# Placing The Einstein Addition Centrally In Special Relativity Theory: An Idea Whose Time Has Come Back

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**ABSTRACT** We challenge the dominant position of the space-time formalism in special relativity in favor of the non-Euclidean style. We show in this expository article that the nonassociative algebra that regulates the general Einstein addition of relativistically admissible velocities gives rise to a grouplike algebraic object, called a *gyrogroup*. It is argued that in the same way that the physics student is expected to know that vector addition is commutative and associative, the modern physics student should be expected to know that Einstein's addition is gyrocommutative and gyroassociative. The prefix "gyro" that we use to emphasize analogies with classical concepts stems from the Thomas gyration. The latter, in turn, is the extension by abstraction of the Thomas precession of special relativity theory. It is hoped that, following this article, the general Einstein velocity addition of relativistically admissible velocities that need not be parallel will find its way into the books on relativity physics. The omission of the general Einstein velocity addition from these books (with only a few exceptions), traced back to Minkowski, is rooted in the competition between two distinct approaches to the study of special relativity that erupted in 1908 soon after the theory was introduced by Einstein in 1905. Following our forthcoming book *Beyond the Einstein Addition Law and its Gyroscopic Thomas Precession: The Theory of Gyrogroups and Gyrovector Spaces* [Kluwer Acad., Boston, 2001], the aim of this paper is to demonstrate that placing the Einstein velocity addition centrally in the special theory of relativity is an idea whose time has come back.

## 1 INTRODUCTION

In an early stage of his *Lectures On Physics* Feynman [1] introduces the algebra that regulates vector addition, emphasizing that vector addition is both commutative and associative. This result must certainly be known to every student of physics. Yet, the analogous result for the Einstein vector addition, asserting that the Einstein addition is both *gyrocommutative* and *gyroassociative*, is unheard of in the class room despite the fact that it is known since 1988 from papers that signal the birth of the theory of gyrogroups and gyrovector spaces, [2] [3] [4] [5], now also available in a book [6].

The aim of this article is to present the nonassociative algebra that regulates the Einstein addition of vectors, promoting it as a part of the lore learned by all modern physics students, thus placing the Einstein velocity addition centrally in the special theory of relativity. As an expository article, proofs are not provided in general, but are available in various published papers and in a book [6] to which the readers are referred. The results are presented in this article within the frame of the abstract real (or complex, in Section 9) inner product space $\mathbb{V}$. However, nothing is lost if the reader always assumes that the abstract real inner product space is the Euclidean 3-space, $\mathbb{V} = \mathbb{R}^3$.

In 1908 Herman Minkowski recast special relativity as a new geometry of spacetime. The resulting Minkowski space is based on his four-dimensional matrix formalism designed to take advantage of the postulated covariance of physical laws with respect to the Lorentz group [7]. In a Minkowski's spacetime, space and time are inseparable, giving rise to the so called *pseudo – Euclidean space* in which the time coordinate is defined to be imaginary. The emergence of Minkowski's four-dimensional formalism as a standard technique in special relativity theory is explored by S. Walter in [7].

In 1910 Varičak [8] discovered that Einsteinian velocities have natural interpretation in the hyperbolic geometry of Bolyai and Lobachevski, thus introducing the non-Euclidean approach to the study of special relativity. The non-Euclidean program of Varičak (1865 – 1942) is described by Walter in [7]. The myth of the unity of space and time, initiated by Minkowski in 1908, breaks down in the non-Euclidean approach to special relativity theory, that contrasts Minkowski's pseudo – Euclidean approach.

The history of the two competing geometrical approaches to relativity physics has been studied quantitatively by Walter [7], calling them

(1) the *space-time formalism*, in which the study of relativity physics is based on the Lorentz transformation law, from which the Einstein addition is derived; and
(2) the *non-Euclidean style*, in which the study of relativity physics is based on the general Einstein velocity addition law and its underlying hyperbolic geometry of Bolyai and Lobachevski, from which the Lorentz transformation is derived.

The crucial distinction between the space-time formalism and the non-Euclidean style lies in the latter's exclusion of the imaginary temporal coordinate. However, there is another important distinction between the two approaches in that the space-time formalism is by definition a four-dimensional structure, while the non-Euclidean style englobes both (i) three-dimensional and (ii) four-dimensional structures. Indeed, (i) the three-dimensional case is illustrated in Sections 2–4 and 7–11, and (ii) the four-dimensional case is illustrated in Section 6. An interesting non-Euclidean style study, linking the bounds of relativistically admissible velocities and areas of hyperbolic triangles has recently been presented in the Journal [9].

Contrasting the success of Minkowski's space-time formalism, Walter explains the source of the failure of the non-Euclidean style. He thus notes in [7] that

> "In contrast to the amount of publicity they received, applications of non-Euclidean geometry to physics by leading practitioners produced slim theoretical results, the value of which was outstripped by the technical intricacy of the methods developed to obtain them."

Among the leading practitioners in the non-Euclidean style Walter mentions Alfred A. Robb, Vladimir Varičak, Gilbert N. Lewis, Edwin B. Wilson, and Émile Borel [7]. A modern study of connections between the theory of relativity and the hyperbolic geometry of Bolyai and Lobachevski is presented, for instance, in Černikov [10] where references to earlier related work are given. From Varičak's acknowledgment of Sommerfeld's 1909 paper [11] it appears that there was a causal link between the latter paper and Varičak's 1910 discovery in [8] of the role that hyperbolic geometry plays in special relativity theory. Thus, it was Sommerfeld's 1909 paper that sparked Varičak's non-Euclidean program for special relativity; see p. 270 in Ref. [12]. Ironically, however, not only did Sommerfeld employ an imaginary temporal coordinate, following the space-time formalism of Minkowski, he deplored the non-Euclidean style in print, as Walter noted on p. 114 in Ref. [7]:

> " ... just after Varičak's first exposé of the non-Euclidean style ([8], 1910), Sommerfeld completed his signal work on the four-dimensional vector calculus for the *Annalen der Physik*. In a footnote to his work, Sommerfeld remarked that the geometrical relations he presented in terms of three real and one imaginary coordinate could be reinterpreted in terms of non-Euclidean geometry. The latter approach, Sommerfeld cautioned (p. 752 in Ref. [13]), could 'hardly be recommended'."

Furthermore, Walter notes in [7] that following the competition between the two geometrical approaches to the understanding of relativity physics:

> "Minkowski neither mentioned the [Einstein] law of velocity addition, nor expressed it in formal terms."

Instead, however [7],

> "Minkowski retained the geometric interpretation of the Lorentz transformations that had accompanied the *now-banished* non-Euclidean interpretation of velocity vectors." [italics added].

It was Minkowski who laid the basis for the mainstream literature on relativity physics. Following Minkowski, therefore, the general Einstein velocity addition law of relativistically admissible velocities that need not be parallel is unheard of in most texts on special and general relativity theory. Rather, it is only the special case of Einstein's addition, corresponding to parallel velocities, which is presented. Among outstanding exceptions we note the general relativity books by Fock [14] and by Sexl and Urbantke [15].

A full-scale treatment of the special theory of relativity, with particular emphasis on the expression and interpretation of its concepts and results in terms of the Minkowski space-time formalism, is presented in Synge's 1956 textbook [16]. It is this book that effectively created the trend toward four-dimensional approaches, suggesting that the current domination of the space-time formalism in textbooks on special relativity theory is a late-20th century phenomenon. Additionally, the boom of research in general relativity is also a late-20th century phenomenon. Since special relativity is considered to be a preliminary subject to general relativity, the promotion of the space-time formalism in special relativity followed the boom of research in general relativity. Connections between the special theory of relativity and the hyperbolic geometry of Bolyai and Lobachevski, like the one explored by Černikov [10], are thus ignored in the mainstream literature. Of course, there are other factors that must be considered in our attempt to understand the current domination of the space-time formalism in textbooks on special relativity.

The task we face in this expository article is to reverse the process initiated by Minkowski, turning the "now-banished" non-Euclidean style into the cornerstone of relativity physics. Following the emergence of the theory of gyrogroups and gyrovector spaces, we now have a better case to demonstrate that the general Einstein addition, which until recently has rested in undeserved obscurity for too long, plays the central role in relativity physics.

The omission of the general Einstein addition in most of the books on relativity physics is contrasted by its presence, in disguise, in the literature on function theory in the unit ball of $\mathbb{C}^n$ [17], as we explain in Section 9.

Following the advent of the theory of gyrogroups and gyrovector spaces [6], the non-Euclidean geometry approach to relativity physics presents a better case. Hence, it is now breaking into the mainstream, as evidenced from [15], where a brief history of the appearance of gyrocommutative gyrogroups in relativity physics is presented on pp. 141 – 142. Taking advantage of gyrogroup-theoretic techniques, now available in [6], this paper tilts the balance away from the space-time formalism in favor of the non-Euclidean geometry approach to relativity physics. As such, this article begins with the presentation of the general Einstein velocity addition and its underlying nonassociative algebra. The underlying nonassociative algebra that regulates Einstein's addition gives rise to the notion of generalized groups, called *gyrogroups*, and generalized vector spaces, called *gyrovector spaces*. Gyrovector spaces, in turn, form the setting for hyperbolic geometry in the same way that vector spaces form the setting for Euclidean geometry. Furthermore, they allow the unification of hyperbolic and Euclidean geometry into a super geometry, called gyrogeometry [18].

The non-Euclidean style has lost the battle against Minkowski's space-time formalism and, thus, disappeared from the mainstream literature. However, following the advent of the theory of gyrogroups and gyrovector spaces, the time has come for the renewal of the use of the non-Euclidean style in the study of relativity physics. This in turn, will produce strong links with two mathematical theories that are presently considered unrelated to Einstein's special theory of relativity. Accordingly, in addition to the obvious link of special relativity with hyperbolic geometry:

- $(i)$ In Section 9 we present a link with the theory of functions in the open ball of the $n$-dimensional complex space $\mathbb{C}^n$ [17]; and
- $(ii)$ In Section 10 we present a link with the geometric viewpoint of complex analysis [19].
- $(iii)$ Finally, since Einstein gyrovector spaces are isomorphic to Möbius gyrovector spaces, we further illustrate graphically the Möbius gyrovector plane with its hyperbolic triangle, hyperbolic Pythagorean theorem, and hyperbolic parallel transport. The gyrovector spaces of Einstein and Möbius form the setting for two models of hyperbolic geometry. Einstein's addition gives rise to the Beltrami model of hyperbolic geometry, which is non-conformal. In contrast, Möbius' addition gives rise to the Poincaré model of hyperbolic geometry, which is conformal and hence also known as the conformal model of hyperbolic geometry.

Thus, while the study of special relativity by its space-time formalism is a necessary preparation for the study of general relativity, it is true that in this way one looses the possibility to make contact with (i) hyperbolic geometry (Sections 7 and 8), (ii) the theory of functions in the ball (Section 9), and (iii) the geometric viewpoint of complex analysis (Section 10).

## 2 THE EINSTEIN ADDITION

Let $\mathbb{V}$ be a real inner product space, and let $\mathbb{V}_c$,

$$\mathbb{V}_c = \{\mathbf{v} \in \mathbb{V} : \|\mathbf{v}\| < c\} \tag{1}$$

be the set of all relativistically admissible velocities in $\mathbb{V}$, that is, all vectors $\mathbf{v} \in \mathbb{V}$ with magnitude $< c$, $c$ being any fixed positive number representing the vacuum speed of light. The set $\mathbb{V}_c$ of all relativistically admissible velocities in $\mathbb{V}$ is thus the open ball $\mathbb{V}_c$ of the space $\mathbb{V}$ with radius $c$, centered at the origin of its space.

We assume that the inner product in $\mathbb{V}$ (also known as a scalar product) is *positive definite* in the sense that $\mathbf{v} \cdot \mathbf{v} = \|\mathbf{v}\|^2 \geq 0$ for all $\mathbf{v} \in \mathbb{V}$ and $\|\mathbf{v}\|^2 > 0$ if $\mathbf{v} \neq \mathbf{0}$. Clearly, a positive definite inner product is *non-degenerate*, that is, if $\mathbf{u}, \mathbf{v} \in \mathbb{V}$ and $\mathbf{u}{\cdot}\mathbf{v} = 0$ for all $\mathbf{v} \in \mathbb{V}$ then $\mathbf{u} = \mathbf{0}$. Indeed, if $\mathbf{u}{\cdot}\mathbf{v} = 0$ for all $\mathbf{v} \in \mathbb{V}$ and any given $\mathbf{u} \in \mathbb{V}$ then, in particular, for $\mathbf{v} = \mathbf{u}$ we have $\|\mathbf{u}\|^2 = 0$ implying $\mathbf{u} = \mathbf{0}$.

Let $\mathbf{u}, \mathbf{v} \in \mathbb{V}_c$ be any two relativistically admissible velocities in $\mathbb{V}$. Their Einstein sum, $\mathbf{u}{\oplus}\mathbf{v}$, is given by the equation

$$\mathbf{u}{\oplus}\mathbf{v} = \frac{1}{1 + \frac{\mathbf{u}\cdot\mathbf{v}}{c^2}} \left\{ \mathbf{u} + \frac{1}{\gamma_{\mathbf{u}}}\mathbf{v} + \frac{1}{c^2}\frac{\gamma_{\mathbf{u}}}{1+\gamma_{\mathbf{u}}}(\mathbf{u}{\cdot}\mathbf{v})\mathbf{u} \right\} \tag{2}$$

where $\gamma_{\mathbf{v}}$ is the Lorentz factor of $\mathbf{v}$,

$$\gamma_{\mathbf{v}} = \frac{1}{\sqrt{1 - \frac{\|\mathbf{v}\|^2}{c^2}}} \tag{3}$$

satisfying the *gamma identity* [20]

$$\gamma_{\mathbf{u}\oplus\mathbf{v}} = \gamma_{\mathbf{u}}\gamma_{\mathbf{v}}\left(1 + \frac{\mathbf{u}\cdot\mathbf{v}}{c^2}\right) \tag{4}$$

It follows from (3) that the reality of $\gamma_{\mathbf{v}}$ is equivalent to the condition $\mathbf{v} \in \mathbb{V}_c$. Hence, it follows from the gamma identity (4) that $\mathbf{u}, \mathbf{v} \in \mathbb{V}_c$ implies $\mathbf{u}{\oplus}\mathbf{v} \in \mathbb{V}_c$ so that Einstein's addition is, indeed, a binary operation in $\mathbb{V}_c$, forming the Einstein *groupoid* $(\mathbb{V}_c, \oplus)$, a groupoid being a non-empty set with a binary operation. $\mathbb{V}_c$ is thus the open $c$-ball of a real inner product space $\mathbb{V}$, equipped with the Einstein velocity addition $\oplus$ and with the positive definite inner product it inherits from its space $\mathbb{V}$.

We note that (2) defines the Einstein sum $\|\mathbf{u}\|{\oplus}\|\mathbf{v}\|$ as well, where the equality $\|\mathbf{u}\|{\cdot}\|\mathbf{v}\| = \|\mathbf{u}\|\|\mathbf{v}\|$ is understood.

In the special case when the velocity vectors are parallel, Einstein's velocity addition (2) takes the simpler form

$$\mathbf{u}{\oplus}\mathbf{v} = \frac{\mathbf{u} + \mathbf{v}}{1 + \frac{\mathbf{u}\cdot\mathbf{v}}{c^2}}, \qquad \mathbf{u}\|\mathbf{v} \tag{5}$$

for $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3_c$ and $\mathbf{u} \parallel \mathbf{v}$, where the operation $\oplus$ is both commutative and associative. Indeed, as Poincaré pointed out in 1906/7 (fn. 34, p. 56 in [21]), Einstein's addition (5) of relativistically admissible parallel velocities is a group operation, thus admitting a useful algebraic structure. More than 80 years later it was discovered that also the general Einstein velocity addition (2) is not algebraically structureless. Rather, it is a gyrocommutative gyrogroup operation [2] [4] [5], as we will see in the sequel.

Einstein's addition (5) of parallel relativistically admissible velocities is found in his 1905 paper that founded special relativity theory [22]. Also the general case of Einstein's velocity addition of relativistically admissible velocities which need not be parallel, (2), is found in Einstein's 1905 paper [23]. We should note that the Euclidean 3-vector algebra was not so widely known in 1905 and, consequently, was not used by Einstein. Einstein, thus, calculated in [23] the behavior of the velocity components parallel and orthogonal to the relative velocity between inertial systems, which is as close as one can get to the vectorial version (2) without vectors. Identity 4, restricted to parallel velocities, is found on p. 36 of Borel's 1914 book [24].

Einstein's addition is noncommutative. In general

$$\mathbf{u}\oplus\mathbf{v} \neq \mathbf{v}\oplus\mathbf{u} \tag{6}$$

$\mathbf{u}, \mathbf{v} \in \mathbb{V}_c$. Moreover, Einstein's addition is also nonassociative. In general

$$(\mathbf{u}\oplus\mathbf{v})\oplus\mathbf{w} \neq \mathbf{u}\oplus(\mathbf{v}\oplus\mathbf{w}) \tag{7}$$

$\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{V}_c$. Hence, the Einstein groupoid $(\mathbb{V}_c, \oplus)$ is not a group.

Physicists and mathematicians tend to think of symmetry as being virtually synonymous with the theory of groups. It therefore seems that some symmetry and mathematical regularity has been lost in the transition from ordinary vector addition, which is a commutative group operation, into Einstein's addition, which is not a group operation. However, we will see in this expository paper that the seemingly lost symmetry and mathematical regularity reappear when we realize that Einstein's addition is, in fact, a gyrocommutative gyrogroup operation. More generally, we will see that despite the fact that a gyrogroup is a nongroup generalized group, it retains the group ability to capture symmetry and mathematical regularity. Moreover, we will see that the structure of the abstract gyrogroup is richer than that of the abstract group, since it involves a gyration, which is an automorphism that possesses its own rich structure.

Most texts on relativity physics present the Einstein addition only for parallel velocities for the sake of simplicity. Restricted to parallel velocities, Einstein's addition is both commutative and associative. Among outstanding exceptions we mention the general relativity books by (i) R.U. Sexl and H.K. Urbantke [15], where the vector Einstein addition (2), as well as its restriction to parallel velocities, is presented; and by (ii) V. Fock [14], where the general Einstein velocity addition is considered at length with its associated space called the Lobachevsky–Einstein velocity space. Furthermore, Fock notes on p. 39 in [14] that the general Einstein velocity addition gives rise to the Riemannian line element of the Beltrami model (also known as the Klein model; see [25]) of hyperbolic geometry. A detailed study of the Riemannian line element of Einstein's metric, and its resulting Beltrami's model of hyperbolic geometry is presented in [6].

Einstein's addition $\oplus$, (2), is a binary operation in the ball $\mathbb{V}_c$ of the real inner product space $\mathbb{V}$. The identity element of $\mathbb{V}_c$ for Einstein's addition is $\mathbf{0} \in \mathbb{V}_c$, and the inverse $\ominus\mathbf{v}$ of $\mathbf{v} \in \mathbb{V}_c$ for Einstein's addition is its negative, $\ominus\mathbf{v} = -\mathbf{v} \in \mathbb{V}_c$. Furthermore, Einstein's addition possesses a *left cancellation law*

$$\mathbf{u}\oplus(\ominus\mathbf{u}\oplus\mathbf{v}) = \mathbf{v} \tag{8}$$

It seems that the right counterpart of (8) does not exist since, in general,

$$(\mathbf{u}\ominus\mathbf{v})\oplus\mathbf{v} \neq \mathbf{u} \tag{9}$$

Fortunately, however, the right cancellation law, in its two mutually dual forms, will naturally emerge in the sequel, (29).

In the limit of large $c$, $c \to \infty$, the ball $\mathbb{V}_c$ expands into the whole of its space $\mathbb{V}$, and Einstein's addition $\oplus$ in $\mathbb{V}_c$ reduces to the ordinary vector addition $+$ in $\mathbb{V}$. The ordinary vector addition $+$ in $\mathbb{V}$ is both commutative and associative, forming the group $(\mathbb{V}, +)$ of vectors. In contrast, Einstein's addition (2) is neither commutative nor associative so that the Einstein groupoid $(\mathbb{V}_c, \oplus)$ is not a group. The precise definition of the group, and of its Einstein's counterpart, the gyrogroup, will be presented in Section 5. It seems that following the breakdown of commutativity and associativity in Einstein's addition some mathematical regularity has been lost in the transition from vector addition in $\mathbb{V}$ into Einstein's vector addition (2) in $\mathbb{V}_c$. This is, however, not the case since, as we will see in the sequel, the Thomas gyration comes to the rescue.

## 3 THE THOMAS GYRATION

For any $\mathbf{u}, \mathbf{v} \in \mathbb{V}_c$ let $\mathrm{gyr}[\mathbf{u}, \mathbf{v}] : \mathbb{V}_c \to \mathbb{V}_c$ be a self-map of $\mathbb{V}_c$ given in terms of Einstein's addition $\oplus$ by the equation

$$\mathrm{gyr}[\mathbf{u}, \mathbf{v}]\mathbf{w} = \ominus(\mathbf{u}\oplus\mathbf{v})\oplus\{\mathbf{u}\oplus(\mathbf{v}\oplus\mathbf{w})\} \tag{10}$$

where $\ominus\mathbf{v} = -\mathbf{v}$ for all $\mathbf{w} \in \mathbb{V}_c$. The self-map $\mathrm{gyr}[\mathbf{u}, \mathbf{v}]$ of $\mathbb{V}_c$, that takes $\mathbf{w} \in \mathbb{V}_c$ into $\ominus(\mathbf{u}\oplus\mathbf{v})\oplus\{\mathbf{u}\oplus(\mathbf{v}\oplus\mathbf{w}) \in \mathbb{V}_c$, is called the *Thomas gyration* generated by $\mathbf{u}$ and $\mathbf{v}$.

Owing to the nonassociativity of Einstein's addition $\oplus$, in general, a Thomas gyration is not the identity map. Interestingly, it keeps the inner product of elements of the ball $\mathbb{V}_c$ invariant, that is,

$$\mathrm{gyr}[\mathbf{u}, \mathbf{v}]\mathbf{a}\cdot\mathrm{gyr}[\mathbf{u}, \mathbf{v}]\mathbf{b} = \mathbf{a}\cdot\mathbf{b} \tag{11}$$

for all $\mathbf{a}, \mathbf{b}, \mathbf{u}, \mathbf{v} \in \mathbb{V}_c$. As such, $\mathrm{gyr}[\mathbf{u}, \mathbf{v}]$ is an *isometry* of $\mathbb{V}_c$, keeping the norm of elements of the ball $\mathbb{V}_c$ invariant,

$$\|\mathrm{gyr}[\mathbf{u}, \mathbf{v}]\mathbf{w}\| = \|\mathbf{w}\| \tag{12}$$

Hence, $\mathrm{gyr}[\mathbf{u}, \mathbf{v}]$ represents a rotation of the ball $\mathbb{V}_c$ about its origin. Interestingly, the inverse of the gyration $\mathrm{gyr}[\mathbf{u}, \mathbf{v}]$ turns out to be the gyration $\mathrm{gyr}[\mathbf{v}, \mathbf{u}]$ for all $\mathbf{u}, \mathbf{v} \in \mathbb{V}_c$. We, accordingly, say in gyrolanguage that $\mathrm{gyr}[\mathbf{u}, \mathbf{v}]$ is a *gyroisometry*. Furthermore, the bijective (one-to-one) self-map $\mathrm{gyr}[\mathbf{u}, \mathbf{v}]$ of $\mathbb{V}_c$ respects the Einstein addition in $\mathbb{V}_c$,

$$\mathrm{gyr}[\mathbf{u}, \mathbf{v}](\mathbf{a}\oplus\mathbf{b}) = \mathrm{gyr}[\mathbf{u}, \mathbf{v}]\mathbf{a}\oplus\mathrm{gyr}[\mathbf{u}, \mathbf{v}]\mathbf{b} \tag{13}$$

so that it is an automorphism of the Einstein groupoid $(\mathbb{V}_c, \oplus)$. We recall that an automorphism of the groupoid $(\mathbb{V}_c, \oplus)$ is a bijective self-map of the groupoid $\mathbb{V}_c$ that respects its binary operation, that is, it satisfies (13). Under bijection composition the automorphisms of a groupoid $(\mathbb{V}_c, \oplus)$ form a group known as the automorphism group, and denoted $\mathrm{Aut}(\mathbb{V}_c, \oplus)$. Being an automorphism, a Thomas gyration is said to be a *gyroautomorphism*.

The gyroautomorphism $\mathrm{gyr}[\mathbf{u}, \mathbf{v}]$ regulates the Einstein addition in the ball $\mathbb{V}_c$, giving rise to the following nonassociative algebraic laws that "repair" the breakdown of commutativity and associativity in Einstein's addition:

$$\begin{aligned}
\mathbf{u}\oplus\mathbf{v} &= \mathrm{gyr}[\mathbf{u}, \mathbf{v}](\mathbf{v}\oplus\mathbf{u}) && \text{Gyrocommutative Law} \\
\mathbf{u}\oplus(\mathbf{v}\oplus\mathbf{w}) &= (\mathbf{u}\oplus\mathbf{v})\oplus\mathrm{gyr}[\mathbf{u}, \mathbf{v}]\mathbf{w} && \text{Left Gyroassociative Law} \\
(\mathbf{u}\oplus\mathbf{v})\oplus\mathbf{w} &= \mathbf{u}\oplus(\mathbf{v}\oplus\mathrm{gyr}[\mathbf{v}, \mathbf{u}]\mathbf{w}) && \text{Right Gyroassociative Law}
\end{aligned} \tag{14}$$

Owing to the gyrocommutative law we recognize the rotation $\mathrm{gyr}[\mathbf{u}, \mathbf{v}]$ as the extension by abstraction of the Thomas precession of special relativity theory.

When the abstract space $\mathbb{V}$ is realized by the Euclidean 3-space $\mathbb{R}^3$, the Thomas gyration is realized by the Thomas precession of special relativity. Following Silberstein's 1914 book [20], the Thomas precession is the unique rotation of $\mathbb{V}_c = \mathbb{R}^3_c$ about its origin, that takes $\mathbf{v}\oplus\mathbf{u}$ to $\mathbf{u}\oplus\mathbf{v}$ about a rotation axis that is perpendicular to the plane that $\mathbf{u}$ and $\mathbf{v}$ span, through an angle smaller than $\pi$. In 1914 the Thomas precession did not have a name, and Silberstein called it in his 1914 book a "certain space-rotation". Only in 1926, following Thomas' discovery of the role that the peculiar precession plays in the spinning electron in atoms [26], it became known as the Thomas precession. Furthermore, the relationship between $\mathbf{u}\oplus\mathbf{v}$ and $\mathbf{v}\oplus\mathbf{u}$ to which the Thomas precession gives rise [27] was not considered a commutative-like law since it was discovered only 74 years later, in 1988 [2], that it is accompanied by an associative-like law (left and right) as well, (14).

Various graphical presentations of the Thomas precession are available in [4] and [6].

Students of relativity physics are commonly taught that special relativistic effects are negligible when the velocities involved are much less than the vacuum speed of light $c$. Yet, the Thomas precession effect in the orbital motion of spinning electrons in atoms is clearly observed in resulting spectral lines despite the speed of electrons in atoms being small compared with the speed of light [6]. One may, therefore, ask whether it is possible to furnish a classical background to the Thomas precession [28]. The Thomas precession identity (10), however, demonstrates that the Thomas precession effect stems from the nonassociativity of Einstein's velocity addition. Hence, it cannot be arrived at from classical mechanics, where velocity addition is associative.

The Thomas precession $\mathrm{gyr}[\mathbf{u}, \mathbf{v}]$ as described in (10) is an abstract operation in the sense that it is defined as an isometry of the ball $\mathbb{V}_c$ of an abstract real inner product space $\mathbb{V}$. We therefore consider $\mathrm{gyr}[\mathbf{u}, \mathbf{v}]$ the generalization by abstraction of the relativistic Thomas precession, calling it the *Thomas gyration* generated by $\mathbf{u}, \mathbf{v} \in \mathbb{V}_c$. The latter term, in turn, suggests the prefix "gyro" that we use extensively to coin terms like gyrogroup, gyrovector, gyrovector space, gyrocommutative and gyroassociative laws, that emphasize analogies with classical counterparts.

A most important property of the Thomas gyration is the so called *loop property* (left and right),

$$\begin{aligned} \mathrm{gyr}[\mathbf{u}\oplus\mathbf{v}, \mathbf{v}] &= \mathrm{gyr}[\mathbf{u}, \mathbf{v}] \qquad \text{Left Loop Property} \\ \mathrm{gyr}[\mathbf{u}, \mathbf{v}\oplus\mathbf{u}] &= \mathrm{gyr}[\mathbf{u}, \mathbf{v}] \qquad \text{Right Loop Property} \end{aligned} \tag{15}$$

for all $\mathbf{u}, \mathbf{v} \in \mathbb{V}_c$.

It follows from the Thomas gyration definition in (10) and from the left cancellation law (8) that gyrations generated by a zero vector vanish, being reduced to the identity map $I$ of $\mathbb{V}_c$. Thus, for all $\mathbf{u}, \mathbf{v} \in \mathbb{V}_c$,

$$\mathrm{gyr}[\mathbf{0}, \mathbf{v}] = \mathrm{gyr}[\mathbf{u}, \mathbf{0}] = I \tag{16}$$

Furthermore, it then follows from the left loop property that

$$\begin{aligned} \mathrm{gyr}[\mathbf{u}, \ominus\mathbf{u}] &= \mathrm{gyr}[\mathbf{u}\oplus(\ominus\mathbf{u}), \ominus\mathbf{u}] \\ &= \mathrm{gyr}[\mathbf{u}\ominus\mathbf{u}, \ominus\mathbf{u}] \\ &= \mathrm{gyr}[\mathbf{0}, \ominus\mathbf{u}] \\ &= I \end{aligned} \tag{17}$$

The Einstein velocity addition, rather than the Lorentz transformation, is a primitive notion in our approach to special relativity. Hence, our novel way of describing the Thomas gyration $\mathrm{gyr}[\mathbf{u}, \mathbf{v}]$ in terms of Einstein's addition,(10), is crucially important in our attempt to present a better case for the superiority of the abandoned non-Euclidean style in the study of special relativity. A totally different way in the search for algebraic tools that regulate the Thomas precession has recently been presented by Rindler and Robinson, proposing in [29] that these tools are bivectors and complexified quaternions. In contrast, our approach for dealing with the Thomas precession and with its generalization by abstraction, the Thomas gyration, requires no new algebraic tools. Rather, it employs the Einstein addition as the basic tool. In order to employ this basic tool effectively, one must understand the algebra that regulates the Einstein addition, that is, the nonassociative algebra of gyrogroups and gyrovector spaces. Gyrovector spaces, in turn, form the setting for hyperbolic geometry in the same way that vector spaces form the setting for Euclidean geometry, thus indicating the superiority of the abandoned non-Euclidean style of special relativity in terms of analogies with classical notions.

We are now in a position to demonstrate the power and elegance of the nonassociative algebra that regulates the Einstein addition, by solving two related equations.

# 4 SOLVING EQUATIONS IN EINSTEIN'S ADDITION

To illustrate the use of the gyrovector algebra of Einstein's addition let us solve the equation

$$\mathbf{a}\oplus\mathbf{x} = \mathbf{b} \tag{18}$$

for the unknown $\mathbf{x}$ and any given $\mathbf{a}, \mathbf{b} \in \mathbb{V}_c$. If $\mathbf{x}$ is a solution, then by the right gyroassociative law, and by (17) and (18), we have

$$\begin{aligned} \mathbf{x} &= \mathbf{0}\oplus\mathbf{x} \\ &= (\ominus\mathbf{a}\oplus\mathbf{a})\oplus\mathbf{x} \\ &= \ominus\mathbf{a}\oplus(\mathbf{a}\oplus\mathrm{gyr}[\mathbf{a}, \ominus\mathbf{a}]\mathbf{x}) \\ &= \ominus\mathbf{a}\oplus(\mathbf{a}\oplus\mathbf{x}) \\ &= \ominus\mathbf{a}\oplus\mathbf{b} \end{aligned} \tag{19}$$

Thus, if a solution exists, it must be given uniquely by

$$\mathbf{x} = \ominus\mathbf{a}\oplus\mathbf{b} \tag{20}$$

Conversely, if $\mathbf{x} = \ominus\mathbf{a}\oplus\mathbf{b}$, then $\mathbf{x}$ is indeed a solution since by the left gyroassociative law and (17) we have

$$\begin{aligned} \mathbf{a}\oplus\mathbf{x} &= \mathbf{a}\oplus(\ominus\mathbf{a}\oplus\mathbf{b}) \\ &= (\mathbf{a}\oplus(\ominus\mathbf{a}))\oplus\mathrm{gyr}[\mathbf{a}, \ominus\mathbf{a}]\mathbf{b} \\ &= \mathbf{0}\oplus\mathbf{b} \\ &= \mathbf{b} \end{aligned} \tag{21}$$

Substituting the solution (20) in its equation (18) and replacing $\mathbf{a}$ by $-\mathbf{a} = \ominus\mathbf{a}$ we obtain the *left cancellation law* for Einstein's addition

$$\ominus\mathbf{a}\oplus(\mathbf{a}\oplus\mathbf{b}) = \mathbf{b} \tag{22}$$

A most interesting application of the loop property and the gyroassociative law appears when we derive the solution of the equation

$$\mathbf{x}\oplus\mathbf{a} = \mathbf{b} \tag{23}$$

for given $\mathbf{a}, \mathbf{b} \in \mathbb{V}_c$. By solving this equation, the importance of the left loop property in making the left gyroassociative law effective is uncovered.

If $\mathbf{x} \in \mathbb{V}_c$ is a solution of (23), then by the left gyroassociative law, the left loop property, and (23) we have

$$\begin{aligned}
\mathbf{x} &= \mathbf{x} \oplus \mathbf{0} \\
&= \mathbf{x} \oplus (\mathbf{a} \oplus (\ominus \mathbf{a})) \\
&= (\mathbf{x} \oplus \mathbf{a}) \oplus \mathrm{gyr}[\mathbf{x}, \mathbf{a}](\ominus \mathbf{a}) \\
&= (\mathbf{x} \oplus \mathbf{a}) \oplus (\ominus \mathrm{gyr}[\mathbf{x}, \mathbf{a}]\mathbf{a}) \\
&= (\mathbf{x} \oplus \mathbf{a}) \ominus \mathrm{gyr}[\mathbf{x}, \mathbf{a}]\mathbf{a} \\
&= \mathbf{b} \ominus \mathrm{gyr}[\mathbf{x}, \mathbf{a}]\mathbf{a} \\
&= \mathbf{b} \ominus \mathrm{gyr}[\mathbf{x} \oplus \mathbf{a}, \mathbf{a}]\mathbf{a} \\
&= \mathbf{b} \ominus \mathrm{gyr}[\mathbf{b}, \mathbf{a}]\mathbf{a}
\end{aligned} \tag{24}$$

where we abbreviate: $\mathbf{a} \ominus \mathbf{b} = \mathbf{a} \oplus (-\mathbf{b}) = \mathbf{a} \oplus (\ominus \mathbf{b})$.

Thus, if a solution of $\mathbf{x} \oplus \mathbf{a} = \mathbf{b}$ exists, it must have the form

$$\mathbf{x} = \mathbf{b} \ominus \mathrm{gyr}[\mathbf{b}, \mathbf{a}]\mathbf{a}. \tag{25}$$

Conversely, one must show that $\mathbf{x} = \mathbf{b} \ominus \mathrm{gyr}[\mathbf{b}, \mathbf{a}]\mathbf{a}$ is indeed a solution of (23). This is a delicate matter, verified in [6].

In order to display analogies with groups, we define a dual binary operation $\boxplus$ by the equation

$$\mathbf{a} \boxplus \mathbf{b} = \mathbf{a} \oplus \mathrm{gyr}[\mathbf{a}, \ominus \mathbf{b}]\mathbf{b} \tag{26}$$

calling it the Einstein *coaddition*, that *coexists* with the Einstein addition $\oplus$. Accordingly, the solution $\mathbf{x}$ in (25) of the equation (23) can be written as

$$\mathbf{x} = \mathbf{b} \ominus \mathrm{gyr}[\mathbf{b}, \mathbf{a}]\mathbf{a} = \mathbf{b} \boxminus \mathbf{a} \tag{27}$$

when we abbreviate

$$\mathbf{a} \boxminus \mathbf{b} = \mathbf{a} \boxplus (-\mathbf{b}) = \mathbf{a} \boxplus (\ominus \mathbf{b}) \tag{28}$$

The Einstein binary cooperation $\boxplus$, called the Einstein coaddition, will prove useful in the algebra of the Lorentz transformation link, (55), (57), presented in Section 6.

Substituting the solution $\mathbf{x}$, (25) and (27), into its equation (23) and replacing $\mathbf{a}$ by $-\mathbf{a} = \ominus \mathbf{a}$ we obtain, by means of the notation in (28), the *right cancellation law* for Einstein's addition

$$(\mathbf{b} \boxplus \mathbf{a}) \ominus \mathbf{a} = \mathbf{b} \tag{29a}$$

The latter can be dualized into the *dual right cancellation law*

$$(\mathbf{b} \oplus \mathbf{a}) \boxminus \mathbf{a} = \mathbf{b} \tag{29b}$$

We thus see that while Einstein's addition $\oplus$ possesses a left cancellation law, (22), we need its coexisting operation, the Einstein coaddition $\boxplus$, in order to have a right cancellation law as well. Furthermore, Einstein's coaddition $\boxplus$ captures other analogies as well, for instance, the one exhibited by identities (56) and (57) in the study of the Lorentz group in Section 6.

Einstein's addition $\oplus$ and coaddition $\boxplus$ are mutually dual in the sense of the following two identities [30] [6]

$$\begin{aligned} \mathbf{a} \boxplus \mathbf{b} &= \mathbf{a} \oplus \mathrm{gyr}[\mathbf{a}, \ominus\mathbf{b}]\mathbf{b} \\ \mathbf{a} \oplus \mathbf{b} &= \mathbf{a} \boxplus \mathrm{gyr}[\mathbf{a}, \mathbf{b}]\mathbf{b} \end{aligned} \tag{30}$$

in which each of the two mutually dual Einstein additions is expressed in terms of the other one and a gyration. Clearly, in order to capture analogies with ordinary vector addition, the two mutually dual Einstein additions, $\oplus$ and $\boxplus$, are needed. Considering the two identities in (30) mutually dual, the two gyrations $\mathrm{gyr}[\mathbf{a}, \mathbf{b}]$ and $\mathrm{gyr}[\mathbf{a}, \ominus\mathbf{b}]$ turn out to be mutually dual as well. The essence of this resulting gyration duality is uncovered in the *gyration/cogyration gyrotransitive laws* of Theorem 7, presented in Section 8 and verified in [6]. Thus, (i) the gyration $\mathrm{gyr}[\mathbf{a}, \ominus\mathbf{b}]$ generated by $\mathbf{a}$ and $\ominus\mathbf{b}$ is also (ii) the cogyration $\mathrm{gyr}[\mathbf{a}, \ominus\mathbf{b}]$ that coexists with the gyration $\mathrm{gyr}[\mathbf{a}, \mathbf{b}]$ generated by $\mathbf{a}$ and $\mathbf{b}$.

The Einstein coaddition $\boxplus$ in (26) can also be written as

$$\mathbf{u} \boxplus \mathbf{v} = C_{\mathbf{u},\mathbf{v}}(\gamma_{\mathbf{u}}\mathbf{u} + \gamma_{\mathbf{v}}\mathbf{v}) \tag{31}$$

where $C_{\mathbf{u},\mathbf{v}}$ is the coefficient

$$C_{\mathbf{u},\mathbf{v}} = \frac{\gamma_{\mathbf{u}} + \gamma_{\mathbf{v}}}{\gamma_{\mathbf{u}}^2 + \gamma_{\mathbf{v}}^2 + \gamma_{\mathbf{u}}\gamma_{\mathbf{v}}(1 + \frac{\mathbf{u}\cdot\mathbf{v}}{c^2}) - 1} \tag{32}$$

that is symmetric in $\mathbf{u}$ and $\mathbf{v}$. Accordingly, the Einstein coaddition $\boxplus$ is commutative. Indeed, more generally, a gyrogroup $(G, \oplus)$ is gyrocommutative if and only if its cooperation $\boxplus$ is commutative [6].

The mutually dual identities in (29) and in (30) herald the emergence of the duality symmetries to which the dual binary additions of Einstein give rise. These, in turn, lead to the novel bifurcation approach to hyperbolic geometry, reported in [30].

## 5 GROUPS AND GYROGROUPS

Motivated by the definition of the group notion in algebra, we take key features of the algebra that regulates the Einstein addition as axioms in the definition of the gyrogroup notion in nonassociative algebra. The definitions of groups and gyrogroups thus follow.

**Definition 1 (Groups and Commutative Groups).** *A group is a groupoid $(G, +)$ whose binary operation satisfies the following axioms. In G there is at least one element, $0$, called a left identity, satisfying*

*(Gr1)* $0 + a = a$ *Left Identity*

*for all $a \in G$. There is an element $0 \in G$ satisfying axiom (Gr1) such that for each $a$ in G there is an element $-a$ in G, called a left inverse of $a$, satisfying*

*(Gr2)* $-a + a = 0$ *Left Inverse*

*Moreover, for all $a, b, c \in G$,*

*(Gr3)* $(a + b) + c = a + (b + c)$ *Associative Law*

*The group $(G+)$ is commutative if for all $a, b \in G$*

*(Gr4)* $a + b = b + a$ *Commutative Law*

**Definition 2 (Gyrogroups and Gyrocommutative Gyrogroups).** *A groupoid $(G, \oplus)$ is a gyrogroup if its binary operation satisfies the following axioms. In G there is at least one element, $0$, called a left identity, satisfying*

*(Gy1)* $0 \oplus a = a$ *Left Identity*

*for all $a \in G$. There is an element $0 \in G$ satisfying axiom (Gy1) such that for each $a$ in G there is an element $\ominus a$ in G, called a left inverse of $a$, satisfying*

*(Gy2)* $\ominus a \oplus a = 0$ *Left Inverse*

*Moreover, for any $a, b, z \in G$ there exists a unique element $\mathrm{gyr}[a, b]z \in G$ such that*

*(Gy3)* $a \oplus (b \oplus z) = (a \oplus b) \oplus \mathrm{gyr}[a, b]z$ *Left Gyroassociative Law*

*If $gyr[a, b]$ denotes the map $gyr[a, b] : G \to G$ given by $z \mapsto gyr[a, b]z$ then*

*(Gy3a)* $\mathrm{gyr}[a, b] \in \mathrm{Aut}(G, \oplus)$ *Gyroautomorphism*

*and $gyr[a, b]$ is called the Thomas gyration, or the gyroautomorphism of G, generated by $a, b \in G$. The operation $\mathrm{gyr} : G \times G \to \mathrm{Aut}(G, \oplus)$ is called* *the gyrooperation of* $G$*. Finally, the gyroautomorphism gyr*$[a,b]$ *generated by any* $a,b\in G$ *satisfies*

*(Gy3b)* $\mathrm{gyr}[a,b]=\mathrm{gyr}[a\oplus b,b]$ *Left Loop Property*

*The gyrogroup* $(G,\oplus)$ *is gyrocommutative if for all* $a,b\in G$

*(Gy4)* $a\oplus b=\mathrm{gyr}[a,b](b\oplus a)$ *Gyrocommutative Law*

First theorems in group theory [31] and in gyrogroup theory [5] [6] establish the existence of a unique identity and a unique inverse, each of which is both left and right.

Einstein's addition turns out to be a gyrocommutative gyrogroup operation and, accordingly, the groupoid consisting of the set of all relativistically admissible velocities with their Einstein's addition forms a gyrocommutative gyrogroup, called the relativity gyrogroup. The gyrogroup notion has thus sprung from the soil of Einstein's special theory of relativity in [2] [3] [32] and [33]. The Einstein relativity gyrogroup is gyrocommutative. The first non-gyrocommutative gyrogroup was discovered in [34]. Gyrogroups, both gyrocommutative and non-gyrocommutative, abound in group theory as shown in [35] [36]. Gyrogroups are related to other nonassociative structures in nonassociative algebra; see for instance [37] [38] [39].

In gyrogroup theory, the Thomas gyration $\mathrm{gyr}[\mathbf{u},\mathbf{v}]$ is directly linked to its gyrogroup operation $\oplus$, (10), from which it results, owing to the nonassociativity of the operation $\oplus$. This, however, is unknown in pregyrogroup relativity. The pregyrogroup overwhelming algebra of the "notorious Thomas precession formula" (in the words of Rindler and Robinson, p. 431 in [29]) is well described by Herbert Goldstein in his book *Classical Mechanics*, pp. 285 – 286 [40]. Discouraging any attempt to simplify the Thomas rotation matrix to the point where its rotation-matrix behavior can be actually demonstrated and applied to related problems, Goldstein notes that

> The decomposition process [describing two successive pure Lorentz transformations as a pure Lorentz transformation preceded, or followed, by a Thomas rotation] can be carried through on the product of two pure Lorentz transformations to obtain explicitly the rotation of the coordinate axes resulting from the two successive boosts [that is, the Thomas rotation; see our gyro-identities (42)]. In general, the algebra involved is *quite forbidding,* more than enough, usually, to *discourage any actual demonstration* of the rotation matrix [italics added].
>
> Herbert Goldstein, Classical Mechanics

In the years 1908 – 1914, the period which experienced a dramatic flowering of creativity in the special theory of relativity, the Croatian physicist and mathematician Vladimir Varičak (1865 – 1942), professor and rector of Zagreb

University, showed in 1910 that this theory has a natural interpretation in hyperbolic geometry [8]. For his chagrin, Varičak had to admit in 1924 (p. 80 in [41]) that the adaption of vector algebra for use in hyperbolic space was just not possible.

The Thomas gyration, in turn, allows the introduction of vectors into hyperbolic geometry, where they are called *gyrovectors*, in such a way that Einstein's velocity addition turns out to be a gyrovector addition. Einstein's addition thus becomes a gyrocommutative, gyroassociative gyrogroup operation in the same way that ordinary vector addition is a commutative, associative group operation. Some gyrogroups of gyrovectors admit scalar multiplication, giving rise to gyrovector spaces in the same way that some groups of vectors that admit scalar multiplication give rise to vector spaces. Furthermore, gyrovector spaces form the setting for hyperbolic geometry in the same way that vector spaces form the setting for Euclidean geometry. In particular, the gyrovector space with gyrovector addition given by Einstein's (Möbius; see Sections 10 and 11) addition forms the setting for the Beltrami (Poincaré) ball model of hyperbolic geometry [6].

Evidence that Einstein's addition $\oplus$ in $\mathbb{V}_c$ is regulated by the Thomas precession thus comes to light in this article, turning the notorious Thomas precession, previously considered the ugly duckling of special relativity theory, into the beautiful swan of gyrogroup and gyrovector space theory, where it has been extended by abstraction into the Thomas gyration. The Thomas gyration operation in gyrogroup theory is an automorphism generator, $\mathrm{gyr}[\cdot,\cdot]$, and the resulting generated gyroautomorphisms of $\mathbb{V}_c$, $\mathrm{gyr}[\mathbf{u},\mathbf{v}]$, are the *Thomas gyrations* generated by gyrogroup elements $\mathbf{u},\mathbf{v} \in \mathbb{V}_c$.

In the following Sections, the link between our gyrogroup approach and hyperbolic geometry, and thereby the connection to the historical non-Euclidean style developed during the period 1908-1912 by Varičak, Robb, Wilson and Lewis, and Borel, is manifest.

# 6 THE GYRO–ALGEBRA OF THE LORENTZ TRANSFORMATION

For simplicity, we realize in this section the abstract real inner product space $\mathbb{V}$ by the Euclidean $n$-space $\mathbb{R}^n$ so that the $c$-ball $\mathbb{V}_c$ of $\mathbb{V}$ is realized by the $c$-ball $\mathbb{R}^n_c$ of $\mathbb{R}^n$. The rotations of $\mathbb{R}^n_c$ are represented by the elements of the rotation group $SO(n)$ of all real $n\times n$ orthogonal matrices with determinant 1.

Let $(t,\mathbf{x})^t$, where exponent $t$ denotes transposition, be a spacetime event, $\mathbf{x}\in\mathbb{R}^n$, $t\in\mathbb{R}^+$, $\mathbb{R}^+$ being the positive ray of the real line, $\mathbb{R}^+=\{r\in\mathbb{R}:r\geq 0\}$. A Lorentz transformation of spacetime coordinates is a coordinate transformation that leaves the norm

$$\left\|\begin{pmatrix} t \\ \mathbf{x}\end{pmatrix}\right\| = \sqrt{t^2-\frac{\|\mathbf{x}\|^2}{c^2}} \tag{33}$$

of a spacetime event $(t,\mathbf{x})^t$ invariant. A Lorentz transformation without rotation is known in the jargon as a *boost*. Let $L(\mathbf{u})$ be a Lorentz boost parametrized by a relativistically admissible velocity $\mathbf{u}\in\mathbb{R}^n_c$. Its application to spacetime coordinates $(t,\mathbf{x})^t=(t,\mathbf{v}t)^t$ is given by the equation

$$\begin{aligned} L(\mathbf{u})\begin{pmatrix} t \\ \mathbf{x}\end{pmatrix} &= \begin{pmatrix} \gamma_{\mathbf{u}}(t+\frac{1}{c^2}\mathbf{u}\cdot\mathbf{x}) \\ (\mathbf{u}\oplus_{\mathrm{E}}\mathbf{v})\gamma_{\mathbf{u}}(1+\frac{1}{c^2}\mathbf{u}\cdot\mathbf{v})t \end{pmatrix} \\ &= \begin{pmatrix} \gamma_{\mathbf{u}}(t+\frac{1}{c^2}\mathbf{u}\cdot\mathbf{x}) \\ \gamma_{\mathbf{u}}\mathbf{u}t+\mathbf{x}+\frac{1}{c^2}\frac{\gamma_{\mathbf{u}}^2}{1+\gamma_{\mathbf{u}}}(\mathbf{u}\cdot\mathbf{x})\mathbf{u} \end{pmatrix} \\ &= \begin{pmatrix} t' \\ \mathbf{x}'\end{pmatrix} \end{aligned} \tag{34}$$

leaving the spacetime norm (33) invariant,

$$\left\|\begin{pmatrix} t \\ \mathbf{x}\end{pmatrix}\right\| = \left\|\begin{pmatrix} t' \\ \mathbf{x}'\end{pmatrix}\right\| \tag{35}$$

The boost is a linear transformation of spacetime coordinates that, in (3+1)-dimensional spacetime, has the matrix representation $L_m(\mathbf{v})$ given by the equation [2] [42]

$$L_m(\mathbf{v}) = \begin{pmatrix} \gamma_{\mathbf{v}} & c^{-2}\gamma_{\mathbf{v}}v_1 & c^{-2}\gamma_{\mathbf{v}}v_2 & c^{-2}\gamma_{\mathbf{v}}v_3 \\ \gamma_{\mathbf{v}}v_1 & 1+c^{-2}\frac{\gamma_{\mathbf{v}}^2}{\gamma_{\mathbf{v}}+1}v_1^2 & c^{-2}\frac{\gamma_{\mathbf{v}}^2}{\gamma_{\mathbf{v}}+1}v_1v_2 & c^{-2}\frac{\gamma_{\mathbf{v}}^2}{\gamma_{\mathbf{v}}+1}v_1v_3 \\ \gamma_{\mathbf{v}}v_2 & c^{-2}\frac{\gamma_{\mathbf{v}}^2}{\gamma_{\mathbf{v}}+1}v_1v_2 & 1+c^{-2}\frac{\gamma_{\mathbf{v}}^2}{\gamma_{\mathbf{v}}+1}v_2^2 & c^{-2}\frac{\gamma_{\mathbf{v}}^2}{\gamma_{\mathbf{v}}+1}v_2v_3 \\ \gamma_{\mathbf{v}}v_3 & c^{-2}\frac{\gamma_{\mathbf{v}}^2}{\gamma_{\mathbf{v}}+1}v_1v_3 & c^{-2}\frac{\gamma_{\mathbf{v}}^2}{\gamma_{\mathbf{v}}+1}v_2v_3 & 1+c^{-2}\frac{\gamma_{\mathbf{v}}^2}{\gamma_{\mathbf{v}}+1}v_3^2 \end{pmatrix} \tag{36}$$

so that

$$L(\mathbf{v})\begin{pmatrix} t \\ \mathbf{x} \end{pmatrix} = L_m(\mathbf{v})\begin{pmatrix} t \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} t' \\ x'_1 \\ x'_2 \\ x'_3 \end{pmatrix} \tag{37}$$

Here $\mathbf{v} = (v_1, v_2, v_3)^t \in \mathbb{R}^3_c, \mathbf{x} = (x_1, x_2, x_3)^t \in \mathbb{R}^3, \mathbf{x}' = (x'_1, x'_2, x'_3)^t \in \mathbb{R}^3$, and $t, t' \in \mathbb{R}^+$.

In the limit of large vacuum speed of light, $c \to \infty$, the Lorentz boost $L(\mathbf{v})$, (37), reduces to the Galilei boost $G(\mathbf{v})$, $\mathbf{v} = (v_1, v_2, v_3) \in \mathbb{R}^3$,

$$\begin{aligned} G(\mathbf{v})\begin{pmatrix} t \\ \mathbf{x} \end{pmatrix} &= \lim_{c\to\infty} L(\mathbf{v})\begin{pmatrix} t \\ \mathbf{x} \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ v_1 & 1 & 0 & 0 \\ v_2 & 0 & 1 & 0 \\ v_3 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} t \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} \\ &= \begin{pmatrix} t \\ x_1 + v_1 t \\ x_2 + v_2 t \\ x_3 + v_3 t \end{pmatrix} \\ &= \begin{pmatrix} t \\ \mathbf{x} + \mathbf{v}t \end{pmatrix} \end{aligned} \tag{38}$$

where $\mathbf{x} = (x_1, x_2, x_3)^t \in \mathbb{R}^3$ and $t \in \mathbb{R}^+$.

The composition of two Galilei boosts is equivalent to a single Galilei boost according to the equation

$$G(\mathbf{u})G(\mathbf{v}) = G(\mathbf{u} + \mathbf{v}) \tag{39}$$

as it is clear from (38).

The composition of Lorentz boosts is more complicated than that of Galilei boosts since, in general, the composition of two Lorentz boosts is not a boost but, rather, a boost preceded (or followed) by a space rotation.

Let $V \in SO(n)$ be a space rotation, and let $E(V)$ be the space rotation of a spacetime event, extended from $V$ by the equation

$$E(V)\begin{pmatrix} t \\ \mathbf{x} \end{pmatrix} = \begin{pmatrix} t \\ V\mathbf{x} \end{pmatrix} \tag{40}$$

$t \in \mathbb{R}^+$, $\mathbf{x} \in \mathbb{R}^3$. Similarly, let $\mathrm{Gyr}[\mathbf{u},\mathbf{v}]$ be the space gyration of a spacetime event, extended from $\mathrm{gyr}[\mathbf{u},\mathbf{v}]$ by the equation

$$\mathrm{Gyr}[\mathbf{u},\mathbf{v}] = E(\mathrm{gyr}[\mathbf{u},\mathbf{v}]) \tag{41}$$

Then, the relativistic analog of (39) is [6]

$$\begin{aligned} L(\mathbf{u})L(\mathbf{v}) &= L(\mathbf{u}\oplus\mathbf{v})\mathrm{Gyr}[\mathbf{u},\mathbf{v}] \\ &= \mathrm{Gyr}[\mathbf{u},\mathbf{v}]L(\mathbf{v}\oplus\mathbf{u}) \end{aligned} \tag{42}$$

Hence, the composition of two successive Lorentz boosts, $L(\mathbf{v})$ followed by $L(\mathbf{u})$,

(1) is equivalent to a single boost, $L(\mathbf{u}\oplus\mathbf{v})$, preceded by the Thomas gyration generated by $\mathbf{u}$ and $\mathbf{v}$; and equally well, it

(2) is equivalent to a single boost, $L(\mathbf{v}\oplus\mathbf{u})$, followed by the Thomas gyration generated by $\mathbf{u}$ and $\mathbf{v}$.

The identities in (42) resolve the Mocanu paradox [43] [44] according to which, paradoxically, the relative velocity that results from two successive boosts by $\mathbf{u}$ and $\mathbf{v}$, $\mathbf{u},\mathbf{v} \in \mathbb{R}^n_c$, is simultaneously $\mathbf{u}\oplus\mathbf{v}$ and $\mathbf{v}\oplus\mathbf{u}$ while, in general, $\mathbf{u}\oplus\mathbf{v} \neq \mathbf{v}\oplus\mathbf{u}$. The identities in (42) show that the two successive boosts by $\mathbf{u}$ and $\mathbf{v}$, without rotations, do generate a rotation, that is, the Thomas precession $\mathrm{gyr}[\mathbf{u},\mathbf{v}]$. Hence, the resulting relative velocity that the two successive boosts generate is no longer rotation free. It is either (i) the composite velocity $\mathbf{u}\oplus\mathbf{v}$ *preceded* by the Thomas rotation $\mathrm{gyr}[\mathbf{u},\mathbf{v}]$, or (ii) the composite velocity $\mathbf{v}\oplus\mathbf{u}$ *followed* by the Thomas rotation $\mathrm{gyr}[\mathbf{u},\mathbf{v}]$, as we see from (42). The question as to whether the "right" composite velocity is $\mathbf{u}\oplus\mathbf{v}$ or $\mathbf{v}\oplus\mathbf{u}$ is meaningless in special relativity since a composite velocity is always accompanied by a Thomas rotation (that vanishes if and only if the two nonzero velocities are parallel). Relations between inertial frames with a relative composite velocity are determined by both relative velocities and relative orientations which are inseparable.

The gyro-decomposition (42) of the product of two successive boosts into a single boost preceded or followed by a space rotation may be compared with other available expressions of the decomposition which are available in the literature; see for instance, [45] [46] [47] [48] and [49].

The Lorentz transformation $L(\mathbf{v},V)$ of spacetime events $(t,\mathbf{x})^t \in \mathbb{R}^+ \times \mathbb{R}^n$ is a boost preceded by a rotation,

$$L(\mathbf{v},V)\begin{pmatrix} t \\ \mathbf{x} \end{pmatrix} = L(\mathbf{v})E(V)\begin{pmatrix} t \\ \mathbf{x} \end{pmatrix} \tag{43}$$

Hence, explicitly, the Lorentz transformation $L(\mathbf{v}, V)$ takes the form

$$L(\mathbf{v}, V)\begin{pmatrix} t \\ \mathbf{x} \end{pmatrix} = \begin{pmatrix} \gamma_{\mathbf{v}}(t + \frac{1}{c^2}\mathbf{v}\cdot V\mathbf{x}) \\ \gamma_{\mathbf{v}}\mathbf{v}t + V\mathbf{x} + \frac{1}{c^2}\frac{\gamma_{\mathbf{v}}^2}{1+\gamma_{\mathbf{v}}}(\mathbf{v}\cdot V\mathbf{x})\mathbf{v} \end{pmatrix} \tag{44}$$

$\mathbf{v} \in \mathbb{R}_c^n$, $V \in SO(n)$, $\mathbf{x} \in \mathbb{R}^n$, $t \in \mathbb{R}^+$, as we see from (34).

In the limit of large vacuum speed of light, $c \to \infty$, the Lorentz transformation $L(\mathbf{v}, V)$ in (44) reduces to its Galilean counterpart $G(\mathbf{v}, V)$,

$$G(\mathbf{v}, V)\begin{pmatrix} t \\ \mathbf{x} \end{pmatrix} = \begin{pmatrix} t \\ \mathbf{v}t + V\mathbf{x} \end{pmatrix} \tag{45}$$

$\mathbf{v} \in \mathbb{R}^n$, $V \in SO(n)$, $\mathbf{x} \in \mathbb{R}^n$, $t \in \mathbb{R}$.

Clearly, the Lorentz transformation $L(\mathbf{v}, I)$ with no rotation, $I$ being the identity element of $SO(n)$, is a Lorentz boost,

$$L(\mathbf{v}, I) = L(\mathbf{v}) \tag{46}$$

for all $\mathbf{v} \in \mathbb{R}_c^n$, illustrated in (37) for $n = 3$. Similarly, for all $\mathbf{v} \in \mathbb{R}^n$

$$G(\mathbf{v}, I) = G(\mathbf{v}) \tag{47}$$

is a Galilei boost, illustrated in (38) for $n = 3$.

The composition of two Galilei transformations is given by parameters composition according to the equation

$$G(\mathbf{u}, U)G(\mathbf{v}, V) = G(\mathbf{u} + U\mathbf{v}, UV) \tag{48}$$

where $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$, and $U, V \in SO(n)$.

In full analogy, the composition of two Lorentz transformations is given by parameters composition according to the equation [6]

$$L(\mathbf{u}, U)L(\mathbf{v}, V) = L(\mathbf{u}\oplus U\mathbf{v}, \mathrm{gyr}[\mathbf{u}, U\mathbf{v}]UV) \tag{49}$$

where $\mathbf{u}, \mathbf{v} \in \mathbb{R}_c^n$, and $U, V \in SO(n)$.

The Galilei transformation composition law (48) is obvious and well known in the literature. In contrast, the Lorentz transformation composition law (49) is a new result that follows from gyrogroup-theoretic considerations [42] [50] [51] [6]. These composition laws in terms of parameters composition demonstrate the usefulness of the velocity – orientation parametrization, $G(\mathbf{v}, V)$ and $L(\mathbf{v}, V)$, of both the Galilei and the Lorentz transformation group.

The Lorentz transformations (44) form a subgroup of the so called full Lorentz group. This subgroup is known as the *homogeneous, proper, orthochronous* Lorentz group $SO(n, 1)$. It is:

(1) *homogeneous*, since each of its elements is a Lorentz transformation that takes the origin of spacetime coordinates into an origin of spacetime coordinates; it is

(2) *proper*, since each of its elements is a Lorentz transformation that is continuously connected to the identity transformation of spacetime; and it is

(3) *orthochronous*, since each of its elements is a Lorentz transformation that preserves the sign of time, that is, it takes positive (negative) time into positive (negative) time.

Identity (49) is thus the group operation of the homogeneous, proper, orthochronous Lorentz group $SO(n, 1)$.

Let

$$\begin{pmatrix} t \\ \mathbf{u}t \end{pmatrix} \qquad and \qquad \begin{pmatrix} t' \\ \mathbf{v}t' \end{pmatrix} \tag{50}$$

be two Galilean spacetime events, $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$, and $t, t' \in \mathbb{R}$, with equal time components,

$$t = t' \tag{51}$$

Then, the unique Galilei transformation without rotation (boost) that links the two spacetime events in (50) – (51) has the velocity parameter $\mathbf{v} - \mathbf{u}$, satisfying

$$G(\mathbf{v} - \mathbf{u}) \begin{pmatrix} t \\ \mathbf{u}t \end{pmatrix} = \begin{pmatrix} t' \\ \mathbf{v}t' \end{pmatrix} \tag{52}$$

as one can readily see from (38).

In the analogous relativistic link, the Galilean negative velocity addition, $-$, becomes the Einstein negative coaddition, $\boxminus$. Specifically, let

$$\begin{pmatrix} t \\ \mathbf{u}t \end{pmatrix} \qquad and \qquad \begin{pmatrix} t' \\ \mathbf{v}t' \end{pmatrix} \tag{53}$$

be two relativistic spacetime events, $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n_c$, and $t, t' \in \mathbb{R}^+$, with equal spacetime norms,

$$t'\sqrt{1 - \frac{\|\mathbf{v}\|^2}{c^2}} = t\sqrt{1 - \frac{\|\mathbf{u}\|^2}{c^2}} \tag{54}$$

Then, the unique Lorentz transformation without rotation (boost) that links the two spacetime events in (53) – (54) has the velocity parameter $\mathbf{v} \boxminus \mathbf{u}$, satisfying [6]

$$L(\mathbf{v} \boxminus \mathbf{u}) \begin{pmatrix} t \\ \mathbf{u}t \end{pmatrix} = \begin{pmatrix} t' \\ \mathbf{v}t' \end{pmatrix} \tag{55}$$

Furthermore, the most general Galilei transformation that links the simultaneous spacetime events in (50) – (51) is $G(\mathbf{v} - R\mathbf{u}, R)$, satisfying

$$G(\mathbf{v} - R\mathbf{u}, R) \begin{pmatrix} t \\ \mathbf{u}t \end{pmatrix} = \begin{pmatrix} t' \\ \mathbf{v}t' \end{pmatrix} \tag{56}$$

for all $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$, and $t, t' \in \mathbb{R}$, where $R \in SO(n)$ is a free rotation parameter.

In full analogy, the most general homogeneous, proper, orthochronous Lorentz transformation that links the equinorm spacetime events in (53) – (54) is $L(\mathbf{v} \boxminus R\mathbf{u}, R)$ satisfying [52] [51] [6]

$$L(\mathbf{v} \boxminus R\mathbf{u}, R) \begin{pmatrix} t \\ \mathbf{u}t \end{pmatrix} = \begin{pmatrix} t' \\ \mathbf{v}t' \end{pmatrix} \tag{57}$$

for all $\mathbf{u}, \mathbf{v} \in \mathbb{R}_c^n$, and $t, t' \in \mathbb{R}^+$, where $R \in SO(n)$ is a free rotation parameter.

The problem of the determination of Lorentz links between given spacetime events was only partially solved by several explorers, as evidenced from [53]. The analogies that the Lorentz link (57) shares with its Galilean counterpart (56) demonstrate (i) the power and elegance of gyrogroup-theoretic techniques, and (ii) the need to employ the two mutually dual Einstein additions, $\oplus$ and $\boxplus$, in order to capture analogies. An example of a study in which the determination of the Lorentz transformations that link two given spacetime events is important is found, for instance, in the study of the electron spin by J. Kupersztych in [54]. One can now complete the study of Kupersztych by incorporating the Lorentz transformation link (57) in which $\mathbf{v} \boxminus R\mathbf{u} = \mathbf{v} \boxplus (-R\mathbf{u})$, where Einstein's coaddition $\boxplus$ is given by (30) – (32).

The parameterization $G(\mathbf{v}, V)$ of the homogeneous, proper Galilei transformation by a velocity parameter $\mathbf{v}$ and an orientation parameter $V$ is natural and well known in the literature. Its composition law in terms of parameters composition, (48), is a group operation known in group theory as a *semidirect product* [31]. The analogous parameterization $L(\mathbf{v}, V)$ of the homogeneous, proper, orthochronous Lorentz transformation can be found in the old, 1914 Silberstein's book on special relativity [20], that dates back to the infancy of relativity theory. This parameterization, however, is no longer found in modern books on special relativity; a unique outstanding exception being [15]. The reason is obvious: in pregyrogroup relativity it seemed that the analogous parameterization $L(\mathbf{v}, V)$ in (44) of the homogeneous, proper, orthochronous Lorentz group by a velocity parameter $\mathbf{v} \in \mathbb{R}_c^3$ and an orientation parameter $V \in SO(3)$ does not capture enough of the interesting detail to allow the answer to important questions such as (i) what is the Lorentz transformation composition law in terms of parameters composition; and (ii) what are all the Lorentz transformations that link two given time-like, future-directed equinorm points.

A modern attempt to restore the velocity-orientation parameterization of the Lorentz group is found in [55] and [15]. Following the discovery of the analogies that the velocity-orientation parameterization, $L(\mathbf{v}, V)$, of the Lorentz group shares with its Galilean counterpart, $G(\mathbf{v}, V)$, this parameterization of the Lorentz group finds its way back to the mainstream literature, and becomes part of the lore learned by modern physics students.

The use of the $SL(2, C)$ algebra and its Pauli's spin matrices in the study of the Lorentz group in (3+1)-dimensional spacetime is well known [40][56]. It is therefore interesting to realize that the $SL(2, C)$ algebra is intimately linked to gyrocommutative gyrogroups, as shown in [57]. The study of gyrogroups on a higher level, and in connection with other areas of mathematics, is found in [37] [58] [39] [38] [35] [36]. The study of gyrogroups and gyrovector spaces can be used as a preparation for the study of the more abstract related areas of differential geometry that are presented in Sabinin's recent book [59].

# 7 THE EINSTEIN GYROVECTOR SPACES

Einstein's addition admits scalar multiplication that possesses useful properties. It is given by the equation [5] [6]

$$
\begin{aligned}
r\otimes\mathbf{v} &= c\frac{(1+\|\mathbf{v}\|/c)^r-(1-\|\mathbf{v}\|/c)^r}{(1+\|\mathbf{v}\|/c)^r+(1-\|\mathbf{v}\|/c)^r}\frac{\mathbf{v}}{\|\mathbf{v}\|} \\
&= c\tanh\left(r\tanh^{-1}\frac{\|\mathbf{v}\|}{c}\right)\frac{\mathbf{v}}{\|\mathbf{v}\|}
\end{aligned}
\tag{58}
$$

where $r\in\mathbb{R}$, $\mathbf{v}\in\mathbb{V}_c$, $\mathbf{v}\neq\mathbf{0}$, and $r\otimes\mathbf{0}=\mathbf{0}$, and with which we use the notation $r\otimes\mathbf{v}=\mathbf{v}\otimes r$.

The Einstein addition and scalar multiplication possess the following properties. For any positive integer $n$ and for all $r,r_1,r_2\in\mathbb{R}$ and $\mathbf{a},\mathbf{b},\mathbf{u},\mathbf{v}\in\mathbb{V}_c$,

$$
\begin{aligned}
n\otimes\mathbf{v} &= \mathbf{v}\oplus\ldots\oplus\mathbf{v} && n\text{ terms} \\
(r_1+r_2)\otimes\mathbf{v} &= r_1\otimes\mathbf{v}\oplus r_2\otimes\mathbf{v} && \text{Scalar Distributive Law} \\
(r_1r_2)\otimes\mathbf{v} &= r_1\otimes(r_2\otimes\mathbf{v}) && \text{Scalar Associative Law} \\
r\otimes(r_1\otimes\mathbf{v}\oplus r_2\otimes\mathbf{v}) &= r\otimes(r_1\otimes\mathbf{v})\oplus r\otimes(r_2\otimes\mathbf{v}) && \text{Monodistributive Law} \\
\|r\otimes\mathbf{v}\| &= |r|\otimes\|\mathbf{v}\| && \text{Homogeneity Property} \\
\frac{|r|\otimes\mathbf{v}}{\|r\otimes\mathbf{v}\|} &= \frac{\mathbf{v}}{\|\mathbf{v}\|} && \text{Scaling Property} \\
\|\mathbf{u}\oplus\mathbf{v}\| &\leq \|\mathbf{u}\|\oplus\|\mathbf{v}\| && \text{Triangle Inequality} \\
\mathrm{gyr}[\mathbf{a},\mathbf{b}](r\otimes\mathbf{v}) &= r\otimes\mathrm{gyr}[\mathbf{a},\mathbf{b}]\mathbf{v} && \text{Gyroautomorphism} \\
\mathrm{gyr}[r_1\otimes\mathbf{v},r_2\otimes\mathbf{v}] &= I && \text{Identity Automorphism}
\end{aligned}
\tag{59}
$$

Remarkably, the classical distributive law is not valid in a gyrovector space. In fact, it is owing to the lack of a distributive law that gyrovector spaces are more general than vector spaces. Ambiguously, the Einstein scalar multiplication $\otimes$ in the homogeneity property, and the Einstein addition $\oplus$ in the triangle inequality are

(1) the Einstein scalar multiplication $\otimes$ and the Einstein addition $\oplus$ in the $c$-ball $\mathbb{V}_c$ of $\mathbb{V}$ on the left; and

(2) the Einstein scalar multiplication $\otimes$ and the Einstein addition $\oplus$ in the non-negative part $[0,\,c)$ of the $c$-ball $\mathbb{R}_c$ of $\mathbb{R}$ on the right.

The triple $(\mathbb{V}_c,\oplus,\otimes)$ is known as the Einstein gyrovector space, $\mathbb{V}_c$ being the $c$-ball of a real inner product space $\mathbb{V}$, (1). Having the gyrovector notion in hand, we can now relativize the Newtonian uniform acceleration

$$\begin{aligned}\mathbf{v}_-(t) &= \mathbf{v}_0 + \mathbf{a}t \\ &= \mathbf{a}t + \mathbf{v}_0\end{aligned} \tag{60}$$

$\mathbf{v}_0, \mathbf{a} \in \mathbb{V}$, $t \in \mathbb{R}$, $\mathbf{v}_0$ and $\mathbf{a}$ being an initial velocity and an acceleration vector. The uniform acceleration (60) is represented geometrically by a line that is a geodesic in the velocity space $\mathbb{V}$ relative to the Euclidean metric $d_-(\mathbf{u}, \mathbf{v})$,

$$d_-(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\| \tag{61}$$

There are various ways to "relativize" the classical notion of uniform acceleration; see for instance [60]. However, a widely accepted one states that in general relativity uniformly accelerated velocities are those which traverse along geodesics in velocity spaces [61]. Guided by analogies, we relativize the classical uniform acceleration that (60) represents into the following collection of two relativistic mutually dual counterparts, represented by the equations

$$\mathbf{v}_{\ominus}(t) = \mathbf{v}_0 \oplus \mathbf{a} \otimes t \tag{62a}$$

$$\mathbf{v}_{\boxminus}(t) = \mathbf{a} \otimes t \oplus \mathbf{v}_0 \tag{62b}$$

$\mathbf{v}_0, \mathbf{a} \in \mathbb{V}_c$, $t \in \mathbb{R}^+$. Noting that Einstein's addition is noncommutative, the two relativistic versions (62) of the Newtonian uniform acceleration (60) are distinct. Let us, accordingly, explore the geometric significance of each of the two distinct, mutually dual Einsteinian uniform accelerations in (62). Calling $\mathbf{v}_{\ominus}(t)$ in (62a) a *gyroline*, and calling $\mathbf{v}_{\boxminus}(t)$ in (62b) a *cogyroline*, we thus explore these two mutually dual gyrolines. They are geodesics relative to their respective mutually dual metrics, the *gyrometric* $d_{\ominus}(\mathbf{u}, \mathbf{v})$ and its dual, *cogyrometric* $d_{\boxminus}(\mathbf{u}, \mathbf{v})$, given by the equations

$$d_{\ominus}(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} \ominus \mathbf{v}\| \tag{63a}$$

$$d_{\boxminus}(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} \boxminus \mathbf{v}\| \tag{63b}$$

$\mathbf{u}, \mathbf{v} \in \mathbb{V}_c$.

The gyrometric $d_{\ominus}(\mathbf{u}, \mathbf{v})$ is called the Einstein metric and, similarly, the cogyrometric $d_{\boxminus}(\mathbf{u}, \mathbf{v})$ is called the Einstein cometric. Interestingly, by methods of differential geometry the Einstein metric can be translated into the Riemannian line element of the Beltrami (also known as the Klein) model of hyperbolic geometry, as we show in [6].

There are two mutually dual families of geodesics that result from the two mutually dual gyrometrics (63). The gyrolines, being geodesics relative to the gyrometric $d_{\ominus}(\mathbf{u}, \mathbf{v})$, are called *gyrogeodesics*. Similarly, the cogyrolines, being geodesics relative to the cogyrometric $d_{\boxminus}(\mathbf{u}, \mathbf{v})$, are called *cogyrogeodesics*.

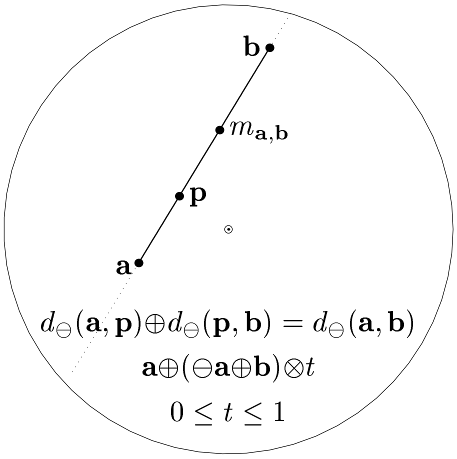


*Figure 1.* The gyroline segment, (65a), linking the two points $\mathbf{a}$ and $\mathbf{b}$ in an Einstein gyrovector space $(\mathbb{V}_c, \oplus, \otimes)$. $\mathbf{p}$ is a generaic point between $\mathbf{a}$ and $\mathbf{b}$, (66), and $m_{\mathbf{a},\mathbf{b}}$ is the midpoint of $\mathbf{a}$ and $\mathbf{b}$, (68).

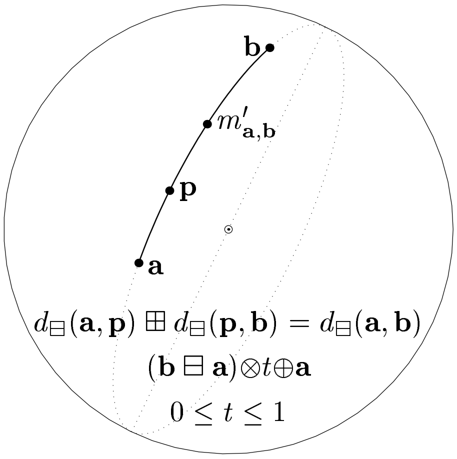


*Figure 2.* The cogyroline segment, (65b), linking the two points $\mathbf{a}$ and $\mathbf{b}$ in an Einstein gyrovector space $(\mathbb{V}_c, \oplus, \otimes)$. $\mathbf{p}$ is a generaic point cobetween $\mathbf{a}$ and $\mathbf{b}$, (70), and $m'_{\mathbf{a},\mathbf{b}}$ is the comidpoint of $\mathbf{a}$ and $\mathbf{b}$, (72).

Further analogies and duality symmetries emerge when we write the Euclidean line (60) in the equivalent form

$$L_-(t) = \mathbf{a} + (-\mathbf{a} + \mathbf{b})t \tag{64}$$

$t \in \mathbb{R}$, that emphasizes the result that it is the unique Euclidean geodesic passing through the two distinct points $\mathbf{a}, \mathbf{b} \in \mathbb{V}$. It passes through the point $\mathbf{a}$ at "time" $t = 0$, and through the point $\mathbf{b}$ at "time" $t = 1$.

In full analogy with (64), the two mutually dual gyrolines in (62) are written as

$$\begin{aligned} L_\ominus(t) &= \mathbf{a}\oplus(\ominus\mathbf{a}\oplus\mathbf{b})\otimes t && \text{(65a)} \\ L_\boxminus(t) &= (\mathbf{b} \boxminus \mathbf{a})\otimes t\oplus\mathbf{a} && \text{(65b)} \end{aligned}$$

$t \in \mathbb{R}$, when we wish to emphasize the result that the gyroline $L_\ominus(t)$ and its dual gyroline, the cogyroline $L_\boxminus(t)$, are the unique ones that pass through the two distinct points $\mathbf{a}, \mathbf{b} \in \mathbb{V}_c$:

(1) The gyroline $L_\ominus(t)$ in (65a), Fig. 1, passes through the point $\mathbf{a}$ at "time" $t = 0$ and, owing to the left cancellation law, (8), it passes through the point $\mathbf{b}$ at "time" $t = 1$. A point $\mathbf{p}$ is *between* $\mathbf{a}$ and $\mathbf{b}$ if

$$\mathbf{p} = \mathbf{a}\oplus(\ominus\mathbf{a}\oplus\mathbf{b})\otimes t_0 \tag{66}$$

for some $0 < t_0 < 1$. It satisfies the *gyrotriangle equality*

$$\|\ominus\mathbf{a}\oplus\mathbf{p}\|\oplus\|\ominus\mathbf{p}\oplus\mathbf{b}\| = \|\ominus\mathbf{a}\oplus\mathbf{b}\| \tag{67}$$

as remarked below the gyrotriangle inequality (75). Furthermore, the point

$$m_{\mathbf{a},\mathbf{b}} = \mathbf{a}\oplus(\ominus\mathbf{a}\oplus\mathbf{b})\otimes\tfrac{1}{2} \tag{68}$$

is the midpoint between $\mathbf{a}$ and $\mathbf{b}$, Fig. 1, satisfying $m_{\mathbf{a},\mathbf{b}} = m_{\mathbf{b},\mathbf{a}}$ and

$$\|\mathbf{a}\ominus m_{\mathbf{a},\mathbf{b}}\| = \|\mathbf{b}\ominus m_{\mathbf{a},\mathbf{b}}\| \tag{69}$$

Similarly,

(2) The cogyroline $L_{\boxminus}(t)$ in (65b), Fig. 2, passes through the point $\mathbf{a}$ at "time" $t = 0$ and, owing to the right cancellation law, (29a), it passes through the point $\mathbf{b}$ at "time" $t = 1$. It is the need to employ the right cancellation law that dictates the necessity to replace the term $\ominus\mathbf{a}\oplus\mathbf{b}$ in (65a) by the term $\boxminus\mathbf{a} \boxplus \mathbf{b} = \mathbf{b} \boxminus \mathbf{a}$ in (65b). The replacement of the term $\ominus\mathbf{a}\oplus\mathbf{b}$ in (65a) by the term $\boxminus\mathbf{a} \boxplus \mathbf{b} = \mathbf{b} \boxminus \mathbf{a}$ in (65b) is thus a matter of necessity rather than choice.

A point $\mathbf{p}$ is *cobetween* $\mathbf{a}$ and $\mathbf{b}$ if

$$\mathbf{p} = (\mathbf{b} \boxminus \mathbf{a})\otimes t_0\oplus\mathbf{a} \tag{70}$$

for some $0 < t_0 < 1$. It satisfies the *cogyrotriangle equality*

$$\| \boxminus \mathbf{a} \boxplus \mathbf{p}\| \boxplus \| \boxminus \mathbf{p} \boxplus \mathbf{b}\| = \| \boxminus \mathbf{a} \boxplus \mathbf{b}\| \tag{71}$$

as remarked below the cogyrotriangle inequality (76). Furthermore, the point

$$m'_{\mathbf{a},\mathbf{b}} = (\mathbf{b} \boxminus \mathbf{a})\otimes\tfrac{1}{2}\oplus\mathbf{a} \tag{72}$$

is the *comidpoint* between $\mathbf{a}$ and $\mathbf{b}$, Fig. 2, satisfying $m'_{\mathbf{a},\mathbf{b}} = m'_{\mathbf{b},\mathbf{a}}$ and

$$\|\mathbf{a} \boxminus m'_{\mathbf{a},\mathbf{b}}\| = \|\mathbf{b} \boxminus m'_{\mathbf{a},\mathbf{b}}\| \tag{73}$$

The Einstein gyrovector space $(\mathbb{V}_c, \oplus, \otimes)$ is bimetric, having the two mutually dual metrics $d_{\ominus}(\mathbf{a}, \mathbf{b})$ and $d_{\boxminus}(\mathbf{a}, \mathbf{b})$ called, respectively, the gyrometric and the cogyrometric of the space. The Euclidean metric $d_{-}(\mathbf{a}, \mathbf{b})$ of a real inner product space $\mathbb{V}$ admits the triangle inequality

$$d_{-}(\mathbf{a}, \mathbf{p}) + d_{-}(\mathbf{p}, \mathbf{b}) \geq d_{-}(\mathbf{a}, \mathbf{b}) \tag{74}$$

for any three points $\mathbf{a}, \mathbf{p}, \mathbf{b} \in \mathbb{V}$, where equality holds if and only if the point $\mathbf{p}$ lies on the Euclidean geodesic segment that links $\mathbf{a}$ to $\mathbf{b}$.

In full analogy, the gyrometric $d_\ominus(\mathbf{a},\mathbf{b})$ admits the gyrotriangle inequality

$$d_\ominus(\mathbf{a},\mathbf{p})\oplus d_\ominus(\mathbf{p},\mathbf{b}) \geq d_\ominus(\mathbf{a},\mathbf{b}) \tag{75}$$

for any three points $\mathbf{a},\mathbf{p},\mathbf{b} \in \mathbb{V}_c$, where equality holds if and only if the point $\mathbf{p}$ lies on the *gyrogeodesic* segment that links $\mathbf{a}$ to $\mathbf{b}$, Fig. 1.

Moreover, the cogyrometric $d_\boxminus(\mathbf{a},\mathbf{b})$ admits the cogyrotriangle inequality

$$d_\boxminus(\mathbf{a},\mathbf{p}) \boxplus d_\boxminus(\mathbf{p},\mathbf{b}) \geq d_\boxminus(\mathbf{a},\mathrm{gyr}[\mathbf{a}\boxminus\mathbf{p},\mathbf{p}\boxminus\mathbf{b}]\mathbf{b}) \tag{76}$$

for any three points $\mathbf{a},\mathbf{p},\mathbf{b} \in \mathbb{V}_c$, where equality holds if and only if the point $\mathbf{p}$ lies on the *cogyrogeodesic* segment that links $\mathbf{a}$ to $\mathbf{b}$, Fig. 2.

It can be shown that the gyrometric with its gyrotriangle inequality (75) is equivalent to a classical metric that admits a classical triangle inequality [62]. This is, however, not the case with the cogyrometric. Accordingly, gyrogeodesics are geodesics in the classical sense of differential geometry, while cogyrogeodesics are not geodesics in the classical sense. Furthermore, it can be shown that geodesics relative to the cogyrometric in the classical sense are incomplete, that is, they do not exist for all "time" $t$. In contrast, the cogyrogeodesics are complete, being valid for all "time" $t \in \mathbb{R}$. Thus, the gyro-correction that appears in the cogyrotriangle inequality (76) is one of the various manifestations of the way nonassociative algebra gyro-corrects itself to achieve perfection.

The Einstein ball gyrovector spaces $(\mathbb{V}_c,\oplus,\otimes)$ form the setting for the Beltrami (also known as the Klein) ball model of hyperbolic geometry, where geodesics in the ball are Euclidean lines [30]. Furthermore, the Einstein ball gyrovector spaces admit hyperbolic trigonometry fully analogous to Euclidean trigonometry, as shown in [6] and [63].

The gyrolines (65a) are the common hyperbolic lines, also called geodesics, in the Beltrami ball model of hyperbolic geometry. The analogies they share with Euclidean lines, (64), suggest the unification of hyperbolic and Euclidean geometry into a super geometry, called gyrogeometry [18]. Euclidean geometry, then, emerges in gyrogeometry as a special, degenerate case corresponding to vanishing gyrations. Further analogies that manifest the unification are provided by the hyperbolic angle. Accordingly, the hyperbolic angle in the Beltrami ball model of hyperbolic geometry is presented in Section 8.

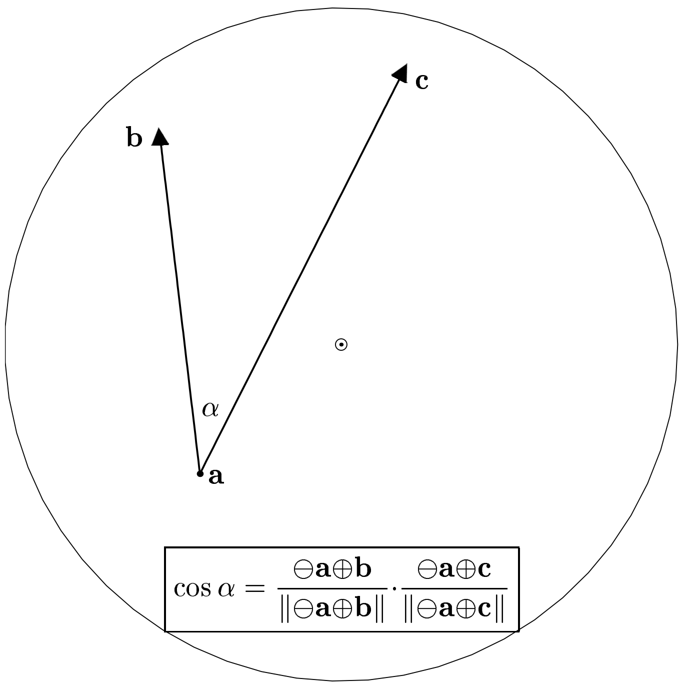


*Figure 3.* The hyperbolic angle $\alpha$ between two gyrovectors rooted at a point $\mathbf{a}$ in an Einstein gyrovector space $(\mathbb{V}_c, \oplus, \otimes)$. The Einstein gyrovector plane $(\mathbb{R}^2_c, \oplus, \otimes)$, shown in the Figure, coincides with the Beltrami (also known as the Klein) disc model of hyperbolic geometry. The traditional way of studying hyperbolic angles in this model is presented on pp. 177 – 180 of [64].

# 8 ANGLES IN EINSTEIN GYROVECTOR SPACES

Einstein's velocity addition admits scalar multiplication. This, in turn, gives rise to geodesics and cogeodesics in the resulting Einstein gyrovector space of relativistically admissible velocities. The geodesics and the cogeodesics are pleasingly expressed in (65) in a form fully analogous to their Euclidean counterpart (64). Having geodesics and cogeodesics in hand, we now introduce *rooted gyrovectors* and *rooted cogyrovectors*. As in Euclidean geometry, a rooted gyrovector $\ominus\mathbf{a}\oplus\mathbf{b}$, rooted at a point $\mathbf{a}$ in an Einstein gyrovector space $(\mathbb{V}_c, \oplus, \otimes)$ is represented graphically, when $\mathbb{V}_c = \mathbb{R}^3_c$, by a directed geodesic segment from the point $\mathbf{a}$ to the point $\mathbf{b}$ of $\mathbb{R}^3_c$. Accordingly, the two rooted gyrovectors $\ominus\mathbf{a}\oplus\mathbf{b}$ and $\ominus\mathbf{a}'\oplus\mathbf{b}'$ that satisfy the equation

$$\ominus\mathbf{a}\oplus\mathbf{b} = \ominus\mathbf{a}'\oplus\mathbf{b}' \tag{77}$$

in an Einstein gyrovector space $(\mathbb{V}_c, \oplus, \otimes)$ are distinct if $\mathbf{a} \neq \mathbf{a}'$ since they are rooted at different points. To reconcile geometry with algebra, two rooted

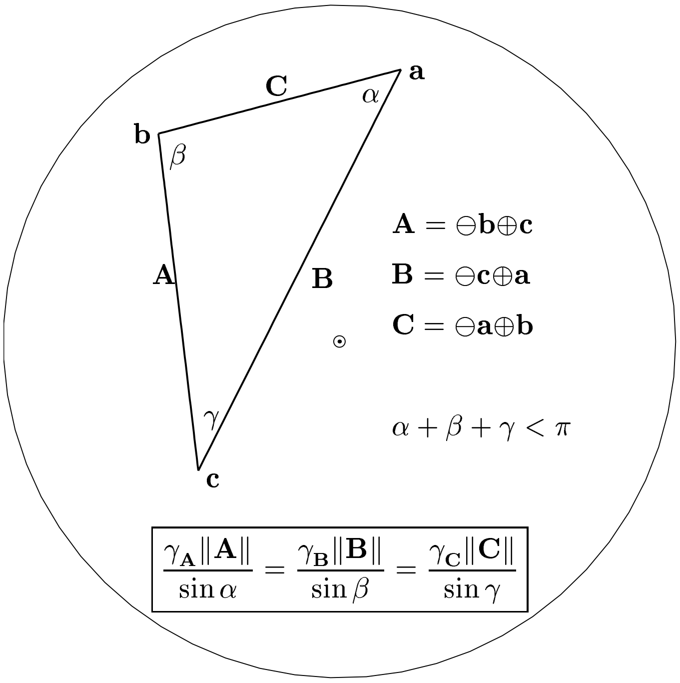


*Figure 4.* The hyperbolic triangle, called a gyrotriangle, $\Delta\mathbf{abc}$ in the Einstein disc gyrovector space or, equivalently, in the Beltrami disc model of hyperbolic geometry. Its vertices are the points $\mathbf{a}, \mathbf{b}, \mathbf{c} \in (\mathbb{R}^2_c, \oplus, \otimes)$ of the Einstein disc gyrovector space, and its sides are geodesic segments which, if directed counterclockwise, are represented by the rooted gyrovectors $\mathbf{A} = \ominus\mathbf{b}\oplus\mathbf{c}$, $\mathbf{B} = \ominus\mathbf{c}\oplus\mathbf{a}$, and $\mathbf{C} = \ominus\mathbf{a}\oplus\mathbf{b}$. The hyperbolic law of sines is valid in any gyrovector space, and is analogous to its Euclidean counterpart.

gyrovectors rooted at different points of their gyrovector space but satisfying (77) are said to be equivalent. The resulting equivalence relation is analogous to the equivalence relation between Euclidean rooted vectors according to which two Euclidean rooted vectors are equivalent if and only if they are parallel and have equal lengths. Similar remarks apply to rooted cogyrovectors as well [6]. Two gyrovectors are shown in Fig. 3, and two cogyrovectors are shown in Fig. 5.

We now wish to study the angles and coangles between rooted gyrovectors and cogyrovectors. Surprisingly, the hyperbolic angle can be recast into a form fully analogous to its Euclidean counterpart, as we present it in the following definition of the hyperbolic angle in the Beltrami ball model of hyperbolic geometry.

**Definition 3 (The Hyperbolic, Einstein Angle).** *Let* $\ominus\mathbf{a}\oplus\mathbf{b}$ *and* $\ominus\mathbf{a}\oplus\mathbf{c}$ *be two rooted gyrovectors emanating from a common point* $\mathbf{a}$ *in an Einstein gyrovector* *space* $(\mathbb{V}_c, \oplus, \otimes)$. *The cosine of the hyperbolic angle* $\alpha$ *that the two rooted gyrovectors generate, Fig. 3, is given by the equation*

$$\cos\alpha = \frac{\ominus\mathbf{a}\oplus\mathbf{b}}{\|\ominus\mathbf{a}\oplus\mathbf{b}\|}\cdot\frac{\ominus\mathbf{a}\oplus\mathbf{c}}{\|\ominus\mathbf{a}\oplus\mathbf{c}\|} \tag{78}$$

As it is well known in hyperbolic geometry [65], the triangle angle sum $\theta = \alpha + \beta + \gamma$, Fig. 4, of a hyperbolic triangle is smaller than $\pi$, the difference $\pi - \theta$ being known as the triangle defect. The hyperbolic triangle defect is important in relativity physics where it gives rise to the Thomas precession [66] [6] and other geometric phase shifts, like Berry's phase [67].

The hyperbolic angle allows hyperbolic trigonometry fully analogous to that of Euclidean geometry [63] [68] [6]. As an example, we present the hyperbolic law of sines in Einstein gyrovector spaces in the following Theorem [6], Fig. 4.

**Theorem 4 (Hyperbolic Law of Sines in Einstein Gyrovector Spaces).** *Let* $\Delta\mathbf{abc}$ *be a triangle in an Einstein gyrovector space* $(\mathbb{V}_c, \oplus, \otimes)$ *with vertices* $\mathbf{a}$*,* $\mathbf{b}$ *and* $\mathbf{c}$ *and with sides* $\mathbf{A}$*,* $\mathbf{B}$ *and* $\mathbf{C}$ *given by*

$$\begin{aligned} \mathbf{A} &= \ominus\mathbf{b}\oplus\mathbf{c} \\ \mathbf{B} &= \ominus\mathbf{c}\oplus\mathbf{a} \\ \mathbf{C} &= \ominus\mathbf{a}\oplus\mathbf{b} \end{aligned} \tag{79}$$

*and let* $\alpha$*,* $\beta$ *and* $\gamma$ *be the respective angles opposite to these sides. Then*

$$\frac{\gamma_{\mathbf{A}}\|\mathbf{A}\|}{\sin\alpha} = \frac{\gamma_{\mathbf{B}}\|\mathbf{B}\|}{\sin\beta} = \frac{\gamma_{\mathbf{C}}\|\mathbf{C}\|}{\sin\gamma} \tag{80}$$

*where* $\gamma_{\mathbf{A}}$*,* $\gamma_{\mathbf{B}}$ *and* $\gamma_{\mathbf{C}}$ *are the Lorentz factors of* $\mathbf{A}$*,* $\mathbf{B}$ *and* $\mathbf{C}$*.*

Some of the duality symmetries that the Einstein dual additions $\oplus$ and $\boxplus$ introduce are seen in (29) and (30). Naturally, these suggest the exploration of the coangles, the dual hyperbolic angles. Dualizing the hyperbolic angle definition we thus have the following definition.

**Definition 5 (Hyperbolic Dual Angles – The Coangles).** *The measure of the hyperbolic dual angle* $\alpha$*, called the coangle* $\alpha$*, between two rooted cogyrovectors* $\mathbf{b} \boxminus \mathbf{a}$ *and* $\mathbf{d} \boxminus \mathbf{c}$ *in a gyrovector space* $(\mathbb{V}_c, \oplus, \otimes)$*, Fig. 5, is given by the equation*

$$\cos\,\alpha = \frac{\boxminus\mathbf{a}\boxplus\mathbf{b}}{\|\boxminus\mathbf{a}\boxplus\mathbf{b}\|}\cdot\frac{\boxminus\mathbf{c}\boxplus\mathbf{d}}{\|\boxminus\mathbf{c}\boxplus\mathbf{d}\|} \tag{81}$$

Surprisingly, the triangle coangles behave like Euclidean angles in the sense of the following hyperbolic $\pi$ – Theorem.

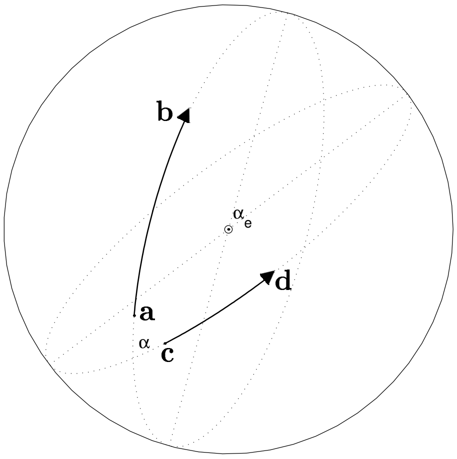


*Figure 5.* The hyperbolic coangle $\alpha$ generated by the two rooted cogyrovectors $\mathbf{b} \boxminus \mathbf{a}$ and $\mathbf{d} \boxminus \mathbf{c}$ in the Einstein disc gyrovector space $(\mathbb{R}^2_c, \oplus, \otimes)$ is shown. Its cosine is given by (81), and is numerically equal to the Euclidean angle $\alpha_e$ generated by the corresponding supporting diameters.

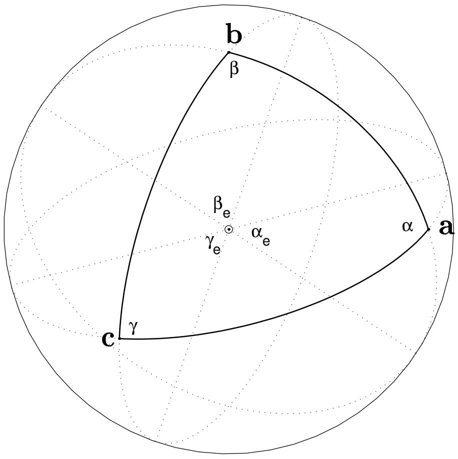


*Figure 6.* The hyperbolic coangle between cogeodesics and its associated Euclidean angle between corresponding supporting diameters are equal. Hence, by inspection, the sum of the hyperbolic coangles of a hyperbolic cotriangle is $\pi$. $\alpha = \alpha_e$, $\beta = \beta_e$, $\gamma = \gamma_e$ and, hence, $\alpha + \beta + \gamma = \pi$.

**Theorem 6 (The Hyperbolic $\pi$ – Theorem).** *Let $\Delta\mathbf{abc}$ be the hyperbolic cotriangle in a gyrovector space $(\mathbb{V}_c, \oplus, \otimes)$, the three vertices of which are the points $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{V}_c$. Then its three coangles $\alpha$, $\beta$, and $\gamma$, Fig. 6, given by*

$$\begin{aligned} \cos\,\alpha &= \frac{\mathbf{b} \boxminus \mathbf{a}}{\|\mathbf{b} \boxminus \mathbf{a}\|} \cdot \frac{\mathbf{c} \boxminus \mathbf{a}}{\|\mathbf{c} \boxminus \mathbf{a}\|} \\ \cos\,\beta &= \frac{\mathbf{a} \boxminus \mathbf{b}}{\|\mathbf{a} \boxminus \mathbf{b}\|} \cdot \frac{\mathbf{c} \boxminus \mathbf{b}}{\|\mathbf{c} \boxminus \mathbf{b}\|} \\ \cos\,\gamma &= \frac{\mathbf{a} \boxminus \mathbf{c}}{\|\mathbf{a} \boxminus \mathbf{c}\|} \cdot \frac{\mathbf{b} \boxminus \mathbf{c}}{\|\mathbf{b} \boxminus \mathbf{c}\|} \end{aligned} \tag{82}$$

*satisfy*

$$\alpha + \beta + \gamma = \pi \tag{83}$$

The definition of the Einstein coangle, and the hyperbolic $\pi$ – Theorem are illustrated graphically in Figs. 5 and 6.

Another pleasing result is presented in the following theorem, revealing that the duality symmetry shared by gyrations and cogyration, encountered in (30), is only the tip of a giant iceberg.

**Theorem 7 (The Gyration/Cogyration Gyrotransitive Laws).** *Let $L_{\ominus}(t)$ and $L_{\boxminus}(t)$ be a gyroline and a cogyroline,* (65)*, Figs. 1 and 2, in a gyrovector space $(\mathbb{V}_c, \oplus, \otimes)$. Let $\{\mathbf{a}_1, \cdots, \mathbf{a}_n\}$ be a set of $n$ co-gyrollinear points lying on the gyroline $L_{\ominus}(t)$ (in any order), and let $\{\mathbf{b}_1, \cdots, \mathbf{b}_n\}$ be a set of $n$ co-cogyrollinear points lying on the cogyroline $L_{\boxminus}(t)$ (in any order). Then we have the gyration gyrotransitive law for gyrolines,*

$$\mathrm{gyr}[\mathbf{a}_1, \ominus\mathbf{a}_2]\mathrm{gyr}[\mathbf{a}_2, \ominus\mathbf{a}_3] \cdots \mathrm{gyr}[\mathbf{a}_{n-1}, \ominus\mathbf{a}_n] = \mathrm{gyr}[\mathbf{a}_1, \ominus\mathbf{a}_n] \tag{84}$$

*and the cogyration gyrotransitive law for cogyrolines,*

$$\mathrm{gyr}[\mathbf{b}_1, \mathbf{b}_2]\mathrm{gyr}[\mathbf{b}_2, \mathbf{b}_3] \cdots \mathrm{gyr}[\mathbf{b}_{n-1}, \mathbf{b}_n] = \mathrm{gyr}[\mathbf{b}_1, \mathbf{b}_n] \tag{85}$$

The proof of Theorems 4, 6 and 7 is found in [6].

# 9 FUNCTION THEORY IN THE UNIT BALL

In the non-Euclidean style of the study of relativity physics it is the Einstein addition, rather than the Minkowski space-time, which is emphasized as a primitive notion. The non-Euclidean style uncovers in special relativity links (i) with the theory of functions in the ball [17], as we show in this section, and (ii) with the geometric viewpoint of complex analysis, as we show in Section 10.

Einstein's addition (2) can readily be complexified into the complex Einstein addition $\oplus_{\text{EC}}$ in the complex ball $\mathbb{V}_c$,

$$\mathbb{V}_c = \{\mathbf{v} \in \mathbb{V} : \|\mathbf{v}\| < c\} \tag{86}$$

of a complex inner product space $\mathbb{V}$. The inner product $\langle \mathbf{u}, \mathbf{v} \rangle$ in $\mathbb{V}$ is denoted by $\mathbf{u}{\cdot}\bar{\mathbf{v}}$ or $\bar{\mathbf{v}}{\cdot}\mathbf{u}$ (we make no attempt to define $\bar{\mathbf{v}}$ alone).

The complex Einstein velocity addition is given by the equation

$$\mathbf{u} \oplus_{\text{EC}} \mathbf{v} = \frac{1}{1 + \frac{\bar{\mathbf{u}}\cdot\mathbf{v}}{c^2}} \left\{ \mathbf{u} + \frac{1}{\gamma_{\mathbf{u}}}\mathbf{v} + \frac{1}{c^2}\frac{\gamma_{\mathbf{u}}}{1+\gamma_{\mathbf{u}}}(\bar{\mathbf{u}}\cdot\mathbf{v})\mathbf{u} \right\} \tag{87}$$

$\gamma_{\mathbf{u}}$ being the Lorentz factor

$$\gamma_{\mathbf{u}} = \frac{1}{\sqrt{1 - \frac{\bar{\mathbf{u}}\cdot\mathbf{u}}{c^2}}} = \frac{1}{\sqrt{1 - \frac{\|\mathbf{u}\|^2}{c^2}}} \tag{88}$$

satisfying the gamma identity

$$\gamma_{\mathbf{u}\oplus_{\text{EC}}\mathbf{v}} = \gamma_{\mathbf{u}}\gamma_{\mathbf{v}} \left| 1 + \frac{\bar{\mathbf{u}}\cdot\mathbf{v}}{c^2} \right| \tag{89}$$

for all $\mathbf{u}, \mathbf{v} \in \mathbb{V}_c$. The groupoid $(\mathbb{V}_c, \oplus_{\text{EC}})$ is a gyrocommutative gyrogroup called the Einstein complex gyrogroup [6].

In the same way that Einstein's addition plays an important role in the study of real spacetime, the complex Einstein addition plays an important role in the study of complex spacetime. The latter, in turn, is useful in some branches of physics as evidenced, for instance, from [69] [70] [71]. Also, the complex Einstein addition may play a novel, central role in complex hyperbolic geometry [72] as well.

Owing to the definition of the Lorentz factor, (88), the complex Einstein addition (87) can be written in the form

$$\mathbf{a} \oplus_{\text{EC}} \mathbf{z} = \frac{1}{1 + \bar{\mathbf{a}}\cdot\mathbf{z}} \left\{ \left[ 1 + \frac{\bar{\mathbf{a}}\cdot\mathbf{z}}{\|\mathbf{a}\|^2}(1 - \sqrt{1 - \|\mathbf{a}\|^2}) \right] \mathbf{a} + \sqrt{1 - \|\mathbf{a}\|^2}\mathbf{z} \right\} \tag{90}$$

Realizing the abstract complex inner product space $\mathbb{V}$ by the $n$-dimensional complex vector space $\mathbb{C}^n$, the right hand side of (90) is recognized, from p. 25

in [17], as the application $\phi_{\mathbf{a}}(-\mathbf{z})$ of the so called Automorphism $\phi_{\mathbf{a}}$ of the ball $\mathbb{B}$ of $\mathbb{C}^n$ to $-\mathbf{z}$, where $\mathbf{a}, \mathbf{z} \in \mathbb{B}$.

Thus, it follows from p. 25 of [17] that (90) can be written equivalently as

$$\phi_{\mathbf{a}}(-\mathbf{z}) = \mathbf{a} \oplus_{\text{EC}} \mathbf{z} \tag{91}$$

so that

$$\phi_{\mathbf{a}}(\mathbf{z}) = \mathbf{a} \ominus_{\text{EC}} \mathbf{z} \tag{92}$$

where $\phi_{\mathbf{a}}$, $\mathbf{a} \in \mathbb{B}$, are the automorphisms of the ball $\mathbb{B}$ studied in [17]. The automorphisms $\phi_{\mathbf{a}}$ play a central role in function theory in the unit ball of $\mathbb{C}^n$ [17]. Interestingly, however, the relationship (92) between the complex Einstein addition $\oplus_{\text{EC}}$ and the automorphisms $\phi_{\mathbf{a}}$, $\mathbf{a} \in \mathbb{B}$, has gone unnoticed in the literature. Recognizing the automorphisms $\phi_{\mathbf{a}}$ of the ball $\mathbb{B} = \mathbb{V}_c$ as left gyrotranslations of the gyrocommutative gyrogroup $(\mathbb{V}_c, \oplus_{\text{EC}})$ allows the introduction of gyrogroup-theoretic techniques into the study of function theory in the ball, and allows the introduction into relativity theory concepts that have been successfully developed in function theory in the ball, as well.

The relationship (92) between the complex Einstein addition and the automorphisms $\phi_{\mathbf{a}}$ of the ball implies readily that these automorphisms $\phi_{\mathbf{a}}$, $\mathbf{a} \in \mathbb{B}$, are *involutive*, that is,

$$\begin{aligned} \phi_{\mathbf{a}}^2(\mathbf{z}) &= \phi_{\mathbf{a}}(\phi_{\mathbf{a}}(\mathbf{z})) \\ &= \phi_{\mathbf{a}}(\mathbf{a} \ominus_{\text{EC}} \mathbf{z}) \\ &= \mathbf{a} \ominus_{\text{EC}} (\mathbf{a} \ominus_{\text{EC}} \mathbf{z}) \\ &= \mathbf{a} \oplus_{\text{EC}} (\ominus_{\text{EC}} \mathbf{a} \oplus_{\text{EC}} \mathbf{z}) \\ &= (\mathbf{a} \ominus_{\text{EC}} \mathbf{a}) \oplus_{\text{EC}} \text{gyr}[\mathbf{a}, \ominus_{\text{EC}} \mathbf{a}] \mathbf{z} \\ &= \mathbf{z} \end{aligned} \tag{93}$$

for all $\mathbf{z} \in \mathbb{B}$, since $\mathbf{a} \ominus_{\text{EC}} \mathbf{a} = \mathbf{0}$ and $\text{gyr}[\mathbf{a}, \ominus_{\text{EC}} \mathbf{a}] = id$ is the identity automorphism. We recall that a non-identity automorphism is *involutive* if its square automorphism is the identity automorphism.

Rudin mentions on p. 30 of his book [17] that the automorphisms $\phi_{\mathbf{a}}$ of $\mathbb{B}$ in a two dimensional complex inner product space $\mathbb{C}^2$ have already been studied by Poincaré in 1907. He however emphasizes that he does not know when they were first written down for $n > 2$ dimensions. Being now identified as the Einstein complex relativistically admissible velocity addition, it seems that it was Einstein's 1905 paper [73] that triggered interest in the automorphisms $\phi_{\mathbf{a}}$ of the unit ball $\mathbb{B}$ of $\mathbb{C}^n$.

Owing to the central role that the automorphisms $\phi_{\mathbf{a}}$ play in the function theory in the unit ball of $\mathbb{C}^n$, this theory may be considered as a topic in the theory of gyrocommutative gyrogroups and gyrovector spaces. Function theory in the unit ball of $\mathbb{C}^n$ is, thus, intimately linked to relativity physics by the identity in (92).

Following the success of Minkowski's spacetime formalism, the general Einstein velocity addition did not find its way into the mainstream literature on relativity physics; only several outstanding exceptions are available. Ironically, however, we now realize that under the disguise of the automorphisms $\phi_{\mathbf{a}}$ of the ball, (92), the complexified general Einstein addition $\oplus_{\text{EC}}$ did find its way into the mainstream literature on function theory in the unit ball of $\mathbb{C}^n$ [17], where it plays a significant role.

## 10 COMPLEX ANALYSIS: THE GEOMETRIC VIEWPOINT

Einstein's addition $\oplus$ in the Einstein ball gyrovector space $\mathbb{V}_c = (\mathbb{V}_c, \oplus, \otimes)$ of a real inner product space $\mathbb{V}$ gives rise to another important binary operation in $\mathbb{V}_c$, known as the Möbius addition $\oplus_{\mathrm{M}}$, according to the equation

$$\begin{aligned}\mathbf{u}\oplus_{\mathrm{M}}\mathbf{v} &= \tfrac{1}{2}\otimes(2\otimes\mathbf{u}\oplus 2\otimes\mathbf{v}) \\ &= \frac{(1+\frac{2}{c^2}\mathbf{u}\cdot\mathbf{v}+\frac{1}{c^2}\|\mathbf{v}\|^2)\mathbf{u}+(1-\frac{1}{c^2}\|\mathbf{u}\|^2)\mathbf{v}}{1+\frac{2}{c^2}\mathbf{u}\cdot\mathbf{v}+\frac{1}{c^4}\|\mathbf{u}\|^2\|\mathbf{v}\|^2}\end{aligned} \tag{94}$$

where $\cdot$ and $\|\cdot\|$ are the inner product and the norm that the ball $\mathbb{V}_c$ inherits from its space $\mathbb{V}$.

Möbius addition satisfies the gamma identity

$$\gamma_{\mathbf{u}\oplus_{\mathrm{M}}\mathbf{v}} = \gamma_{\mathbf{u}}\gamma_{\mathbf{v}}\sqrt{1+\frac{2}{c^2}\mathbf{u}\cdot\mathbf{v}+\frac{1}{c^4}\|\mathbf{u}\|^2\|\mathbf{v}\|^2} \tag{95}$$

where $\gamma_{\mathbf{u}}$ is the Lorentz factor $\gamma_{\mathbf{u}} = (1-\|\mathbf{u}\|^2/c^2)^{-1/2}$.

The resulting triple $(\mathbb{V}_c, \oplus_{\mathrm{M}}, \otimes)$ is a gyrovector space called the Möbius gyrovector space of the ball $\mathbb{V}_c$ which, by means of (94), is isomorphic to (that is, algebraically indistinguishable from) the Einstein gyrovector space $(\mathbb{V}_c, \oplus, \otimes)$. Indeed, from the point of view of geometry, the gyrovector spaces of Einstein and of Möbius form the setting for two distinct models of hyperbolic geometry. These are the ball model of Beltrami and the ball model of Poincaré.

To justify calling $\oplus_{\mathrm{M}}$ the Möbius addition in $\mathbb{V}_c$, we present the Möbius transformation of the unit open disc $\mathbb{D}$ of a complex plane $\mathbb{C}$,

$$\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\} \tag{96}$$

and the binary operation in the disc to which it gives rise. The Möbius transformation proves useful in string theory, and it plays an important role in the geometric viewpoint of complex analysis [19] [74]. It is, thus, interesting to realize that in a classical topic such as the Möbius transformation of the disc $\mathbb{D}$ there can be found remarkable innovations that stem from recent discoveries about the Einstein velocity addition in relativity physics.

The most general Möbius transformation of the complex open unit disc $\mathbb{D}$ [75][76][77][78]

$$z \mapsto e^{i\theta}\frac{z_0+z}{1+\bar{z}_0 z} = e^{i\theta}(z_0\oplus_{\mathrm{M}} z) \tag{97}$$

$z_0 \in \mathbb{D}$, $\theta \in \mathbb{R}$, is a source of inspiration in complex analysis [74] and geometry [19] [79]. Suggestively, it inspires us to define the Möbius addition $\oplus_{\mathrm{M}}$ in the disc, enabling the generic Möbius transformation (97) of the disc to be viewed as a Möbius left gyrotranslation

$$z \mapsto z_0 \oplus_{\text{M}} z = \frac{z_0 + z}{1 + \bar{z}_0 z} \tag{98}$$

followed by a rotation. The term "gyrotranslation" emphasizes the result that the Möbius addition $\oplus_{\text{M}}$ that we define in (98) is a (gyrocommutative) gyrogroup operation [30]. It conforms with our gyrolanguage in which we use the prefix "gyro" to emphasize analogies with classical results. We will show in (103) below that the binary operation $\oplus_{\text{M}}$ in $\mathbb{D}$, given by (98), is a special case of the binary operation $\oplus_{\text{M}}$ in $\mathbb{V}_c$, given by (94). This will clearly justify the use of the term "Mönius addition" to describe the binary operation $\oplus_{\text{M}}$ in $\mathbb{V}_c$ that we defined in Equation (94).

The Poincaré hyperbolic distance function $d(a, b)$ in $\mathbb{D}$ is given by the equation [77]

$$d(a, b) = \left| \frac{a - b}{1 - \bar{a}b} \right| = |a \ominus_{\text{M}} b| \tag{99}$$

$a, b \in \mathbb{D}$, where we abbreviate $a \ominus_{\text{M}} b = a \oplus_{\text{M}} (-b)$. It satisfies the Möbius triangle inequality

$$d(a, c) \le d(a, b) \oplus_{\text{M}} d(b, c) \tag{100}$$

that involves the Möbius addition $\oplus_{\text{M}}$ of two real numbers in the disc $\mathbb{D}$ [62].

We will now justify calling $\oplus_{\text{M}}$ in (94) a Möbius addition by demonstrating its identity relation with $\oplus_{\text{M}}$ in (98) in the two-dimensional special case.

Identifying vectors in the Euclidean plane $\mathbb{R}^2$ with complex numbers in the usual way we have

$$\mathbf{u} = (u_1, u_2) = u_1 + iu_2 = u \tag{101}$$

The inner product and the norm in $\mathbb{R}^2$ then become the real numbers

$$\begin{aligned} \mathbf{u}\cdot\mathbf{v} &= \mathrm{Re}(\bar{u}v) = \frac{\bar{u}v + u\bar{v}}{2} \\ \|\mathbf{u}\| &= |u| \end{aligned} \tag{102}$$

where $\bar{u}$ is the complex conjugate of $u$, and $|u|$ is the modulus of $u$.

Under the translation (101) – (102) of elements of the disc $\mathbb{R}^2_{c=1}$ of the Euclidean plane $\mathbb{R}^2$ to elements of the complex unit disc $\mathbb{D}$, the Möbius addition

(94), with $c = 1$ for simplicity, takes the form

$$\begin{aligned}
\mathbf{u} \oplus_{\mathrm{M}} \mathbf{v} &= \frac{(1 + 2\mathbf{u}\cdot\mathbf{v} + \|\mathbf{v}\|^2)\mathbf{u} + (1 - \|\mathbf{u}\|^2)\mathbf{v}}{1 + 2\mathbf{u}\cdot\mathbf{v} + \|\mathbf{u}\|^2\|\mathbf{v}\|^2} \\
&= \frac{(1 + \bar{u}v + u\bar{v} + |v|^2)u + (1 - |u|^2)v}{1 + \bar{u}v + u\bar{v} + |u|^2|v|^2} \\
&= \frac{(1 + u\bar{v})(u + v)}{(1 + \bar{u}v)(1 + u\bar{v})} \\
&= \frac{u + v}{1 + \bar{u}v} \\
&= u \oplus_{\mathrm{M}} v
\end{aligned} \tag{103}$$

for all $\mathbf{u}, \mathbf{v} \in \mathbb{R}^2_{c=1}$, and all $u, v \in \mathbb{D}$. In (103) we thus recognize the Möbius addition $\oplus_{\mathrm{M}}$ in the open unit disc $\mathbb{D}$ of $\mathbb{C}$, (98), as a special case of the Möbius addition (94) in the ball $\mathbb{V}_c$.

The Möbius gyrovector spaces $(\mathbb{V}_c, \oplus_{\mathrm{M}}, \otimes)$ form the setting for the Poincaré ball model of hyperbolic geometry in the same way that the Einstein gyrovector spaces $(\mathbb{V}_c, \oplus, \otimes)$ form the setting for the Beltrami ball model of hyperbolic geometry, and in the same way that vector spaces form the setting for the Euclidean space model of Euclidean geometry. Thus, in particular, the Einstein gyrolines and cogyrolines (65) are translated to corresponding Möbius gyrolines and cogyrolines,

$$L_{\ominus_{\mathrm{M}}}(t) = \mathbf{a} \oplus_{\mathrm{M}} (\ominus_{\mathrm{M}} \mathbf{a} \oplus_{\mathrm{M}} \mathbf{b}) \otimes t \tag{104a}$$

$$L_{\boxminus_{\mathrm{M}}}(t) = (\mathbf{b} \boxminus_{\mathrm{M}} \mathbf{a}) \otimes t \oplus_{\mathrm{M}} \mathbf{a} \tag{104b}$$

$t \in \mathbb{R}$, shown in Figs. 7 and 8.

A Möbius gyrovector space $(\mathbb{V}_c, \oplus_{\mathrm{M}}, \otimes)$ is bimetric, possessing the gyrometric and the cogyrometric

$$d_{\ominus_{\mathrm{M}}}(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} \ominus_{\mathrm{M}} \mathbf{v}\| \tag{105a}$$

$$d_{\boxminus_{\mathrm{M}}}(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} \boxminus_{\mathrm{M}} \mathbf{v}\| \tag{105b}$$

$\mathbf{u}, \mathbf{v} \in \mathbb{V}_c$. In the special case of $\mathbb{V}_c = \mathbb{R}^2_{c=1}$, the gyrometric (105a) reduces to the well known Poincaré metric of the disc [77].

The Möbius gyrolines (104a) are geodesics relative to the Poincaré metric (105a), and the Möbius cogyrolines (104b) are cogeodesics relative to the Poincaré cometric (105b). The Möbius gyrolines and cogyrolines are shown in Figs. 7 and 8 for the open unit disc $\mathbb{V}_c = \mathbb{R}^2_{c=1}$. The Möbius gyroline in

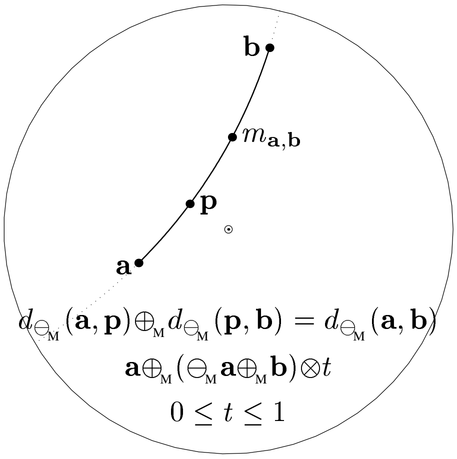


*Figure 7.* The gyroline segment, (104a), linking the two points $\mathbf{a}$ and $\mathbf{b}$ in a Möbius gyrovector space $(\mathbb{V}_c, \oplus_M, \otimes)$. $\mathbf{p}$ is a generic point between $\mathbf{a}$ and $\mathbf{b}$, (66), and $m_{\mathbf{a},\mathbf{b}}$ is the midpoint of $\mathbf{a}$ and $\mathbf{b}$, (68), where $\oplus = \oplus_M$ in (66) and (68).

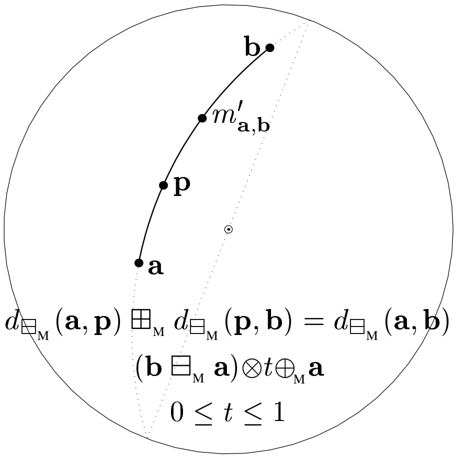


*Figure 8.* The cogyroline segment, (104b), linking the two points $\mathbf{a}$ and $\mathbf{b}$ in a Möbius gyrovector space $(\mathbb{V}_c, \oplus_M, \otimes)$. $\mathbf{p}$ is a generic point cobetween $\mathbf{a}$ and $\mathbf{b}$, (70), and $m'_{\mathbf{a},\mathbf{b}}$ is the comidpoint of $\mathbf{a}$ and $\mathbf{b}$, (72), where $\oplus = \oplus_M$ and $\boxplus = \boxplus_M$ in (70) and (72).

Fig. 7 is recognized as the well known geodesic of the Poincaré disc model of hyperbolic geometry [65]. It is a circular arc that intersects the boundary of the disc orthogonally.

Unlike the Beltrami ball model of hyperbolic geometry, governed by Einstein's gyrovector spaces, the Poincaré ball model of hyperbolic geometry, governed by Möbius gyrovector spaces, is conformal. Accordingly, the measure of hyperbolic angles between intersecting gyrolines is equal to the measure of Euclidean angles between corresponding intersecting tangent lines. It is therefore the Poincaré model of hyperbolic geometry, regulated by the Möbius addition, rather than the Beltrami model, regulated by the Einstein addition, that we select in Section 11 for the illustration of the use of hyperbolic geometry in terms of analogies with Euclidean geometry.

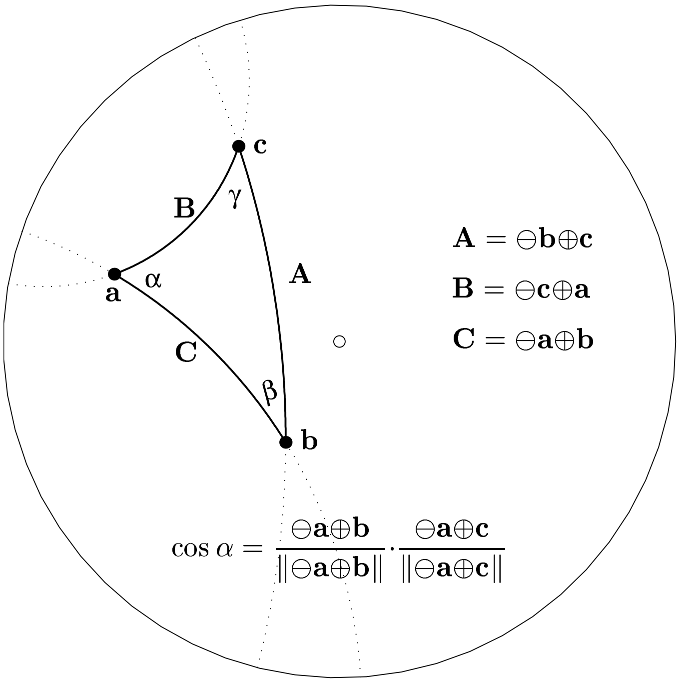


*Figure 9.* A Möbius triangle $\Delta\mathbf{abc}$ in the Möbius gyrovector plane $\mathbb{D} = (\mathbb{R}^2_c, \oplus, \otimes)$ is shown. Its sides are formed by geometric gyrovectors that link its vertices, in full analogy with Euclidean triangles. Its hyperbolic angles satisfy an identity which is analogous to its Euclidean counterpart. The Möbius gyrovector plane coincides with the Poincaré disc model of hyperbolic geometry.

## 11 ILLUSTRATING THE MÖBIUS GYROVECTOR SPACE

In this section the binary operation $\oplus$ always denotes the Möbius addition, $\oplus = \oplus_{\text{M}}$. To illustrate the elegant, powerful ability of the gyro-formalism to capture analogies

(1) We present graphically in Fig. 9 a hyperbolic triangle with its hyperbolic sides and hyperbolic angles in the Möbius gyrovector plane $(\mathbb{R}^2_{c=1}, \oplus, \otimes)$, that one may contrast with the hyperbolic triangle in the Einstein gyrovector plane in Fig. 4;

(2) We present graphically in Fig. 10 the hyperbolic Pythagorean Theorem in the Möbius gyrovector plane $(\mathbb{R}^2_{c=1}, \oplus, \otimes)$, studied in [62] and [6]; and

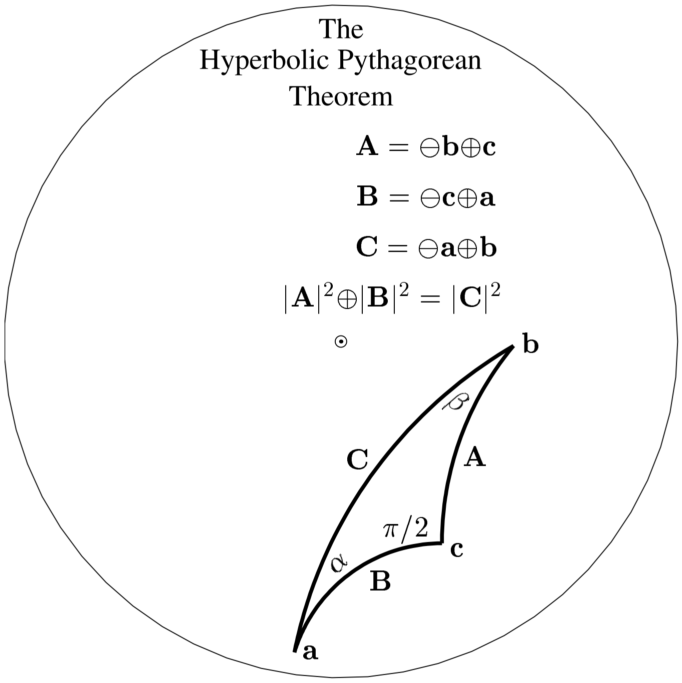


*Figure 10.* The hyperbolic Pythagorean theorem for Möbius right-angled hyperbolic triangles in the Möbius gyrovector plane $\mathbb{D} = (\mathbb{R}^2_{c=1}, \oplus, \otimes)$ in a form fully analogous to its Euclidean counterpart [62].

(3) We present graphically in Fig. 11 the nonassociative algebra of the hyperbolic parallel transport along geodesics in the Möbius gyrovector plane $(\mathbb{R}^2_c, \oplus, \otimes)$, studied in [6].

The gyro-formalism, some aspects of which are illustrated graphically in Figs. 9, 10 and 11 for the disc $\mathbb{R}^2_c$ of the Euclidean two-space $\mathbb{R}^2$, remains valid in the ball $\mathbb{V}_c$ of any real inner product space $\mathbb{V}$. Typically to the study of hyperbolic geometry as the geometry of gyrovector spaces, the hyperbolic geometry concepts shown in Figs. 9, 10 and 11 turn out to be fully analogous to their Euclidean counterparts. In contrast, prior to the emergence of gyrogroup and gyrovector space theory

(1) hyperbolic angles were calculated by methods of differential geometry rather than by methods of (nonassociative) algebra;

(2) the Hyperbolic Pythagorean Theorem appeared in the literature in a form which shares no geometrically visual analogies with its Euclidean counterpart; and

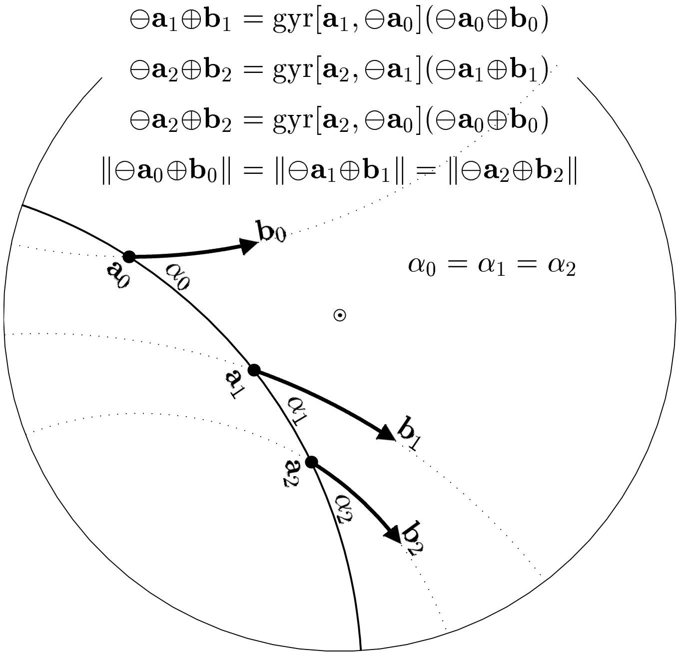


*Figure 11.* The Poincaré $n$-dimensional ball model of hyperbolic geometry turns out to be the $n$-dimensional Möbius gyrovector space. The latter, in turn, is isomorphic to the $n$-dimensional Einstein gyrovector space. In gyro-formalism the nonassociative algebra of the hyperbolic parallel transport of a geodesic segment $(\mathbf{a}_0, \mathbf{b}_0)$ to the geodesic segment $(\mathbf{a}_1, \mathbf{b}_1)$ and then, further, to the geodesic segment $(\mathbf{a}_2, \mathbf{b}_2)$ along the Möbius geodesic that links $\mathbf{a}_0$ and $\mathbf{a}_1$ and $\mathbf{a}_2$ in a Möbius gyrovector space $(\mathbb{V}_c, \oplus, \otimes)$ is fully analogous to the algebra of its Euclidean counterpart, as shown here and explained in [6]. The special case when $\mathbb{V}_c = \mathbb{R}^2_c$ is shown here graphically.

(3) parallel transport in classical hyperbolic geometry could only be achieved by methods of differential geometry rather than by methods of (nonassociative) algebra.

We may note that in pregyrogroup hyperbolic geometry one can find a theorem commonly known as the hyperbolic Pythagorean theorem, which is different from the one presented in Fig. 10. However, unlike the one in Fig. 10, the common hyperbolic Pythagorean theorem in the literature on hyperbolic geometry does not share visual analogies with the Euclidean Pythagorean theorem. As a result, some authors believed that a truly hyperbolic Pythagorean theorem does not exist. Thus, for instance, Wallace and West assert [80] “the Pythagorean theorem is strictly Euclidean” since “in the hyperbolic model the Pythagorean theorem is not valid!” Gyrovector space theory thus reveals that a truly hyperbolic Pythagorean theorem does exist. The hyperbolic Pythagorean theorem in Einstein gyrovector spaces is presented in [81].

The importance of parallel transport in relativity physics and quantum mechanics, and in their underlying hyperbolic geometry, is well known; see, for instance, [82] [83]. The definition of parallel transport in gyrovector spaces follows.

**Definition 8 (Parallel Transport Along Geodesics).** *Let* $\ominus\mathbf{a}_0\oplus\mathbf{b}_0$ *be a rooted gyrovector in a gyrovector space* $(\mathbb{V}_c, \oplus, \otimes)$ *with tail* $\mathbf{a}_0$ *and head* $\mathbf{b}_0$*, and let* $\mathbf{a}_1 \in \mathbb{V}_c$ *be a point of the space. The displacement of the gyrovector* $\ominus\mathbf{a}_0\oplus\mathbf{b}_0$ *to the gyrovector* $\ominus\mathbf{a}_1\oplus\mathbf{b}_1$*, Fig. 11, is said to be a parallel transport, or a parallel translation, of* $\ominus\mathbf{a}_0\oplus\mathbf{b}_0$ *to* $\ominus\mathbf{a}_1\oplus\mathbf{b}_1$ *along the geodesic that links* $\mathbf{a}_0$ *and* $\mathbf{a}_1$ *if* $\mathbf{b}_1$ *is given by*

$$\mathbf{b}_1 = \mathbf{a}_1\oplus\mathrm{gyr}[\mathbf{a}_1, \ominus\mathbf{a}_0](\ominus\mathbf{a}_0\oplus\mathbf{b}_0) \tag{106}$$

The following theorem, verified in [6], insures that the parallel transport in Definition 8 coincides with its counterpart in differential geometry.

**Theorem 9 (The Parallel Transport Theorem).** *Let* $\mathbf{a}_0, \mathbf{b}_0, \mathbf{a}_1 \in \mathbb{V}_c$ *be three given points of a gyrovector space* $(\mathbb{V}_c, \oplus, \otimes)$*, and let the gyrovector* $\ominus\mathbf{a}_0\oplus\mathbf{b}_0$ *be parallel transported to the gyrovector* $\ominus\mathbf{a}_1\oplus\mathbf{b}_1$ *along the geodesic that links the points* $\mathbf{a}_0$ *and* $\mathbf{a}_1$*, Fig. 11. Then*

$$\|\ominus\mathbf{a}_1\oplus\mathbf{b}_1\| = \|\ominus\mathbf{a}_0\oplus\mathbf{b}_0\| \tag{107}$$

*and*

$$\alpha_1 = \alpha_0 \tag{108}$$

*as shown in Fig. 11.*

The parallel transport of the rooted gyrovector $\ominus\mathbf{a}_0\oplus\mathbf{b}_0$, rooted at the point $\mathbf{a}_0$ to the rooted gyrovector $\ominus\mathbf{a}_1\oplus\mathbf{b}_1$, rooted at the point $\mathbf{a}_1$, where the points $\mathbf{a}_0$, $\mathbf{b}_0$, and $\mathbf{a}_1$, are given is accomplished by calculating the point $\mathbf{b}_1$ in Fig. 11 according to the equation, (106),

$$\mathbf{b}_1 = \mathbf{a}_1\oplus\mathrm{gyr}[\mathbf{a}_1, \ominus\mathbf{a}_0](\ominus\mathbf{a}_0\oplus\mathbf{b}_0) \tag{109}$$

which, by Section 4, is equivalent to the equation

$$\ominus\mathbf{a}_1\oplus\mathbf{b}_1 = \mathrm{gyr}[\mathbf{a}_1, \ominus\mathbf{a}_0](\ominus\mathbf{a}_0\oplus\mathbf{b}_0) \tag{110}$$

shown in Fig. 11.

Similarly, the parallel transport of the rooted gyrovector $\ominus\mathbf{a}_1\oplus\mathbf{b}_1$, rooted at the point $\mathbf{a}_1$ to the rooted gyrovector $\ominus\mathbf{a}_2\oplus\mathbf{b}_2$, rooted at the point $\mathbf{a}_2$, where the points $\mathbf{a}_1$, $\mathbf{b}_1$, and $\mathbf{a}_2$, are given is accomplished by calculating the point $\mathbf{b}_2$ in Fig. 11 according to the equation

$$\mathbf{b}_2 = \mathbf{a}_2 \oplus \mathrm{gyr}[\mathbf{a}_2, \ominus\mathbf{a}_1](\ominus\mathbf{a}_1 \oplus \mathbf{b}_1) \tag{111}$$

which, by Section 4, is equivalent to the equation

$$\ominus\mathbf{a}_2 \oplus \mathbf{b}_2 = \mathrm{gyr}[\mathbf{a}_2, \ominus\mathbf{a}_1](\ominus\mathbf{a}_1 \oplus \mathbf{b}_1) \tag{112}$$

shown in Fig. 11.

Also, similarly, the parallel transport of the rooted gyrovector $\ominus\mathbf{a}_0 \oplus \mathbf{b}_0$, rooted at the point $\mathbf{a}_0$ to the rooted gyrovector $\ominus\mathbf{a}_2 \oplus \mathbf{b}_2$, rooted at the point $\mathbf{a}_2$, where the points $\mathbf{a}_0$, $\mathbf{b}_0$, and $\mathbf{a}_2$, are given is accomplished by calculating the point $\mathbf{b}_2$ in Fig. 11 according to the equation

$$\mathbf{b}_2 = \mathbf{a}_2 \oplus \mathrm{gyr}[\mathbf{a}_2, \ominus\mathbf{a}_0](\ominus\mathbf{a}_0 \oplus \mathbf{b}_0) \tag{113}$$

which, by Section 4, is equivalent to the equation

$$\ominus\mathbf{a}_2 \oplus \mathbf{b}_2 = \mathrm{gyr}[\mathbf{a}_2, \ominus\mathbf{a}_0](\ominus\mathbf{a}_0 \oplus \mathbf{b}_0) \tag{114}$$

shown in Fig. 11.

Interestingly, (114) follows from the substitution of (110) into (112) by means of the gyration gyrotransitive law (84) for gyrolines (exercise).

Thus, in Fig. 11,

(1) the two successive parallel transports of the rooted gyrovector $\ominus\mathbf{a}_0 \oplus \mathbf{b}_0$ to the rooted gyrovector $\ominus\mathbf{a}_1 \oplus \mathbf{b}_1$ and then to the rooted gyrovector $\ominus\mathbf{a}_2 \oplus \mathbf{b}_2$, represented respectively by (110) and (112), are equivalent to

(2) the single parallel transport of the rooted gyrovector $\ominus\mathbf{a}_0 \oplus \mathbf{b}_0$ to the rooted gyrovector $\ominus\mathbf{a}_2 \oplus \mathbf{b}_2$, represented by (114).

Some interesting results about the gyro-structure of the Poincaré disc model of hyperbolic geometry may be found in [84] and [85].

**Acknowledgments:** The history of the two competing geometrical approaches to the study of special relativity, as described by Scott Walter in [7], plays an important role in the present expository article. I am grateful to Scott Walter for useful comments about (i) the historical non-Euclidean style in the study of special relativity theory, developed during the period 1908–1912 by Varičak, Robb, Wilson and Lewis, and Borel; and (ii) the current domination of the space-time formalism in textbooks on special relativity theory.

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