant under left gyrotranslations. Interestingly, however, the sum of the defects of the three angles of a hyperbolic triangle is invariant under left gyrotranslations.

Another indication that geodesics and their dual geodesics are equally significant for mutually dual reasons is provided by the *gyrotransitive law* (5.14) of successive gyrations along geodesics and its dual law (5.15), that they respectively obey. Let $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \cdots, \mathbf{a}_n\}$ be a set of any n points lying on a geodesic in any order, and similarly, let $\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3, \cdots, \mathbf{b}_n\}$ be a set of any n points lying on a dual geodesic in any order. Then

$$\mathrm{gyr}[\mathbf{a}_1;-\mathbf{a}_2]\,\mathrm{gyr}[\mathbf{a}_2;-\mathbf{a}_3]\ \cdots\ \mathrm{gyr}[\mathbf{a}_{n-1};-\mathbf{a}_n] = \mathrm{gyr}[\mathbf{a}_1;-\mathbf{a}_n] \tag{5.14}$$

and

$$\mathrm{gyr}[\mathbf{b}_1;\mathbf{b}_2]\,\mathrm{gyr}[\mathbf{b}_2;\mathbf{b}_3]\ \cdots\ \mathrm{gyr}[\mathbf{b}_{n-1};\mathbf{b}_n] = \mathrm{gyr}[\mathbf{b}_1;\mathbf{b}_n] \tag{5.15}$$

Identity (5.15) is verified in Eq. (7.3b) of Ref. 32 and the proof of identity (5.14) is similar. Interestingly, the duality between gyr[$\mathbf{x};\mathbf{y}$] and gyr[$\mathbf{x};-\mathbf{y}$] exhibited by Eqs. (5.14) and (5.15) is also clear from Eqs. (5.4) and (5.5), as well as from Eqs. (4.5a) and (4.5b) of Ref. 3.

We have selected the Poincaré ball model of hyperbolic geometry for the presentation of the Hyperbolic Pythagorean Theorem since, within Euclidean geometry, this model of hyperbolic geometry exhibits Euclidean angles. The Poincaré measure of an angle is given by the Euclidean measure of the angle formed by Euclidean tangent rays.(24) From that point of view, the best model of hyperbolic geometry for the presentation of geometric objects that involve dual angles, rather than angles, is the Weierstrass whole space model $V_\infty=(V_\infty, \oplus_W, \odot_W)$ of hyperbolic geometry whose underlying real inner product space is $V_\infty=(V_\infty,+,\cdot)$.(3) In this model dual geodesics are Euclidean straight lines, and the measure of dual angles between dual geodesics is equal to the Euclidean measure of the angle between the corresponding straight lines. Geodesics and dual geodesics in the Weierstrass gyrovector spaces are shown in Figs. 6 – 10 of Ref. 3 where the gyrogroup operation $\oplus_W$ and its dual operation $\boxplus_W$, as well as its scalar multiplication $\odot_W$ are presented. Thus, for instance, the first equation in (5.13a), expressed in a Weierstrass gyrovector space, describes a hyperbolic dual angle α whose measure equals its Euclidean counterpart. In symbols,

$$\alpha = \cos^{-1}\Big(\frac{-\mathbf{a}\boxplus_W\mathbf{b}}{\|-\mathbf{a}\boxplus_W\mathbf{b}\|}\cdot\frac{-\mathbf{a}\boxplus_W\mathbf{c}}{\|-\mathbf{a}\boxplus_W\mathbf{c}\|}\Big) = \cos^{-1}\Big(\frac{-\mathbf{a}+\mathbf{b}}{\|-\mathbf{a}+\mathbf{b}\|}\cdot\frac{-\mathbf{a}+\mathbf{c}}{\|-\mathbf{a}+\mathbf{c}\|}\Big) \tag{5.16}$$

for all $\mathbf{a},\mathbf{b},\mathbf{c}\in V_\infty$, since, in particular,

$$\frac{-\mathbf{a}\boxplus_W\mathbf{b}}{\|-\mathbf{a}\boxplus_W\mathbf{b}\|} = \frac{-\mathbf{a}+\mathbf{b}}{\|-\mathbf{a}+\mathbf{b}\|} \tag{5.17a}$$

or, equivalently,

$$\mathbf{a}\boxplus_W\mathbf{b} = \frac{B_{\mathbf{a}}+B_{\mathbf{b}}}{B_{\mathbf{a}}B_{\mathbf{b}}+1-\mathbf{a}\cdot\mathbf{b}/s^2}\,(\mathbf{a}+\mathbf{b}) \tag{5.17b}$$

$B_{\mathbf{a}}^2=1+\|\mathbf{a}\|^2/s^2$, in a Weierstrass gyrovector space. Eq. (5.16) explains why the sum of the three dual angles of a dual triangle in the Weierstrass model is π, Eq. (5.13b). Unexpectedly, however, while Eqs. (5.16) – (5.17) are valid only in the Weierstrass model of hyperbolic geometry, Eq. (5.13b) is model independent thus possessing hyperbolic geometric significance.

In gyrovector space formalism the duality between the two binary operations $\oplus$ and $\boxplus$, Eqs. (2.4) and (5.7), in a Möbius gyrovector space is obvious from Eqs. (5.1) – (5.6). In contrast, this duality is not apparent in Riemannian geometry. Thus, in particular,

(I) the Riemannian metric $d\mathbf{x}_\bigcirc^2$ corresponding to the hyperbolic metric $d_\bigcirc$, Eq. (5.8), in the disk $\mathbb{R}_s^2$ of the Euclidean plane $\mathbb{R}^2$ of the x_1x_2-plane is,(43)

$$d\mathbf{x}_\bigcirc^2 = \frac{dx_1^2+dx_2^2}{[1-\frac{1}{s^2}(x_1^2+x_2^2)]^2} \tag{5.18a}$$

whose Gaussian curvature,[44] is a negative constant,

$$K_{\bigcirc} = -\frac{4}{s^2} \tag{5.18b}$$

and

(II) the Riemannian metric $d\mathbf{x}_{\square}^2$ corresponding to the dual hyperbolic metric $d_{\square}$, Eq. (5.9), in the disk $\mathbb{R}_s^2$ of the Euclidean plane $\mathbb{R}^2$ is given by the quadratic differential form

$$d\mathbf{x}_{\square}^2 = \frac{Edx_1^2 + 2Fdx_1dx_2 + Gdx_2^2}{[1 - \frac{1}{s^4}(x_1^2 + x_2^2)^2]^2} \tag{5.19a}$$

where

$$\begin{aligned} E &= [x_1^2 + (s - x_2)^2]\,[x_1^2 + (s + x_2)^2]/s^4 \\ G &= [(s - x_1)^2 + x_2^2]\,[(s + x_1)^2 + x_2^2]/s^4 \\ F &= 4x_1x_2/s^2 \end{aligned}$$

and

$$EG - F^2 = [1 - \frac{1}{s^4}(x_1^2 + x_2^2)^2]^2$$

having a variable positive Gaussian curvature,[44]

$$K_{\square} = \frac{8}{s^2}\,\frac{1}{[1 + \frac{1}{s^2}(x_1^2 + x_2^2)]^4} \tag{5.19b}$$

In the limit of large s, $s \to \infty$, the Riemannian metric $d\mathbf{x}_{\bigcirc}^2$ and its dual metric $d\mathbf{x}_{\square}^2$ both reduce to the Euclidean metric $d\mathbf{x}^2 = dx_1^2 + dx_2^2$ and the two corresponding curvatures $K_{\square}$ and $K_{\bigcirc}$ vanish. The hyperbolic metric $d\mathbf{x}_{\bigcirc}^2$ is conformal, being proportional to the plane Euclidean metric at each point. Hence, the actual angles for this metric coincide with Euclidean angles. We see no indication in Riemannian geometry that the two Riemannian metrics $d\mathbf{x}_{\bigcirc}^2$ and $d\mathbf{x}_{\square}^2$ are dual in any sense. In contrast, gyrovector space theory clearly exposes their duality symmetries in Eqs. (5.1) – (5.6).

Riemann was aware of the possible application of his geometry to physics. In his inaugural address in 1854 on the occasion of joining the University Faculty of Göttingen he said that the value of his geometry can possibly be to liberate us from preconceived ideas, should ever the time come that in the exploration of the laws of physics the concepts of Euclidean geometry may have to be abandoned.[45] These prophetic words were literally fulfilled fifty years after his death by the Einstein theory of general relativity.[10]

Of particular interest in the literature on differential geometry is the case when Gauss curvature is constant, as this is the only known case which permits free mobility of figures on the surface without influencing their inner connections. It is therefore important to realize that despite the fact that the Gaussian curvature $K_{\square}$ is non-constant, dual geodesic segments can freely be rotated (obvious) and right-gyrotranslated on the surface of this curvature by a family of right gyrotranslations that will be specified in Eq. (5.21a) below.

Let $\mathbf{b} \boxminus \mathbf{a} = -\mathbf{a} \boxplus \mathbf{b}$ be the dual geometric vector represented by a directed dual geodesic segment from **a** to **b** in a gyrovector space $(V_s, \oplus, \odot)$. Furthermore, let

$$m_{\mathbf{a},\mathbf{b}} = \frac{1}{2} \odot (\mathbf{a} \boxminus \mathbf{b}) \oplus \mathbf{b} \tag{5.20}$$

be the midpoint of **a** and **b** in V_s, satisfying $m_{\mathbf{a},\mathbf{b}} = m_{\mathbf{b},\mathbf{a}}$.[32] Then, two distinct right gyrotranslations of the two edges **a** and **b** of the dual geometric vector $-\mathbf{a} \boxplus \mathbf{b}$, specified in Eq. (5.21a), can freely move it without rotation to any point of V_s. These motions without rotations are given by the identity

$$-(\mathbf{a} \oplus \mathrm{gyr}[\mathbf{a}, m_{\mathbf{a},\mathbf{b}}]\mathbf{x}) \boxplus (\mathbf{b} \oplus \mathrm{gyr}[\mathbf{b}, m_{\mathbf{b},\mathbf{a}}]\mathbf{x}) = -\mathbf{a} \boxplus \mathbf{b} \tag{5.21a}$$

which is valid for any $\mathbf{x} \in V_s$. Eq. (5.21a) presents full analogy with Euclidean geometry where any translation of a geometric vector leaves it intact,

$$-(\mathbf{a}+\mathbf{x}) + (\mathbf{b}+\mathbf{x}) = -\mathbf{a} + \mathbf{b} \tag{5.21b}$$

Thus, motions on a surface with the non-constant curvature $K_{\square}$ are possible. They are given by rotations and specific right gyrotranslations. The specified right gyrotranslations of a pair of points **a** and **b** involve an arbitrary $\mathbf{x} \in V_s$ which must be rotated (i) by $\mathrm{gyr}[\mathbf{a}, m_{\mathbf{a},\mathbf{b}}]$ when applied to **a**, and (ii) by $\mathrm{gyr}[\mathbf{b}, m_{\mathbf{b},\mathbf{a}}]$ when applied to **b**.

Finally we may remark that, following Cartan, a generalization of the hyperbolic Pythagorean theorem as well as other results of the present article, to some symmetric spaces is possible. Elie Cartan generalized the situation in (2.3) to *Riemannian globally symmetric spaces of noncompact type*, proving that these spaces are exactly all quotients G/K, where G is a noncompact semisimple and K is a maximal compact subgroup. As in (2.3), G has a *Cartan decomposition* $G = e^p K = PK$. Cartan's theory is presented in Ref. 46. By methods of Ref. 25 concerning transversals it can be shown that the factor P in the Cartan decomposition $G = PK$ turns out to be a gyrocommutative gyrogroup. The gyrogroup operation $\oplus$ in the gyrogroup P is determined by the action of P on itself, and the gyroautomorphisms $\mathrm{gyr}[a\,;b]$, expressible in terms of the binary operation $\oplus$ by Eq. (2.1), are Thomas gyrations. Some related results about the Cartan decomposition and its resulting gyrocommutative gyrogroup in any bounded symmetric domain are presented in Ref. 47.

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33. See, for instance, Ian N. Sneddon, ed., *Encyclopedic Dictionary of Mathematics for Engineers and Applied Scientists,* p. 320 (Pergamon, New York, 1976).

34. See Section 7 in Abraham A. Ungar, "The holomorphic automorphism group of the complex disk," *Aequat Math.* **47,** 240-254 (1994); and, for instance, H.S. Bear, "Distance-decreasing functions on the hyperbolic plane," *Mich. Math. J.,* **39,** 271-279 (1992). Detailed explanation why the distance function $2\tanh^{-1}\|\mathbf{x}\ominus\mathbf{y}\|$ is preferred in the literature over our Poincaré distance functions $\|\mathbf{x}\ominus\mathbf{y}\|$, as well as a presentation of the traditional hyperbolic Pythagorean theorem, may be found in Hans Schwerdtfeger, *Geometry of Complex Numbers, Circle Geometry, Möbius Transformations, Non-Euclidean Geometry,* pp. 141, 146-147, 149-151 (Dover, New York, 1979). See also Michael Henle, *Modern Geometries The Analytic Approach,* p. 93 (Prentice Hall, NJ, 1997); and Christian Grosche, "The path integral on the Poincaré disk, the Poincaré upper half plane and on the hyperbolic strip," *Fortschr. Phys.* **38,** 531-569 (1990).

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36. It is customary in the literature to select $s=1$ without loss of generality. However, there are circumstances when the retention of s as a free positive parameter is clearly useful; see e.g., Olli Lehto and K. I. Virtanen, *Quasiconformal Mappings in the Plane,* trans. from the German by K. W. Lucas, 2nd ed. p. 66 (Springer, Berlin, NY, 1973).

37. See Eq. (3.3) in Lars V. Ahlfors, "Old and new in Möbius groups," *Ann. Acad. Sci. Fenn. Ser. A I Math.* **9,** 93-105 (1984); and p. 25 in Lars V. Ahlfors, Möbius *Transformations in Several Dimensions,* Lecture Notes (University of Minnesota, Minneapolis, 1981).

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41. Edward Andalafte and Raymond Freese, "Weak homogeneity of metric Pythagorean orthogonality," *J. Geom.* **56,** 3-8 (1996), and references therein. In particular, see R.C. James, Orthogonality in normed linear spaces," *Duke Math. J.* **12,** 291-302 (1945). In that paper the Pythagorean orthogonality of vectors **x** and **y** in a normed linear space is defined by the equation $\|\mathbf{x}+\mathbf{y}\|^2 = \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2$. This relation is said to be homogeneous if Pythagorean orthogonality between **x** and **y** implies Pythagorean orthogonality between **x** and $r\mathbf{y}$ for all $r \in I\!R$. The homogeneity of Pythagorean orthogonality is important since it characterizes inner product spaces. In full analogy, Eq. (4.2) indicates that hyperbolic-Pythagorean orthogonality in gyrovector spaces is homogeneous, that is, if **a** and **b** are hyperbolic-Pythagorean orthogonal in a gyrovector space, then so are **a** and $r \odot \mathbf{b}$ for all $r \in I\!R$.

42. Michael K. Kinyon and Abraham A. Ungar, "The complex unit disk," preprint.

43. See Exercise 10 in Saul Stahl, *The Poincaré Half-Plane A Gateway to Modern Geometry,* p. 205 (Jones and Bartlett, Boston, 1993).

44. The curvature K of an arbitrary Riemann metric $Edx^2 + 2Fdxdy + Gdy^2$ is defined on p. 201 in Saul Stahl, *The Poincaré Half-Plane A Gateway to Modern Geometry* (Jones and Bartlett, Boston, 1993).

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FIGURE CAPTIONS

Fig. 1 *The Hyperbolic Pythagorean Theorem in the* Poincaré *unit ball.* $\Delta\mathbf{abc}$ is any hyperbolic right angled triangle in the Poincaré unit ball model of hyperbolic geometry in any dimension, with vertices $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and sides A, B, C. The two sides A and B are orthogonal. The hyperbolic lengths of the sides relative to the Poincaré metric of the unit ball are, respectively, $\|A\|$, $\|B\|$, $\|C\|$, satisfying the hyperbolic Pythagorean identity (1.1). The special case when the Poincaré unit ball of the abstract real inner product space is realized by the unit disk $\mathbb{R}^2_{s=1}$ of the Euclidean plane $\mathbb{R}^2$ is shown graphically.

Fig. 2 The common hyperbolic geodesics in the Poincaré ball model of hyperbolic geometry are given by $\mathbf{v}_g(t) = \mathbf{v}_0 \oplus \mathbf{a} \odot t$ in full analogy with the Euclidean geodesics, given by $\mathbf{v}_0 + \mathbf{a}t$, $t \in \mathbb{R}$. A geodesic in the Poincaré disk model of hyperbolic geometry is shown. The vector $\mathbf{a}$ is Euclidean-parallel to the tangent line of the geodesic at the point $\mathbf{v}_0$.[42] Here $\oplus$ is the Möbius addition in the Poincaré disk, given by Eq. (2.2) with $V_s = \mathbb{R}^2_{s=1}$ being the Poincaré unit disk in the Euclidean plane $V_\infty = \mathbb{R}^2$.

Fig. 3 The dual hyperbolic geodesics in the Poincaré ball model of hyperbolic geometry are given by $\mathbf{v}_d(t) = \mathbf{a} \odot t \oplus \mathbf{v}_0$, $t \in \mathbb{R}$. A dual geodesic in the Poincaré disk model of hyperbolic geometry is shown. It is a circular arc that intersects the boundary of the disk at two antipodal points. The vector $\mathbf{a}$ is Euclidean-parallel to the supporting diameter,[32] an observation leading to the $\pi-$Theorem in Section 5. As in Fig. 2, $\oplus$ is the Möbius addition in the Poincaré disk $\mathbb{R}^2_{s=1}$.

Fig. 4 The hyperbolic angle α between two hyperbolic rays emanated from a point $\mathbf{a}$. As in Euclidean geometry, the angle α between directed rays is independent of the choice of the points $\mathbf{b}$ and $\mathbf{c}$, other than $\mathbf{a}$, on the directed rays, Eq. (3.5). Unlike Euclidean geometry, however, the point $\mathbf{a}$ is unique in the sense that it cannot be replaced by two distinct points on each of the two rays that define α.

Fig. 5 The hyperbolic triangle $\Delta\mathbf{abc}$ in the Poincaré disk model: Its vertices are $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$, and its sides, if directed counterclockwise, are $A = -\mathbf{b} \oplus \mathbf{c}$, $B = -\mathbf{c} \oplus \mathbf{a}$ and $C = -\mathbf{a} \oplus \mathbf{b}$. Its angle α is given by Eq. (3.3). The angular defect of the triangle in plane hyperbolic geometry equals the rotation angle of the Thomas gyration (or, rotation) $\mathrm{gyr}[\mathbf{a} \ominus \mathbf{b}; -\mathbf{a} \oplus \mathbf{c}]$, as explained in Ref. 3 for plane hyperbolic geometry.

Fig. 6 *The Hyperbolic Pythagorean Theorem.* Left gyrotranslations and rotations of a hyperbolic right angled triangle with right angle at the origin generate other hyperbolic right angled triangles. The hyperbolic triangle $\Delta\mathbf{abc}$ in the Poincaré unit disk model, $\mathbb{R}^2_{s=1}$, of hyperbolic geometry has vertices $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$, and corresponding sides, $A = -\mathbf{b} \oplus \mathbf{c}$, $B = -\mathbf{c} \oplus \mathbf{a}$ and $C = -\mathbf{a} \oplus \mathbf{b}$, two of which, A and B, are orthogonal. It satisfies the hyperbolic Pythagorean identity $\|A\|^2 \oplus \|B\|^2 = \|C\|^2$, expressing the square of the hyperbolic length of the hypotenuse of a hyperbolic right angled triangle as the Einstein sum of the squares of the hyperbolic lengths of the other two sides. The Hyperbolic Pythagorean Theorem in its present form, thus, completes the long road from Pythagoras to Einstein, a path that has been emphasized by several authors.[9, 10]