BEYOND THE EINSTEIN ADDITION LAW AND ITS GYROSCOPIC THOMAS PRECESSION: THE THEORY OF GYROGROUPS AND GYROVECTOR SPACES

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Preface

"I cannot define coincidence [in mathematics]. But I shall argue that coincidence can always be elevated or organized into a superstructure which performs a unification along the coincidental elements. The existence of a coincidence is strong evidence for the existence of a covering theory."

—Philip J. Davis [Dav81]

Alluding to the Thomas gyration, this book presents the Theory of gyrogroups and gyrovector spaces, taking the reader to the immensity of hyperbolic geometry that lies beyond the Einstein special theory of relativity.

Soon after its introduction by Einstein in 1905 [Ein05], special relativity theory (as named by Einstein ten years later) became overshadowed by the appearance of general relativity. Subsequently, the exposition of special relativity followed the lines laid down by Minkowski, in which the role of hyperbolic geometry is not emphasized. This can doubtlessly be explained by the strangeness and unfamiliarity of hyperbolic geometry [Bar98] [Wal99b] [Ung01d].

The aim of this book is to reverse the trend of neglecting the role of hyperbolic geometry in the special theory of relativity, initiated by Minkowski, by emphasizing the central role that hyperbolic geometry plays in the theory. We will find in this book that the special theory of relativity and hyperbolic geometry cross-pollinate to produce important new results:

(1) The relativistic gyroscopic precession, known as the Thomas precession, turns out to play an important role in the foundations of hyperbolic geometry, uncovering hitherto unnoticed analogies with Euclidean geometry; and

(2) guided by the analogies that hyperbolic geometry shares with Euclidean geometry, uncovered by means of the Thomas precession, we discover related analogies that the Lorentz transformation shares with the Galilei transformation. The emerging analogies reverse a trend initiated by Minkowski: The relativistic spacetime emerges as a derived concept rather than a primitive (that is, unexplained) notion.

Exploring the special theory of relativity and its structure beyond the Einstein addition law and its gyroscopic Thomas precession, we take advantage, in this book, of the idea that hyperbolic geometry governs velocities in relativity physics in the same way that Euclidean geometry governs velocities in prerelativity physics, allowing us to partially restore the classical picture. In particular, we take advantage of the result that the Einstein velocity addition of relativistically admissible velocities is a gyrocommutative gyrogroup operation in the same way that the Galilei velocity addition (that is, ordinary vector addition) is a commutative group operation. Furthermore, Einstein's addition is a gyrocommutative gyrogroup operation that admits scalar multiplication, thus giving rise to a gyrovector space. Gyrovector spaces, in turn, form the setting for hyperbolic geometry in the same way that vector spaces form the setting for Euclidean geometry.

The primary purpose of this book is, accordingly, to provide readers with a self-contained account of the two topics in the subtitle.

A gyrogroup is a grouplike algebraic structure that provides a most natural generalization of the group notion. Historically, the first gyrogroup was discovered by the author in the algebraic structure that underlies Einstein's velocity addition [Ung88a]. However, the best way to introduce the gyrogroup notion by example is offered by the Möbius transformation group of the disc.

Ahlfors' book [Ahl73], *Conformal Invariants: Topics in Geometric Function Theory*, begins with the presentation of the *polar decomposition* of the most general Möbius self-transformation of the complex open unit disc $\mathbb{D}$

$$z \mapsto e^{i\theta} \frac{z_0 + z}{1 + \bar{z}_0 z}$$

which we write as

$$z \mapsto e^{i\theta} \frac{z_0 + z}{1 + \bar{z}_0 z} = e^{i\theta}(z_0 \oplus z)$$

$z_0 \in \mathbb{D}$, $\theta \in \mathbb{R}$. The polar decomposition of the Möbius transformation group of the disc is a most beautiful result that has a great appeal to the Poincaré disc model of hyperbolic geometry, as we will see in this book. Suggestively, we define the Möbius addition $\oplus$ in the disc, allowing the generic Möbius transformation of the disc to be viewed as a Möbius *left gyrotranslation*

$$\boxed{z \mapsto z_0 \oplus z = \frac{z_0 + z}{1 + \bar{z}_0 z}}$$

followed by a rotation. The prefix "gyro" that we use to emphasize analogies with classical notions, stems from the Thomas gyration, which will soon become clear. The resulting Möbius addition in the disc $\mathbb{D}$ is neither commutative nor associative. To 'repair' the breakdown of commutativity in the Möbius addition we associate it with the *gyration* (or, rotation) generated by $a, b \in \mathbb{D}$,

$$\boxed{\mathrm{gyr}[a,b] = \frac{a \oplus b}{b \oplus a} = \frac{1 + a\bar{b}}{1 + \bar{a}b}}$$

giving rise to the gyrocommutative law of Möbius addition,

$$a \oplus b = \mathrm{gyr}[a,b](b \oplus a) \qquad \text{Gyrocommutative law}$$

Following the gyration definition the gyrocommutative law is not terribly surprising, but we are not finished.

Coincidentally, the gyration that repairs the breakdown of commutativity in the Möbius addition repairs the breakdown of associativity as well, giving rise to identities that capture analogies. Thus, for instance, for all $a, b, c \in \mathbb{D}$ we have

$$\begin{aligned}
&a \oplus (b \oplus c) = (a \oplus b) \oplus \mathrm{gyr}[a,b]c && \text{Left Gyroassociative Law} \\
&(a \oplus b) \oplus c = a \oplus (b \oplus \mathrm{gyr}[b,a]c) && \text{Right Gyroassociative Law} \\
&\mathrm{gyr}[a,b] = \mathrm{gyr}[a \oplus b, b] && \text{Left Loop Property} \\
&\mathrm{gyr}[a,b] = \mathrm{gyr}[a, b \oplus a] && \text{Right Loop Property}
\end{aligned}$$

The Möbius addition is clearly regulated by its associated gyration and, accordingly, the Möbius addition and its associated gyration are inextricably linked. Furthermore, Möbius addition $\oplus$ comes with a dual addition, called the Möbius coaddition $\boxplus$, with which it shares duality symmetries like

$$\begin{aligned}
a \boxplus b &= a \oplus \mathrm{gyr}[a, \ominus b]b \\
a \oplus b &= a \boxplus \mathrm{gyr}[a,b]b
\end{aligned}$$

and which allows us to capture analogies, like the following left and right cancellation laws in the disc $\mathbb{D}$,

$$\begin{aligned}
x \oplus (\ominus x \oplus a) &= a \\
(a \ominus x) \boxplus x &= a
\end{aligned}$$

and the ones shown in Table 0.1.

Where there are coincidences there is significance. The emerging coincidences to which the gyration gives rise uncover an interesting algebraic structure that merits extension by abstraction, leading to the grouplike structure called a gyrogroup.

Gyrogroups are generalized groups that share remarkable analogies with groups. In full analogy with groups:

(1) Gyrogroups are classified into gyrocommutative gyrogroups and non-gyrocommutative gyrogroups.

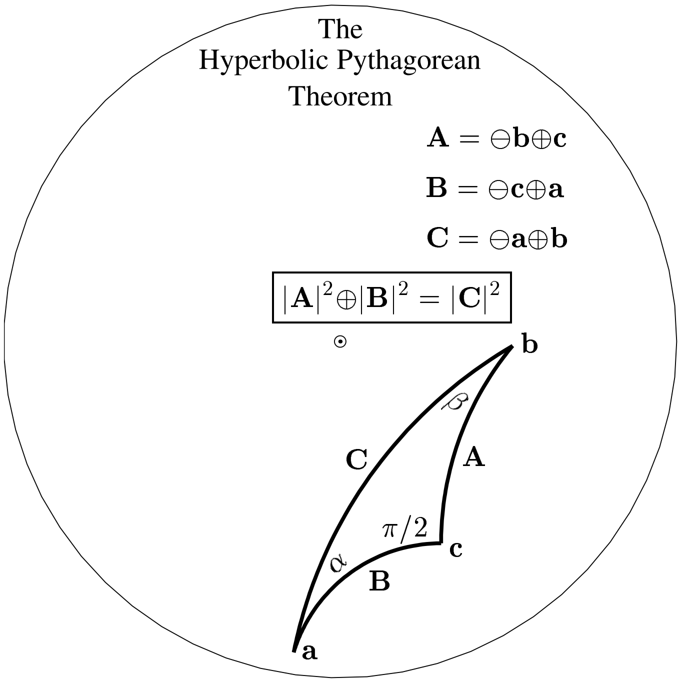

Figure 0.1. Artful Application of the Möbius Transformation in Hyperbolic Geometry, I. The hyperbolic Pythagorean theorem for Möbius right angled hyperbolic triangles in the complex open unit disc $\mathbb{D}$ in a form visually analogous to its Euclidean counterpart [Ung99].

(2) Some gyrocommutative gyrogroups admit scalar multiplication, turning them into gyrovector spaces.

(3) Gyrovector spaces, in turn, provide the setting for hyperbolic geometry just as vector spaces provide the setting for Euclidean geometry, thus enabling the two geometries to be unified.

(4) Moreover, the resulting analogies that the motions of Euclidean geometry (that is, a commutative group of translations and a group of rotations) and the motions of hyperbolic geometry (that is, a gyrocommutative gyrogroup of left gyrotranslations and a group of rotations) share induce analogies that the Galilei transformation and the Lorentz transformation share.

The flavor of this book is illustrated by Figs. 0.1 and 0.2. To demonstrate the ability of gyro-formalism to capture analogies:

(1) We present graphically in Fig. 0.1 the hyperbolic Pythagorean Theorem in the Möbius gyrovector plane $(\mathbb{D}, \oplus, \otimes)$, which we will study in Chapters 4 and 6; and

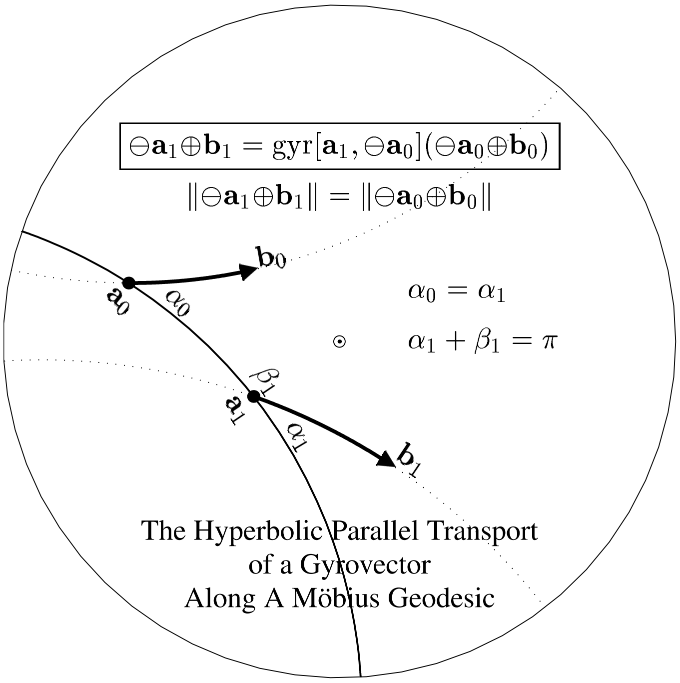

Figure 0.2. Artful Application of the Möbius Transformation in Hyperbolic Geometry, II. The Poincaré n-dimensional ball model of hyperbolic geometry turns out to be the n-dimensional Möbius gyrovector space. In Chapter 7 we will find that in gyro-formalism the nonassociative algebra of the hyperbolic parallel transport of a gyrovector $(-\mathbf{a}_0\oplus\mathbf{b}_0)$ rooted at $\mathbf{a}_0$ to the gyrovector $(-\mathbf{a}_1\oplus\mathbf{b}_1)$ rooted at $\mathbf{a}_1$ along the Möbius geodesic that links $\mathbf{a}_0$ and $\mathbf{a}_1$ in a Möbius gyrovector space $(\mathbb{R}_c^2, \oplus, \otimes)$ is fully analogous to the algebra of its Euclidean counterpart. The special case of $n = 2$ is shown here graphically.

(2) We present graphically in Fig. 0.2 the algebra of the hyperbolic parallel transport along geodesics in the Möbius gyrovector plane $(\mathbb{R}_c^2, \oplus, \otimes)$, which we will study in Chapter 7.

Typically to the study of hyperbolic geometry as the geometry of gyrovector spaces, the hyperbolic geometry concepts shown in Figs. 0.1 and 0.2 turn out to be fully analogous to their Euclidean counterparts. In contrast, prior to the emergence of gyrogroup and gyrovector space theory:

(1) The Hyperbolic Pythagorean Theorem appeared in the literature in a form which shares no visual analogies with its Euclidean counterpart.

(2) Parallel transport in classical hyperbolic geometry is achieved by methods of differential geometry rather than by methods of nonassociative algebra.

Notion	Euclidean	Hyperbolic
Addition	$+$	$\oplus$ and $\boxplus$
Commutativity	$\mathbf{a}+\mathbf{b}=\mathbf{b}+\mathbf{a}$	$\mathbf{a}\oplus\mathbf{b}=\mathrm{gyr}[\mathbf{a},\mathbf{b}](\mathbf{b}\oplus\mathbf{a})$ $\mathbf{a}\boxplus\mathbf{b}=\mathbf{b}\boxplus\mathbf{a}$
Associativity	$\mathbf{a}+(\mathbf{b}+\mathbf{c})=(\mathbf{a}+\mathbf{b})+\mathbf{c}$	$\mathbf{a}\oplus(\mathbf{b}\oplus\mathbf{c})=(\mathbf{a}\oplus\mathbf{b})\oplus\mathrm{gyr}[\mathbf{a},\mathbf{b}]\mathbf{c}$ $(\mathbf{a}\oplus\mathbf{b})\oplus\mathbf{c}=\mathbf{a}\oplus(\mathbf{b}\oplus\mathrm{gyr}[\mathbf{b},\mathbf{a}]\mathbf{c})$
Distance	$\|\mathbf{a}-\mathbf{b}\|$	$\|\mathbf{a}\ominus\mathbf{b}\|$
Pythagorean Theorem	$\|\mathbf{A}\|^2+\|\mathbf{B}\|^2=\|\mathbf{C}\|^2$	$\|\mathbf{A}\|^2\oplus\|\mathbf{B}\|^2=\|\mathbf{C}\|^2$ in the Poincaré unit ball model
Gyrovector Transport	$-\mathbf{a}_1+\mathbf{b}_1=-\mathbf{a}_0+\mathbf{b}_0$	$\ominus\mathbf{a}_1\oplus\mathbf{b}_1=\ominus\mathbf{a}_0\oplus\mathbf{b}_0$
Parallel Transport	$-\mathbf{a}_1+\mathbf{b}_1=-\mathbf{a}_0+\mathbf{b}_0$	$\ominus\mathbf{a}_1\oplus\mathbf{b}_1=\mathrm{gyr}[\mathbf{a}_1,\ominus\mathbf{a}_0](\ominus\mathbf{a}_0\oplus\mathbf{b}_0)$
Triangle Inequality	$\|-\mathbf{a}+\mathbf{b}\|-\|-\mathbf{b}+\mathbf{c}\|$ $\le\|-\mathbf{a}+\mathbf{c}\|$ $\le\|-\mathbf{a}+\mathbf{b}\|+\|-\mathbf{b}+\mathbf{c}\|$	$\|\ominus\mathbf{a}\oplus\mathbf{b}\|\ominus\|\ominus\mathbf{b}\oplus\mathbf{c}\|$ $\le\|\ominus\mathbf{a}\oplus\mathbf{c}\|$ $\le\|\ominus\mathbf{a}\oplus\mathbf{b}\|\oplus\|\ominus\mathbf{b}\oplus\mathbf{c}\|$
Geodesics	$\mathbf{a}+(-\mathbf{a}+\mathbf{b})t$ $-\infty<t<\infty$	$\mathbf{a}\oplus(\ominus\mathbf{a}\oplus\mathbf{b})\otimes t$ $-\infty<t<\infty$
Midpoint $(\mathbf{a},\mathbf{b})$	$\mathbf{a}+\frac{1}{2}(-\mathbf{a}+\mathbf{b})=\frac{1}{2}(\mathbf{a}+\mathbf{b})$	$\mathbf{a}\oplus\frac{1}{2}\otimes(\ominus\mathbf{a}\oplus\mathbf{b})=\frac{1}{2}\otimes(\mathbf{a}\boxplus\mathbf{b})$
Angle cosines between rays	$\cos\alpha=\frac{-\mathbf{a}+\mathbf{b}}{\|-\mathbf{a}+\mathbf{b}\|}\cdot\frac{-\mathbf{a}+\mathbf{c}}{\|-\mathbf{a}+\mathbf{c}\|}$	$\cos\alpha=\frac{\ominus\mathbf{a}\oplus\mathbf{b}}{\|\ominus\mathbf{a}\oplus\mathbf{b}\|}\cdot\frac{\ominus\mathbf{a}\oplus\mathbf{c}}{\|\ominus\mathbf{a}\oplus\mathbf{c}\|}$
Equations	$\mathbf{a}+\mathbf{x}=\mathbf{b}$ $\mathbf{x}=-\mathbf{a}+\mathbf{b}$ $\mathbf{y}+\mathbf{a}=\mathbf{b}$ $\mathbf{y}=\mathbf{b}-\mathbf{a}$	$\mathbf{a}\oplus\mathbf{x}=\mathbf{b}$ $\mathbf{x}=\ominus\mathbf{a}\oplus\mathbf{b}$ $\mathbf{y}\oplus\mathbf{a}=\mathbf{b}$ $\mathbf{y}=\mathbf{b}\ominus\mathrm{gyr}[\mathbf{b},\mathbf{a}]\mathbf{a}=\mathbf{b}\boxminus\mathbf{a}$
Cancellation Laws	$\mathbf{x}+(-\mathbf{x}+\mathbf{a})=\mathbf{a}$ $(\mathbf{a}-\mathbf{x})+\mathbf{x}=\mathbf{a}$	$\mathbf{x}\oplus(\ominus\mathbf{x}\oplus\mathbf{a})=\mathbf{a}$ $(\mathbf{a}\ominus\mathbf{x})\boxplus\mathbf{x}=\mathbf{a}$ $(\mathbf{a}\boxminus\mathbf{x})\oplus\mathbf{x}=\mathbf{a}$

Table 0.1. Euclidean-Hyperbolic Analogies for the Möbius Addition $\oplus$ and Coaddition $\boxplus$ in gyrovector spaces

These and some other remarkable analogies that we will study in this book are presented in Table 0.1.

One of the attractive features of this book is that the prerequisites are minimal, encouraging readers to perform their own research at an early stage once they master the manipulation of analogies to which the Thomas gyration gives rise. The theory of gyrogroups and gyrovector spaces recasts hyperbolic geometry and aspires to further conquests in the immense domain of mathematical physics and hyperbolic geometry.