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From Einstein Addition to Thomas Gyration: Taming The Thomas Precession

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Thomas precession is traditionally studied as an isolated phenomenon in relativity physics. To demonstrate the central role that Thomas precession plays in the special theory of relativity we uncover the road from Einstein addition to Thomas precession. Thomas precession is thus tamed into the so called Thomas gyration that, in turn, captures analogies between Newtonian and Einsteinian velocity spaces, and between their respective Euclidean and hyperbolic geometry. In particular, the analogies to which Thomas precession gives rise indicate that Einstein addition is regulated by the hyperbolic geometry of Bolyai and Lobachevski just as ordinary vector addition is regulated by Euclidean geometry. The path from Einstein addition to Thomas precession thus gives students a great way to get a glimpse into the fascinating world of hyperbolic geometry and the role it plays in relativity physics. There are numerous reasons for the introduction of hyperbolic geometry into the mathematics and physics curricula. But, one certainly is that by working with the special theory of relativity in the context of hyperbolic geometry students will gain a better geometric insight into relativity physics, as we demonstrate in this article.

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I. INTRODUCTION

In the small arena of Terrestrial and planetary measurements, Euclidean geometry is the shoe that fits the foot. Over the vast reaches of intergalactic space-time, Lorentzian geometry, the geometry of general relativity, appears to be what is wanted. Neglecting gravitation, the geometry needed is the hyperbolic geometry of Bolyai and Lobachevski, the geometry of special relativity. Indeed, an interesting use of hyperbolic geometry in special relativity has recently been presented by Criado and Alamo in Refs. [1] and [2]. Costella *et al* have recently emphasized the central role that the Thomas precession or rotation, plays in special relativity [3]. A subsequent interesting supporting comment was presented by Rebilas [4].

The aim of this article is to demonstrate that the most natural and rewarding way to study and introduce Einstein addition and its Thomas precession into the undergraduate and graduate curricula is provided by hyperbolic geometry, the geometry that underlies the special theory of relativity. Accordingly, we will show that it is the hyperbolic geometry of Bolyai and Lobachevski that regulates both Einstein addition and its associated Thomas precession.

As a mathematical prerequisite we assume nothing more than familiarity with the elementary operations of 3-dimensional vectors, and with straight lines and angles of plane analytic geometry. This prerequisite makes our presentation of the road from Einstein addition to Thomas precession in the context of hyperbolic geometry as painlessly as possible and accessible to a wider audience, including the undergraduate physics student.

In Section III we employ hyperbolic functions to describe the historical way in which hyperbolic geometry emerged in special relativity. Readers who are not familiar with hyperbolic functions may skip reading this section with no loss of understanding of the rest of this article.

Counterintuitively, Einstein addition is neither commutative nor associative. Moreover, the seemingly isolated phenomenon of the Thomas precession confuses the undergraduate students who are not familiar with hyperbolic geometry. Accordingly, we will show in this article that the seemingly structureless Einstein addition and the seemingly isolated Thomas precession reappear in the context of hyperbolic geometry as a unified phenomenon with rich structure of mathematical regularity. It is hoped that following this article the need to bring hyperbolic geometry, as presented in Ref. [5], to the core of the curriculum will become evident.

A novel application to relativity physics of our way to study Einstein addition and its associated Thomas precession in the context of hyperbolic geometry is presented in Section IX. In an unexpected way it explains the notion of the *relativistic center of momentum* in terms of novel analogies that the pairs

$$\begin{pmatrix} \textit{Euclidean Geometry} \\ \textit{Newtonian Mechanics} \end{pmatrix} \leftrightarrow \begin{pmatrix} \textit{Hyperbolic Geometry} \\ \textit{Einsteinian Mechanics} \end{pmatrix} \tag{1}$$

share. Remarkably, these novel analogies stem from the relativistic mass correction, according to which the mass of moving objects is velocity dependent [6]. In particular, we show that the common relativistic mass correction for moving objects has an interpretation in hyperbolic geometry.

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Einsteinian velocities and space rotations parametrize the Lorentz transformation group of relativistic mechanics in a way fully analogous to the well-known use of Newtonian velocities and space rotations to parametrize the Galilei transformation group of classical mechanics. This and other related analogies the Lorentz transformation group shares with its Galilean counterpart are presented in Section X in terms of Einstein addition and its associated Thomas precession. Following these analogies, students who intuitively understand the Galilei transformation group can straightforwardly extend their intuitive understanding to the Lorentz transformation group as well.

II. EINSTEIN ADDITION

Attempts to measure the absolute velocity of the earth through the hypothetical ether had failed. The most famous of these experiments is one performed by Michelson and Morley in 1887 [7]. It was 18 years later before the null results of these experiments were finally explained by Einstein in terms of a new velocity addition law that bears his name, that he introduced in his 1905 paper that founded the special theory of relativity [8, 9].

Contrasting Newtonian velocities, which are vectors in the Euclidean 3-space $\mathbb{R}^3$, Einsteinian velocities must be relativistically admissible, that is, their magnitude must not exceed the vacuum speed of light. It will prove useful to introduce a notation, $\oplus$, for the addition of Einsteinian velocities, which is similar to the common notation, $+$, that we use for vector addition of Newtonian velocities. The notation $\oplus$ for Einstein addition will help us in this article to treat Einstein addition and vector addition on essentially the same footing. Vector addition will emerge in this article as a special case of Einstein addition, corresponding to a vanishing Thomas precession.

Let c be the vacuum speed of light, and let

$$\mathbb{R}^3_c = \{\mathbf{v} \in \mathbb{R}^3 : \|\mathbf{v}\| < c\} \tag{2}$$

be the c-ball of all relativistically admissible velocities. It is the ball of radius c, centered at the origin of the Euclidean 3-space $\mathbb{R}^3$, consisting of all vectors $\mathbf{v}$ in $\mathbb{R}^3$ with magnitude $\|\mathbf{v}\|$ smaller than c. Einstein addition $\oplus$ in the ball, for velocities that need not be parallel, is given by the equation [5, 10],

$$\mathbf{u}\oplus\mathbf{v} = \frac{1}{1+\frac{\mathbf{u}\cdot\mathbf{v}}{c^2}}\left\{\mathbf{u} + \frac{1}{\gamma_{\mathbf{u}}}\mathbf{v} + \frac{1}{c^2}\frac{\gamma_{\mathbf{u}}}{1+\gamma_{\mathbf{u}}}(\mathbf{u}\cdot\mathbf{v})\mathbf{u}\right\} \tag{3}$$

for all $\mathbf{u},\mathbf{v}\in\mathbb{R}^3_c$, where $\gamma_{\mathbf{u}}$ is the Lorentz factor

$$\gamma_{\mathbf{u}} = \frac{1}{\sqrt{1-\frac{\|\mathbf{u}\|^2}{c^2}}} \tag{4}$$

Einstein addition (3) of relativistically admissible velocities is found in his 1905 paper (see p. 141 in Ref. [9]) where the magnitudes, (11), of the two sides of Einstein addition (3) are presented. One has to remember here that the Euclidean 3-vector algebra was not so widely known in 1905 and, consequently, was not used by Einstein. Einstein calculated in Ref. [8] the behavior of the velocity components parallel and orthogonal to the relative velocity between inertial systems, which is as close as one can get without vectors to the vectorial version that we present in (3).

In the Newtonian limit, $c \to \infty$, the ball $\mathbb{R}^3_c$ of admissible velocities expands to the whole of its space $\mathbb{R}^3$, as we see from (2), and Einstein addition $\oplus$ reduces to ordinary vector addition $+$ in $\mathbb{R}^3$, as we see from (3) and (4).

For the sake of simplicity, some normalize the vacuum speed of light to $c = 1$. We, however, prefer to leave it as a free positive parameter enabling Einstein addition to be reduced to the ordinary vector addition in the Newtonian limit, when $c \to \infty$. Thus, in the Newtonian limit the *groupoid* of Einsteinian velocities $(\mathbb{R}^3_c, \oplus)$ reduces to the group of Newtonian velocities $(\mathbb{R}^3, +)$. A groupoid in mathematics is a non-empty set with a binary operation. Einstein addition $\oplus$ is a binary operation in the ball $\mathbb{R}^3_c$, giving rise to the groupoid $(\mathbb{R}^3_c, \oplus)$, just as ordinary vector addition $+$ is a binary operation in the Euclidean 3-space $\mathbb{R}^3$, giving rise to the group $(\mathbb{R}^3, +)$. A group is a groupoid in which the binary operation is associative, and has a neutral element and an inverse element. As an example, the group $(\mathbb{R}^3, +)$ of Newtonian velocities is the Euclidean 3-space $\mathbb{R}^3$ with its ordinary vector addition $+$. The neutral element is the zero vector $\mathbf{0}$, and the inverse element of $\mathbf{v}\in\mathbb{R}^3$ is $-\mathbf{v}$. Vector addition in $\mathbb{R}^3$ is both commutative and associative since for all $\mathbf{u}, \mathbf{v}, \mathbf{w}\in\mathbb{R}^3$ we have

$$\begin{aligned} \mathbf{u}+\mathbf{v} &= \mathbf{v}+\mathbf{u} \\ \mathbf{u}+(\mathbf{v}+\mathbf{w}) &= (\mathbf{u}+\mathbf{v})+\mathbf{w} \end{aligned} \tag{5}$$

Unlike the group $(\mathbb{R}^3, +)$ of Newtonian velocities, we will see in Section IV that the groupoid $(\mathbb{R}^3_c, \oplus)$ of Einsteinian velocities is not a group.

III. THE EMERGENCE OF HYPERBOLIC GEOMETRY IN SPECIAL RELATIVITY

In this section we present the historical way in which hyperbolic geometry emerged in special relativity soon after it was introduced by Einstein in 1905. The rest of the paper is independent of this section so that readers may skip it if they are interested in the new rather than the old.

The Lorentz factor (4) is linked to Einstein addition (3) by the two equivalent identities

$$\gamma_{\mathbf{u}\oplus\mathbf{v}} = \gamma_{\mathbf{u}}\gamma_{\mathbf{v}}\left(1+\frac{\mathbf{u}\cdot\mathbf{v}}{c^2}\right) \tag{6}$$

$$\gamma_{\mathbf{u}\ominus\mathbf{v}} = \gamma_{\mathbf{u}}\gamma_{\mathbf{v}}\left(1-\frac{\mathbf{u}\cdot\mathbf{v}}{c^2}\right) \tag{7}$$

first studied by Sommerfeld [11], and Varičak [12].

We use the notation $\ominus\mathbf{v} = -\mathbf{v}$ for the negative Einsteinian velocity, and $\mathbf{u}\ominus\mathbf{v} = \mathbf{u}\oplus(-\mathbf{v})$ for the Einstein subtraction in the ball $\mathbb{R}^3_c$ so that, for instance, $\mathbf{v}\ominus\mathbf{v} = \mathbf{0}$.

The usefulness of hyperbolic functions in special relativity is well known [13]. The rapidity $\phi_\mathbf{v}$ of a relativistically admissible velocity $\mathbf{v}$ is defined by the equation

$$\phi_\mathbf{v} = \tanh^{-1}\frac{\|\mathbf{v}\|}{c} \tag{8}$$

so that,

$$\begin{aligned} \cosh\phi_\mathbf{v} &= \gamma_\mathbf{v} \\ \sinh\phi_\mathbf{v} &= \gamma_\mathbf{v}\frac{\|\mathbf{v}\|}{c} \end{aligned} \tag{9}$$

In the years 1910–1914, the period which experienced a dramatic flowering of creativity in the special theory of relativity, the Croatian physicist and mathematician Vladimir Varičak (1865–1942), professor and rector of Zagreb University, showed in January 1910 [12], that this theory has a natural interpretation in hyperbolic geometry [14] [15].

Indeed, in terms of rapidities the second identity in (7) can be written as

$$\cosh\phi_{\mathbf{u}\ominus\mathbf{v}} = \cosh\phi_\mathbf{u}\cosh\phi_\mathbf{v} - \sinh\phi_\mathbf{u}\sinh\phi_\mathbf{v}\cos A \tag{10}$$

where, according to J.F. Barrett [16], the angle A has been interpreted by Sommerfeld [11], and Varičak [12], as a hyperbolic angle in the relativistic "triangle of velocities" in the Beltrami ball model of hyperbolic geometry. The role of Carathéodory in this approach to special relativity and hyperbolic geometry has recently been described by J.F. Barrett [16], emphasizing that (10) is the "cosine rule" in hyperbolic geometry.

Like vector addition, Einstein addition has a neutral element, the zero vector $\mathbf{0}$, and the inverse of a vector $\mathbf{v}$ is the vector $\ominus\mathbf{v} = -\mathbf{v}$ in the ball $\mathbb{R}^3_c$. Unlike vector addition, however, Einstein addition is neither commutative nor associative. Thus, in the context of Euclidean geometry Einstein addition does not possess the structure one would expect from "addition". This is, however, not the case in the context of hyperbolic geometry, as we will see.

IV. ON THE ALGEBRA OF EINSTEIN ADDITION

Einstein addition (3) is not commutative since, in general, $\mathbf{u}\oplus\mathbf{v} \neq \mathbf{v}\oplus\mathbf{u}$ in the ball $\mathbb{R}^3_c$ of all relativistically admissible velocities. Moreover, Einstein addition (3) is not associative since, in general, $\mathbf{u}\oplus(\mathbf{v}\oplus\mathbf{w}) \neq (\mathbf{u}\oplus\mathbf{v})\oplus\mathbf{w}$ for relativistically admissible velocities $\mathbf{u}, \mathbf{v}$ and $\mathbf{w}$ in the ball. Paradoxically, if the relative velocity between two inertial frames is the composite velocity of $\mathbf{u}$ and $\mathbf{v}$, one may ask whether the "correct" relative velocity is $\mathbf{u}\oplus\mathbf{v}$ or $\mathbf{v}\oplus\mathbf{u}$. This paradox will be resolved in (82).

We may note that the two distinct velocities $\mathbf{u}\oplus\mathbf{v}$ and $\mathbf{v}\oplus\mathbf{u}$ in the ball $\mathbb{R}^3_c$ have equal magnitudes [17],

$$\begin{aligned} \|\mathbf{u}\oplus\mathbf{v}\|^2 &= \|\mathbf{v}\oplus\mathbf{u}\|^2 \\ &= \left\|\frac{\mathbf{u}+\mathbf{v}}{1+\frac{\mathbf{u}\cdot\mathbf{v}}{c^2}}\right\|^2 - \frac{1}{c^2}\left\|\frac{\mathbf{u}\times\mathbf{v}}{1+\frac{\mathbf{u}\cdot\mathbf{v}}{c^2}}\right\|^2 \end{aligned} \tag{11}$$

so that there exists a rotation that takes $\mathbf{v}\oplus\mathbf{u}$ into $\mathbf{u}\oplus\mathbf{v}$. Indeed, the unique rotation that takes $\mathbf{v}\oplus\mathbf{u}$ into $\mathbf{u}\oplus\mathbf{v}$ about a rotation axis perpendicular to the plane of $\mathbf{u}$ and $\mathbf{v}$ through an angle $< \pi$ is the well known Thomas rotation, or precession, generated by $\mathbf{u}$ and $\mathbf{v}$ [18]. It was already known to Silberstein in 1914 [19], as we will emphasize in Section VI. Graphical illustration of the Thomas precession generated by two relativistic velocities is presented in [20].

For parallel velocities in $\mathbb{R}^3_c$ Einstein addition (3) specializes to the binary operation

$$\mathbf{u}\oplus\mathbf{v} = \frac{\mathbf{u}+\mathbf{v}}{1+\frac{\mathbf{u}\cdot\mathbf{v}}{c^2}}, \qquad \mathbf{u}\|\mathbf{v} \tag{12}$$

which is both commutative and associative. Owing to its simplicity and mathematical regularity, the special case (12) of Einstein addition (3) for parallel relativistically admissible velocities is commonly found in the literature on relativity physics. In contrast, Einstein addition (3) for relativistically admissible velocities that need not be parallel seems to have no algebraic structure and, accordingly, it is almost unheard of in texts on relativity physics. Among outstanding exceptions are Refs. [10] and [21].

Fortunately, Thomas gyration comes to the rescue of the breakdown of commutativity and associativity in Einstein addition, as we will see in Section V. Consequently, Einstein addition will emerge as a gyrocommutative, gyroassociative binary operation with rich algebraic structure.

V. THOMAS GYRATION

In this section we define a self-map of the ball $\mathbb{R}^3_c$ of all relativistically admissible velocities. It is denoted $\mathrm{gyr}[\mathbf{u},\mathbf{v}]$ and called the Thomas gyration generated by $\mathbf{u},\mathbf{v}\in\mathbb{R}^3_c$, This self-map of the ball is generated by any two velocities $\mathbf{u}$ and $\mathbf{v}$ in the ball. In Section VI we will find that the Thomas gyration $\mathrm{gyr}[\mathbf{u},\mathbf{v}]$ is realized experimentally as the familiar Thomas precession of special relativity. The relativistic effect known as the Thomas precession will, then, appear as a mathematical object called Thomas gyration, that we can harness for work in the study of relativity physics. As an example, the way we harness the Thomas gyration for solving the equation $\mathbf{x}\oplus\mathbf{a} = \mathbf{b}$ for the unknown $\mathbf{x}$ in the ball $\mathbb{R}^3_c$ of Einsteinian velocities will be shown in (28).

Let $\mathrm{gyr}[\mathbf{u},\mathbf{v}]$ be the self-map of the ball $\mathbb{R}^3_c$ generated by $\mathbf{u},\mathbf{v}\in\mathbb{R}^3_c$ according to the equation

$$\mathrm{gyr}[\mathbf{u},\mathbf{v}]\mathbf{w} = \ominus(\mathbf{u}\oplus\mathbf{v})\oplus(\mathbf{u}\oplus(\mathbf{v}\oplus\mathbf{w})) \tag{13}$$

Thus, for instance, for $\mathbf{u} = \mathbf{v} = \mathbf{0}$ we have

$$\mathrm{gyr}[\mathbf{0},\mathbf{0}]\mathbf{w} = \mathbf{w} \tag{14}$$

so that $\mathrm{gyr}[\mathbf{0},\mathbf{0}]$ is the identity map of the ball $\mathbb{R}^3_c$.

It can be shown that

$$\mathrm{gyr}[\mathbf{u},\mathbf{0}]\mathbf{w} = \mathrm{gyr}[\mathbf{0},\mathbf{v}]\mathbf{w} = \mathbf{w} \tag{15}$$

and that, more generally,

$$\mathrm{gyr}[\mathbf{u},\mathbf{v}]\mathbf{w} = \mathbf{w}, \qquad \mathbf{u}\|\mathbf{v} \tag{16}$$

whenever $\mathbf{u}$ and $\mathbf{v}$ are parallel in the ball $\mathbb{R}^3_c$ of relativistic velocities.

To verify (16) we note that Einstein addition for parallel velocities, (12), is associative, so that for $\mathbf{u}\|\mathbf{v}$ we have from the definition of $\mathrm{gyr}[\mathbf{u},\mathbf{v}]$ in (13),

$$\begin{aligned}\mathrm{gyr}[\mathbf{u},\mathbf{v}]\mathbf{w} &= \ominus(\mathbf{u}\oplus\mathbf{v})\oplus(\mathbf{u}\oplus(\mathbf{v}\oplus\mathbf{w}))\\ &= (\ominus(\mathbf{u}\oplus\mathbf{v})\oplus(\mathbf{u}\oplus\mathbf{v}))\oplus\mathbf{w}\\ &= \mathbf{0}\oplus\mathbf{w}\\ &= \mathbf{w}\end{aligned} \tag{17}$$

as desired.

As a special case of (16), and for later reference, we note that

$$\mathrm{gyr}[\mathbf{u},\ominus\mathbf{u}]\mathbf{w} = \mathbf{w} \tag{18}$$

for all $\mathbf{u},\mathbf{w}\in\mathbb{R}^3_c$ since $\mathbf{u}$ and $\ominus\mathbf{u} = -\mathbf{u}$ are parallel.

Owing to the breakdown of associativity in Einstein addition, the self-map $\mathrm{gyr}[\mathbf{u},\mathbf{v}]$ of the ball $\mathbb{R}^3_c$ is, in general, not the identity map. It turns out to be an element of the group $SO(3)$. The group $SO(3)$ is the group of all 3×3 real orthogonal matrices with determinant 1. Indeed, the map $\mathrm{gyr}[\mathbf{u},\mathbf{v}]$ can be written as a 3×3 real orthogonal matrix with determinant 1, as shown in [17]. More specifically, the map $\mathrm{gyr}[\mathbf{u},\mathbf{v}]$ represents a rotation of the Euclidean 3-space $\mathbb{R}^3$ about its origin since it preserves the inner product in the ball. Indeed, we have

$$\mathrm{gyr}[\mathbf{u},\mathbf{v}]\mathbf{a}\cdot\mathrm{gyr}[\mathbf{u},\mathbf{v}]\mathbf{b} = \mathbf{a}\cdot\mathbf{b} \tag{19}$$

and, hence,

$$\|\mathrm{gyr}[\mathbf{u},\mathbf{v}]\mathbf{a}\| = \|\mathbf{a}\| \tag{20}$$

for all $\mathbf{a},\mathbf{b},\mathbf{u},\mathbf{v}\in\mathbb{R}^3_c$, where $\cdot$ and $\|\ \|$ are the inner product and the norm that the ball $\mathbb{R}^3_c$ inherits from its space $\mathbb{R}^3$.

Furthermore, $\mathrm{gyr}[\mathbf{u},\mathbf{v}]$ turns out to be an *automorphism* of the relativistic *groupoid* $(\mathbb{R}^3_c,\oplus)$ in the following sense.

(1) A groupoid $(\mathbb{R}^3_c,\oplus)$ is a non-empty set $\mathbb{R}^3_c$ with a binary operation $\oplus$.

(2) An automorphism $\mathrm{gyr}[\mathbf{u},\mathbf{v}]$, $\mathbf{u},\mathbf{v}\in\mathbb{R}^3_c$, of the groupoid $(\mathbb{R}^3_c,\oplus)$ is a bijective (one-to-one) self-map of $\mathbb{R}^3_c$ that preserves its binary operation $\oplus$. Indeed,

  (i) we have

$$(\mathrm{gyr}[\mathbf{u},\mathbf{v}])^{-1} = \mathrm{gyr}[\mathbf{v},\mathbf{u}] \tag{21}$$

  where $(\mathrm{gyr}[\mathbf{u},\mathbf{v}])^{-1}$ is the inverse of $\mathrm{gyr}[\mathbf{u},\mathbf{v}]$, so that $\mathrm{gyr}[\mathbf{u},\mathbf{v}]$ is bijective; and

  (ii) we have

$$\mathrm{gyr}[\mathbf{u},\mathbf{v}](\mathbf{a}\oplus\mathbf{b}) = \mathrm{gyr}[\mathbf{u},\mathbf{v}]\mathbf{a}\oplus\mathrm{gyr}[\mathbf{u},\mathbf{v}]\mathbf{b} \tag{22}$$

  for all $\mathbf{u},\mathbf{v},\mathbf{a},\mathbf{b}\in\mathbb{R}^3_c$, so that $\mathrm{gyr}[\mathbf{u},\mathbf{v}]$ preserves the Einstein addition $\oplus$ in the ball $\mathbb{R}^3_c$ of all relativistically admissible velocities.

The automorphism $\mathrm{gyr}[\mathbf{u},\mathbf{v}]$, $\mathbf{u},\mathbf{v}\in\mathbb{R}^3_c$, is called a *Thomas gyration*. It is the Thomas gyration generated by $\mathbf{u}$ and $\mathbf{v}$. The latter, in turn, is the mathematical abstraction of the relativistic effect known as the Thomas precession [22], as we will explain in Section VI.

Remarkably, Thomas gyration "repairs" the breakdown of commutativity and associativity in Einstein addition, giving rise to their following gyro-counterparts,

$$\begin{aligned}\mathbf{u}\oplus\mathbf{v} &= \mathrm{gyr}[\mathbf{u},\mathbf{v}](\mathbf{v}\oplus\mathbf{u}) && \text{Gyrocommutative Law}\\ \mathbf{u}\oplus(\mathbf{v}\oplus\mathbf{w}) &= (\mathbf{u}\oplus\mathbf{v})\oplus\mathrm{gyr}[\mathbf{u},\mathbf{v}]\mathbf{w} && \text{Left Gyroassociative Law}\\ (\mathbf{u}\oplus\mathbf{v})\oplus\mathbf{w} &= \mathbf{u}\oplus(\mathbf{v}\oplus\mathrm{gyr}[\mathbf{v},\mathbf{u}]\mathbf{w}) && \text{Right Gyroassociative Law}\end{aligned} \tag{23}$$

for all $\mathbf{u},\mathbf{v},\mathbf{w}\in\mathbb{R}^3_c$. The gyrocommutative and the gyroassociative laws of Einstein addition share obvious analogies with the common commutative and associative laws of vector addition, allowing the classical picture of velocity addition to be restored.

Moreover, Thomas gyration possesses the elegant left and right loop property

$$\begin{aligned} &\mathrm{gyr}[\mathbf{u}\oplus\mathbf{v},\mathbf{v}] = \mathrm{gyr}[\mathbf{u},\mathbf{v}] && \text{Left Loop Property} \\ &\mathrm{gyr}[\mathbf{u},\mathbf{v}\oplus\mathbf{u}] = \mathrm{gyr}[\mathbf{u},\mathbf{v}] && \text{Right Loop Property} \end{aligned} \tag{24}$$

for all $\mathbf{u},\mathbf{v} \in \mathbb{R}^3_c$. The loop property proves useful in (28) below and, more generally, in Ref. [5].

The identities in (19) – (24) are verified by lengthy but straightforward algebra that one can readily calculate by using a computer software for symbolic manipulation like MATHEMATICA or MAPLE. Hence, the task of proving these identities can be given to students as exercises. For students who wish to verify (21) we may note that the automorphism identity in (21) is equivalent to the vector identity

$$\mathrm{gyr}[\mathbf{v},\mathbf{u}]\mathrm{gyr}[\mathbf{u},\mathbf{v}]\mathbf{w} = \mathbf{w} \tag{25}$$

in the ball of all relativistically admissible velocities $\mathbb{R}^3_c$ for all $\mathbf{u},\mathbf{v},\mathbf{w}\in\mathbb{R}^3_c$. A similar remark applies to the identities in (24) as well. For instance, to verify the first identity in (24), which is an identity between maps, one has to verify the identity

$$\mathrm{gyr}[\mathbf{u}\oplus\mathbf{v},\mathbf{v}]\mathbf{w} = \mathrm{gyr}[\mathbf{u},\mathbf{v}]\mathbf{w} \tag{26}$$

in the ball $\mathbb{R}^3_c$, for all $\mathbf{u},\mathbf{v},\mathbf{w} \in \mathbb{R}^3_c$, which is an identity between vectors. Noting the definition of $\oplus$ and gyr in (3) and (13), one can verify (26) by straightforward algebra with the help of a computer software for symbolic manipulation.

The prefix "gyro" that stems from Thomas gyration is extensively used to emphasize analogies with classical terms as, for instance, gyrocommutative, gyroassociative binary operations in gyrogroups [23], and gyrovector spaces [5, 24]; see fn. 36 in Ref. [20]. The history of the use of the resulting notion of the gyrogroup in relativity physics is presented on pp. 141 – 142 of Ref. [10].

We can now harness the Thomas precession for work in relativity physics as well as in hyperbolic geometry, as the following illustrative example of solving the equation

$$\mathbf{x}\oplus\mathbf{a} = \mathbf{b} \tag{27}$$

for Einsteinian velocities demonstrates.

Assuming that $\mathbf{x}$ satisfies (27), we wish to solve this equation for the unknown $\mathbf{x}$ and any given relativistically admissible velocities $\mathbf{a}$ and $\mathbf{b}$. Employing the left gyroassociative law and the left loop property in (23) – (24), we have the chain of equations

$$\begin{aligned} \mathbf{x} &= \mathbf{x}\oplus\mathbf{0} \\ &= \mathbf{x}\oplus(\mathbf{a}\ominus\mathbf{a}) \\ &= (\mathbf{x}\oplus\mathbf{a})\oplus\mathrm{gyr}[\mathbf{x},\mathbf{a}](\ominus\mathbf{a}) \\ &= (\mathbf{x}\oplus\mathbf{a})\ominus\mathrm{gyr}[\mathbf{x},\mathbf{a}]\mathbf{a} \\ &= (\mathbf{x}\oplus\mathbf{a})\ominus\mathrm{gyr}[\mathbf{x}\oplus\mathbf{a},\mathbf{a}]\mathbf{a} \\ &= \mathbf{b}\ominus\mathrm{gyr}[\mathbf{b},\mathbf{a}]\mathbf{a} \end{aligned} \tag{28}$$

thus verifying that the solution of the equation $\mathbf{x}\oplus\mathbf{a} = \mathbf{b}$ in the ball $\mathbb{R}^3_c$ of Einsteinian velocities is the velocity $\mathbf{x} = \mathbf{b}\ominus\mathrm{gyr}[\mathbf{b},\mathbf{a}]\mathbf{a}$ in the ball. Remarkably, the solution involves a Thomas precession.

In a similar way, also the equation

$$\mathbf{a}\oplus\mathbf{x} = \mathbf{b} \tag{29}$$

can be solved in $\mathbb{R}^3_c$ for the unknown $\mathbf{x}$ and any given $\mathbf{a},\mathbf{b}\in\mathbb{R}^3_c$. Assuming that $\mathbf{x}$ satisfies (29), and employing the right gyroassociative law and (18), we have the chain of equations

$$\begin{aligned} \mathbf{x} &= \mathbf{0}\oplus\mathbf{x} \\ &= (\ominus\mathbf{a}\oplus\mathbf{a})\oplus\mathbf{x} \\ &= \ominus\mathbf{a}\oplus(\mathbf{a}\oplus\mathrm{gyr}[\mathbf{a},\ominus\mathbf{a}]\mathbf{x}) \\ &= \ominus\mathbf{a}\oplus(\mathbf{a}\oplus\mathbf{x}) \\ &= \ominus\mathbf{a}\oplus\mathbf{b} \end{aligned} \tag{30}$$

thus verifying that the solution of the equation $\mathbf{a}\oplus\mathbf{x} = \mathbf{b}$ in $\mathbb{R}^3_c$ is the velocity $\mathbf{x} = \ominus\mathbf{a}\oplus\mathbf{b}$. Unlike (28), the solution in (30) does not involve a Thomas precession.

Being able to solve the relativistic velocity equations (27) and (29) in (28) and (30), astronauts who want to journey around the universe can now manipulate relativistic velocities in a way fully analogous to the manipulation of Newtonian velocities.

Guided by analogies with classical results, the solution (30) of the equation in (29) seems more "natural" than the solution (28) of the equation in (27). It will prove useful to make the solution (28) appear "natural" as well. For this purpose we define the "Einstein coaddition" (or, dual addition) $\boxplus$ by the equation

$$\mathbf{a} \boxplus \mathbf{b} = \mathbf{a}\oplus\mathrm{gyr}[\mathbf{a},\ominus\mathbf{b}]\mathbf{b} \tag{31}$$

Then, if we use the notation $\mathbf{a} \boxminus \mathbf{b} = \mathbf{a} \boxplus (\ominus\mathbf{b})$, we have the following desired result,

$$\begin{aligned} \mathbf{a} \boxminus \mathbf{b} &= \mathbf{a} \boxplus (\ominus\mathbf{b}) \\ &= \mathbf{a}\oplus\mathrm{gyr}[\mathbf{a},\mathbf{b}](\ominus\mathbf{b}) \\ &= \mathbf{a}\oplus(\ominus\mathrm{gyr}[\mathbf{a},\mathbf{b}]\mathbf{b}) \\ &= \mathbf{a}\ominus\mathrm{gyr}[\mathbf{a},\mathbf{b}]\mathbf{b} \end{aligned} \tag{32}$$

allowing the solution (28) of the relativistic velocity equation in (27) to be written as

$$\mathbf{x} = \mathbf{b}\ominus\mathrm{gyr}[\mathbf{b},\mathbf{a}]\mathbf{a} = \mathbf{b} \boxminus \mathbf{a} \tag{33}$$

which now shares a notational analogy with its classical counterpart.

Unexpected elegant analogies to which the Einstein coaddition gives rise will be uncovered in (50), (54) – (55) and in (96) – (97). The mathematical beauty that Einstein coaddition introduces is revealed by the duality symmetries that it captures, like the one exhibited in the following pair of identities,

$$\begin{aligned} \mathbf{a} \boxplus \mathbf{b} &= \mathbf{a}\oplus\mathrm{gyr}[\mathbf{a},\ominus\mathbf{b}]\mathbf{b} \\ \mathbf{a}\oplus\mathbf{b} &= \mathbf{a} \boxplus \mathrm{gyr}[\mathbf{a},\mathbf{b}]\mathbf{b} \end{aligned} \tag{34}$$

for all $\mathbf{a},\mathbf{b}\in\mathbb{R}^3_c$. This and other duality symmetries that stem from Einstein coaddition are presented in Ref. [5].

VI. THOMAS PRECESSION

In this section we present the Thomas precession of special relativity, and explain why the mathematical object called "Thomas gyration" that we presented in Section V is realized experimentally by the Thomas precession of special relativity.

Owing to the gyrocommutative law in (23), Thomas gyration is recognized as the familiar Thomas precession, since the gyrocommutative law was already known to Silberstein in 1914 in the following sense [19]. The Thomas precession generated by $\mathbf{u},\mathbf{v}\in\mathbb{R}^3_c$ is the unique rotation that takes $\mathbf{v}\oplus\mathbf{u}$ into $\mathbf{u}\oplus\mathbf{v}$ about an axis perpendicular to the plane of $\mathbf{u}$ and $\mathbf{v}$ through an angle $< \pi$ [18], as explained below (11). Obviously, Silberstein did not use the terms "Thomas precession" and "gyrocommutative law" which have been coined later, respectively, following Thomas' 1926 paper [25], and in 1991 [23].

An excellent description of the 3-space rotation, which since 1926 is named after Thomas, is found in Silberstein's 1914 book, Ref. [19]. In 1914 the Thomas precession did not have a name, and Silberstein called it in his 1914 book a "certain space-rotation" (p. 169 in Ref. [19]). An early study of the Thomas rotation, made by the famous mathematician Emile Borel in 1913, is described in his 1914 book, Ref. [26], and more recently, in Ref. [27]. According to Belloni and Reina [28], Sommerfeld's route to the Thomas precession dates back to 1909. The only knowledge Thomas had in 1925 about the peculiar relativistic gyroscopic precession, however, came from De Sitter's formula describing the relativistic corrections for the motion of the moon, found in Eddington's book [29], which was just published at that time [5].

The physical significance of the peculiar rotation in special relativity emerged in 1925 when Thomas relativistically re-computed the precessional frequency of the doublet separation in the fine structure of the atom, and thus rectified a missing factor of $1/2$. This correction has come to be known as the *Thomas half*. Thomas' discovery of the relativistic precession of the electron spin on Christmas 1925 thus led to the understanding of the significance of the relativistic effect which became known as the *Thomas precession*. Llewellyn Hilleth Thomas died in Raleigh, NC, on April 20, 1992. A paper dedicated to the centenary of the birth of Llewellyn H. Thomas (1902 – 1992) has recently appeared [30].

Once identified as $\mathrm{gyr}[\mathbf{u},\mathbf{v}]$, it is clear from its definition in (13) that Thomas precession owes its existence solely to the nonassociativity of Einstein addition of Einsteinian velocities. Accordingly, Thomas precession has no classical counterpart since the addition of Newtonian velocities is associative.

Physics students are commonly taught that special relativistic effects are negligible when the velocities involved are much less than the vacuum speed of light c. Yet, Thomas precession effect in the orbital motion of spinning electrons in atoms is clearly observed in resulting spectral lines despite the speed of electrons in atoms being small compared with the speed of light. One may, therefore, ask whether it is possible to furnish a classical background to Thomas precession [31]. It is therefore important to realize that Thomas precession has no echo in classical mechanics since the addition of Newtonian velocities is associative.

VII. THE RELATIVISTIC GYROVECTOR SPACE

Modeled on Einstein's relativistic velocity groupoid $(\mathbb{R}^3_c,\oplus)$ and guided by analogies with group theory, the definition of the abstract gyrogroup and the resulting theory of gyrogroups and gyrovector spaces are presented in Ref. [5]. Einstein addition admits scalar multiplication $\otimes$ turning the Einsteinian velocity gyrogroup $(\mathbb{R}^3_c,\oplus)$ into a gyrovector space $(\mathbb{R}^3_c,\oplus,\otimes)$ in the same way that the common vector addition admits scalar multiplication that turns the Newtonian velocity group $(\mathbb{R}^3,+)$ into the vector space $(\mathbb{R}^3,+,\cdot)$. Hence, by analogy, in the same way that we may consider $(\mathbb{R}^3,+,\cdot)$ as a Newtonian velocity vector space, we consider $(\mathbb{R}^3_c,\oplus,\otimes)$ as an Einsteinian velocity gyrovector space.

The Einstein scalar multiplication that Einstein addition admits is given by the equation

$$\begin{aligned} r\otimes\mathbf{v} &= c\frac{(1+\|\mathbf{v}\|/c)^r-(1-\|\mathbf{v}\|/c)^r}{(1+\|\mathbf{v}\|/c)^r+(1-\|\mathbf{v}\|/c)^r}\frac{\mathbf{v}}{\|\mathbf{v}\|} \\ &= c\tanh\left(r\tanh^{-1}\frac{\|\mathbf{v}\|}{c}\right)\frac{\mathbf{v}}{\|\mathbf{v}\|} \end{aligned} \tag{35}$$

where r is any real number, $r\in\mathbb{R}$, $\mathbf{v}\in\mathbb{R}^3_c$, $\mathbf{v}\neq\mathbf{0}$, and $r\otimes\mathbf{0}=\mathbf{0}$, and with which we use the notation $\mathbf{v}\otimes r = r\otimes\mathbf{v}$.

Einstein addition and scalar multiplication possess the following properties. For any positive integer n, and for all real numbers $r, r_1, r_2 \in\mathbb{R}$ and admissible velocities $\mathbf{v}\in\mathbb{R}^3_c$,

$$\begin{aligned} n\otimes\mathbf{v} &= \mathbf{v}\oplus\ldots\oplus\mathbf{v} && n \text{ terms} \\ (r_1+r_2)\otimes\mathbf{v} &= r_1\otimes\mathbf{v}\oplus r_2\otimes\mathbf{v} && \text{Scalar Distributive Law} \\ (r_1 r_2)\otimes\mathbf{v} &= r_1\otimes(r_2\otimes\mathbf{v}) && \text{Scalar Associative Law} \end{aligned}$$

We may note that, unlike vector spaces, the Einstein gyrovector space does not possess a distributive law since, in general,

$$r\otimes(\mathbf{u}\oplus\mathbf{v}) \neq r\otimes\mathbf{u}\oplus r\otimes\mathbf{v} \tag{36}$$

for $r\in\mathbb{R}$ and $\mathbf{u},\mathbf{v}\in\mathbb{R}^3_c$.

As an example, the Einstein half is given by the equation

$$\tfrac{1}{2}\otimes\mathbf{v} = \frac{\gamma_{\mathbf{v}}}{1+\gamma_{\mathbf{v}}}\mathbf{v} \tag{37}$$

so that, accordingly,

$$\begin{aligned}2\otimes(\tfrac{1}{2}\otimes\mathbf{v}) &= 2\otimes\frac{\gamma_{\mathbf{v}}}{1+\gamma_{\mathbf{v}}}\mathbf{v}\\ &= \frac{\gamma_{\mathbf{v}}}{1+\gamma_{\mathbf{v}}}\mathbf{v}\oplus\frac{\gamma_{\mathbf{v}}}{1+\gamma_{\mathbf{v}}}\mathbf{v}\\ &= \mathbf{v}\end{aligned} \tag{38}$$

as expected from the scalar associative law of the Einstein gyrovector space.

Remarkably, the Einstein gyrovector space $(\mathbb{R}^3_c, \oplus, \otimes)$ of Einsteinian velocities with its hyperbolic distance function given by the equation

$$d_{\oplus}(\mathbf{u},\mathbf{v}) = \|\mathbf{u}\ominus\mathbf{v}\| \tag{39}$$

$\mathbf{u},\mathbf{v}\in\mathbb{R}^3_c$, forms the setting for the Beltrami ball model of 3-dimensional hyperbolic geometry just as the vector space $(\mathbb{R}^3, +, \cdot)$ of Newtonian velocities with its Euclidean distance function

$$d_{+}(\mathbf{u},\mathbf{v}) = \|\mathbf{u}-\mathbf{v}\| \tag{40}$$

forms the setting for the standard model of 3-dimensional Euclidean geometry. Applications of gyrovector spaces in relativity physics and hyperbolic geometry are found in Ref. [5], and applications in quantum computation geometry are found in Refs. [24] and [30]. The connection between Einstein addition and the Beltrami ball model of 3-dimensional hyperbolic geometry has been noted by Fock on p. 39 of Ref. [21], and is demonstrated below, in (41) – (43), in a simple way.

To see the connection between Einstein addition and the Beltrami ball model of hyperbolic geometry we use the notation

$$ds^2 = \|\mathbf{x}\ominus(\mathbf{x}+d\mathbf{x})\|^2 \tag{41}$$

describing the Riemannian line element ds^2 as the squared Einsteinian distance (39) between a point $\mathbf{x}\in\mathbb{R}^3_c$ and its infinitesimally close neighboring point $\mathbf{x}+d\mathbf{x}\in\mathbb{R}^3_c$. It links the Riemannian line element ds^2 with both (i) Einstein addition $\oplus$ and subtraction $\ominus$ and (ii) the Einstein hyperbolic distance function (39). In the two dimensional case we have $\mathbf{x} = (x_1, x_2)$ and $d\mathbf{x} = (dx_1, dx_2)$ in (41). Taylor series expansion of ds^2, viewed as a function of the two variables dx_1 and dx_2, about the origin $(0, 0)$ gives

$$ds^2 = Edx_1^2 + 2Fdx_1dx_2 + Gdx_2^2 + \dots \tag{42}$$

where, if we use the notation $r^2 = x_1^2 + x_2^2$,

$$\begin{aligned}E &= c^2\frac{c^2-x_2^2}{(c^2-r^2)^2}\\ F &= c^2\frac{x_1x_2}{(c^2-r^2)^2}\\ G &= c^2\frac{c^2-x_1^2}{(c^2-r^2)^2}\end{aligned} \tag{43}$$

In differential geometry the functions E, F and G of (x_1, x_2) are known as the *coefficients of the first fundamental form* in the Cartesian coordinates (x_1, x_2); see p. 93 in Ref. [32].

The triple $(g_{11}, g_{12}, g_{22}) = (E, F, G)$ with $g_{21} = g_{12}$ is known in differential geometry as the metric tensor g_{ij} [33]. It turns out to be the metric tensor of the Beltrami disc model of hyperbolic geometry; see, for instance, p. 220 in Ref. [34]. Hence, ds^2 in (41) – (42) is the Riemannian line element of the Beltrami disc model of hyperbolic geometry, linked to Einstein addition (3) and to the Einstein hyperbolic distance function (39). More links between Einstein addition and the Beltrami model, as well as analogies with classical results will be uncovered in Section VIII.

VIII. HYPERBOLIC GEODESICS, MIDPOINTS AND ANGLES

As illustrative examples we present basic expressions in analytic hyperbolic geometry, (45) – (46) and (51), that shed light on the analogies that our approach in this article uncovers.

A point $\mathbf{v}$ in the Einstein gyrovector space $(\mathbb{R}^3_c, \oplus, \otimes)$ or the Einstein gyrovector plane $(\mathbb{R}^2_c, \oplus, \otimes)$, shown in Fig. 1, represents all the inertial frames $\Sigma_{\mathbf{v}}$ with relativistically admissible velocity $\mathbf{v}$ relative to the rest frame $\Sigma_{\mathbf{0}}$. Accordingly, the relativistic velocity of frame $\Sigma_{\mathbf{v}}$ relative to frame $\Sigma_{\mathbf{u}}$ is $\ominus\mathbf{u}\oplus\mathbf{v}$, and the relativistic velocity of frame $\Sigma_{\mathbf{u}}$ relative to frame $\Sigma_{\mathbf{v}}$ is $\ominus\mathbf{v}\oplus\mathbf{u}$. Remarkably, these two Einsteinian reciprocal velocities are not reciprocal in the classical sense since they are related by the identity

$$\ominus\mathbf{u}\oplus\mathbf{v} = \ominus(\mathbf{u}\ominus\mathbf{v}) = \ominus\mathrm{gyr}[\mathbf{u},\ominus\mathbf{v}](\ominus\mathbf{v}\oplus\mathbf{u}) \tag{44}$$

that involves a Thomas precession. Identity (44) is obtained by employing the *automorphic inverse property* of Einstein addition, according to which $\ominus(\mathbf{a}\oplus\mathbf{b}) = \ominus\mathbf{a}\ominus\mathbf{b}$, and the gyrocommutative law. The classical counterpart of (44), $-\mathbf{u}+\mathbf{v} = -(-\mathbf{v}+\mathbf{u})$, is known as the principle of reciprocity.

Astrophysicists and space travelers who wish to measure angles between relativistic velocities must invoke the Einstein hyperbolic angle measure presented in (51) below and in Fig. 2.

For any points $\mathbf{u},\mathbf{v}\in\mathbb{R}^3_c$, and the real parameter $t\in\mathbb{R}$ that may represent "time", $-\infty < t < \infty$, the curve

$$\mathbf{u}\oplus(\ominus\mathbf{u}\oplus\mathbf{v})\otimes t \tag{45}$$

called a *gyroline*, traces a geodesic line in the Beltrami ball model of hyperbolic geometry, as shown in Fig. 1. It is the unique geodesic passing through the points $\mathbf{u}$ and $\mathbf{v}$. It passes through the point $\mathbf{u}$ at "time" $t = 0$, and through the point $\mathbf{v}$ at "time" $t = 1$. Geodesics in the Beltrami ball model of hyperbolic geometry are the analog of straight lines in Euclidean geometry, representing

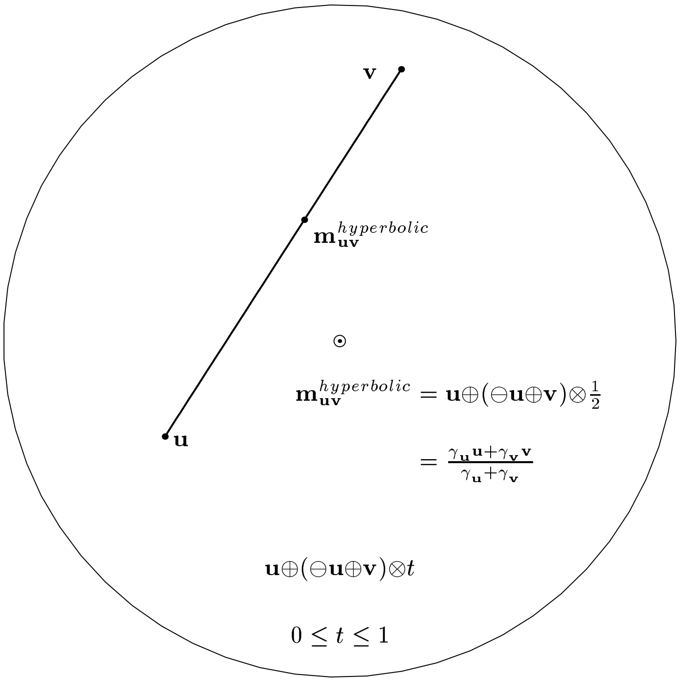


FIG. 1: Points of the disc $\mathbb{R}_c^2=\{\mathbf{v}\in\mathbb{R}^2 : \|\mathbf{v}\| < c\}$ represent two-dimensional relativistically admissible velocities $\mathbf{v}$ relative to some inertial rest frame with velocity $\mathbf{0}$. The disc $\mathbb{R}_c^2$ is endowed with Einstein addition $\oplus$ and scalar multiplication $\otimes$, giving rise to the Einstein gyrovector disc $(\mathbb{R}_c^2, \oplus, \otimes)$, that turns out to be the Beltrami disc model of two-dimensional hyperbolic geometry. The geodesic segment joining the points $\mathbf{u}$ and $\mathbf{v}$ in the Einstein relativistic gyrovector disc $(\mathbb{R}_c^2, \oplus, \otimes)$ and the hyperbolic midpoint $\mathbf{m}_{\mathbf{uv}}^{hyperbolic}$ between $\mathbf{u}$ and $\mathbf{v}$ are shown. The expressions that generate the geodesic segment and its midpoint exhibit analogies that (1i) the hyperbolic geodesic shares with (1ii) its Euclidean counterpart; that (2i) the hyperbolic midpoint shares with (2ii) its Euclidean counterpart; and, as we will see in Section IX, that (3i) the Einsteinian velocity interpretation of the hyperbolic midpoint as a relativistic center of momentum shares with (3ii) the Newtonian velocity interpretation of the Euclidean midpoint as a classical center of momentum.

paths that minimize arc lengths measured by the distance function (39). These geodesics are therefore also called *gyrolines* in order to emphasize analogies.

For $t = 1/2$ in (45) we have the hyperbolic midpoint $\mathbf{m}_{\mathbf{uv}}$,

$$\mathbf{m}_{\mathbf{uv}} = \mathbf{u}\oplus(\ominus\mathbf{u}\oplus\mathbf{v})\otimes\tfrac{1}{2} \tag{46}$$

satisfying

$$\mathbf{m}_{\mathbf{uv}} = \mathbf{m}_{\mathbf{vu}} \tag{47}$$

and

$$d_\oplus(\mathbf{u}, \mathbf{m}_{\mathbf{uv}}) = d_\oplus(\mathbf{v}, \mathbf{m}_{\mathbf{uv}}) \tag{48}$$

The mathematical proof of (47) – (48) is found in [5], where it is shown that the hyperbolic midpoint $\mathbf{m}_{\mathbf{uv}}$ can also be written as

$$\mathbf{m}_{\mathbf{uv}} = \frac{\gamma_{\mathbf{u}}\mathbf{u} + \gamma_{\mathbf{v}}\mathbf{v}}{\gamma_{\mathbf{u}} + \gamma_{\mathbf{v}}} \tag{49}$$

and as

$$\mathbf{m}_{\mathbf{uv}} = \tfrac{1}{2}\otimes(\mathbf{u} \boxplus \mathbf{v}) \tag{50}$$

In Section IX we will uncover the relativistic mechanical interpretation of the hyperbolic midpoint $\mathbf{m}_{\mathbf{uv}}$, resulting in the emergence of a novel, elegant analogy that it shares with the Euclidean midpoint and its classical mechanical interpretation.

The extension of the analogy uncovered in (45) to angles is straightforward. For any points $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}_c^3$ the hyperbolic angle $\alpha = \angle\mathbf{vuw}$, generated by the intersecting geodesics that respectively pass through the points $\mathbf{u}, \mathbf{v}$ and $\mathbf{u}, \mathbf{w}$, and therefore intersect at the point $\mathbf{u}$, is given by the equation

$$\cos\alpha = \frac{\ominus\mathbf{u}\oplus\mathbf{v}}{\|\ominus\mathbf{u}\oplus\mathbf{v}\|}\cdot\frac{\ominus\mathbf{u}\oplus\mathbf{w}}{\|\ominus\mathbf{u}\oplus\mathbf{w}\|} \tag{51}$$

shown in Fig. 2. Thus, for instance, the three hyperbolic angles of a hyperbolic triangle $\Delta\mathbf{uvw}$, Fig. 4, are given in terms of Einstein addition and scalar product by the equations

$$\begin{aligned}
\cos\alpha &= \frac{\ominus\mathbf{u}\oplus\mathbf{v}}{\|\ominus\mathbf{u}\oplus\mathbf{v}\|}\cdot\frac{\ominus\mathbf{u}\oplus\mathbf{w}}{\|\ominus\mathbf{u}\oplus\mathbf{w}\|} \\
\cos\beta &= \frac{\ominus\mathbf{v}\oplus\mathbf{u}}{\|\ominus\mathbf{v}\oplus\mathbf{u}\|}\cdot\frac{\ominus\mathbf{v}\oplus\mathbf{w}}{\|\ominus\mathbf{v}\oplus\mathbf{w}\|} \\
\cos\gamma &= \frac{\ominus\mathbf{w}\oplus\mathbf{v}}{\|\ominus\mathbf{w}\oplus\mathbf{u}\|}\cdot\frac{\ominus\mathbf{w}\oplus\mathbf{v}}{\|\ominus\mathbf{w}\oplus\mathbf{v}\|}
\end{aligned} \tag{52}$$

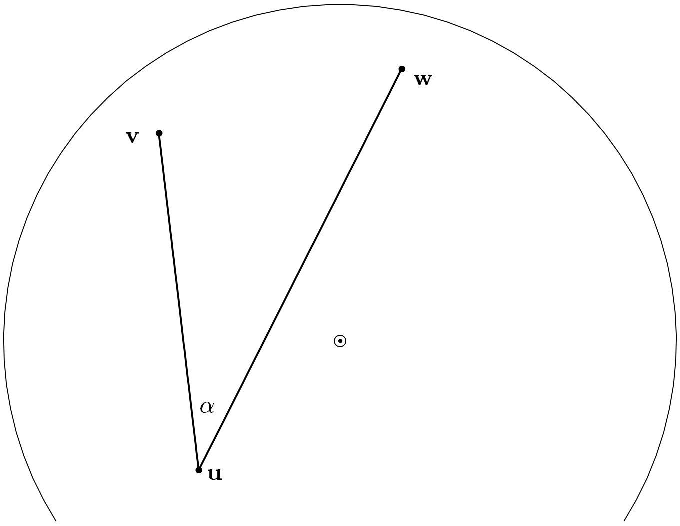
As expected in hyperbolic geometry, the triangle angle sum is less than π,

$$\alpha + \beta + \gamma < \pi \tag{53}$$

Surprisingly, Thomas precession is able to “repair” the lack of equality in (53), as we will see in (54) – (55).

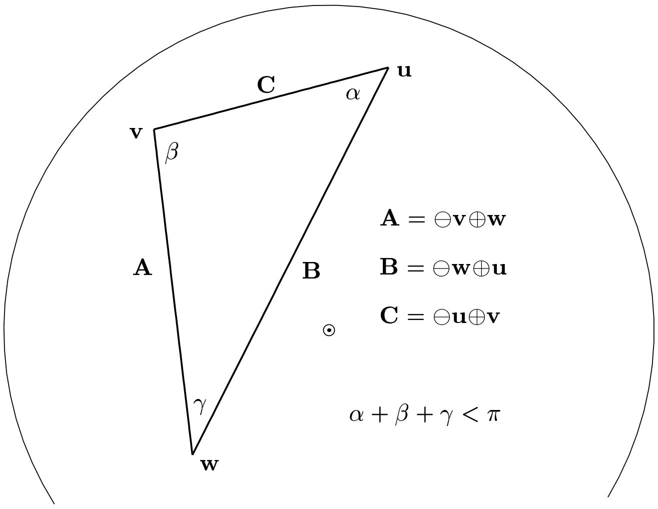
Fig. 2 shows three points, $\mathbf{u}, \mathbf{v}, \mathbf{w}$, in the Einstein gyrovector plane $(\mathbb{R}_c^2, \oplus, \otimes)$. They represent three inertial frames, $\Sigma_{\mathbf{u}}$, $\Sigma_{\mathbf{v}}$, $\Sigma_{\mathbf{w}}$, with relativistic velocities $\mathbf{u}, \mathbf{v}, \mathbf{w}$ relative to a rest frame $\Sigma_{\mathbf{0}}$. The velocities of frames $\Sigma_{\mathbf{v}}$ and $\Sigma_{\mathbf{w}}$ relative to frame $\Sigma_{\mathbf{u}}$ are, respectively, $\ominus\mathbf{u}\oplus\mathbf{v}$ and $\ominus\mathbf{u}\oplus\mathbf{w}$. Astrophysicists and space travelers who wish to measure the angle between these relative velocities will have to invoke the Einstein hyperbolic angle measure (51) shown in Figs. 2 and 3.

The Einstein hyperbolic angle measure (51) coincides with the common hyperbolic angle measure in the Beltrami ball model of hyperbolic geometry. To gain experience and confidence with the use of this angle measure students may be encouraged to corroborate numerically the *Einstein hyperbolic sine rule* shown in Fig. 3, which



$$\boxed{\cos\alpha = \frac{\ominus\mathbf{u}\oplus\mathbf{v}}{\|\ominus\mathbf{u}\oplus\mathbf{v}\|}\cdot\frac{\ominus\mathbf{u}\oplus\mathbf{w}}{\|\ominus\mathbf{u}\oplus\mathbf{w}\|}}$$

FIG. 2: The Einstein hyperbolic angle measure in an Einstein gyrovector space.



$$\boxed{\frac{\gamma_{\mathbf{A}}\|\mathbf{A}\|}{\sin\alpha} = \frac{\gamma_{\mathbf{B}}\|\mathbf{B}\|}{\sin\beta} = \frac{\gamma_{\mathbf{C}}\|\mathbf{C}\|}{\sin\gamma}}$$

FIG. 3: The Einstein hyperbolic sine rule in an Einstein gyrovector space.

exhibit obvious analogies with its Euclidean counterpart. The mathematical proof of the Einstein hyperbolic sine rule is found in Ref. [5].

To see once more the mysterious ability of Thomas precession to capture analogies with classical results, we rewrite (52) with Einstein addition $\oplus$, (3), replaced by Einstein coaddition $\boxplus$, (34), obtaining the following hyperbolic triangle *coangles* of the hyperbolic triangle $\Delta\mathbf{uvw}$,

$$\begin{aligned}
\cos\alpha' &= \frac{\boxminus\mathbf{u}\boxplus\mathbf{v}}{\|\boxminus\mathbf{u}\boxplus\mathbf{v}\|}\cdot\frac{\boxminus\mathbf{u}\boxplus\mathbf{w}}{\|\boxminus\mathbf{u}\boxplus\mathbf{w}\|}\\
\cos\beta' &= \frac{\boxminus\mathbf{v}\boxplus\mathbf{u}}{\|\boxminus\mathbf{v}\boxplus\mathbf{u}\|}\cdot\frac{\boxminus\mathbf{v}\boxplus\mathbf{w}}{\|\boxminus\mathbf{v}\boxplus\mathbf{w}\|}\\
\cos\gamma' &= \frac{\boxminus\mathbf{w}\boxplus\mathbf{v}}{\|\boxminus\mathbf{w}\boxplus\mathbf{u}\|}\cdot\frac{\boxminus\mathbf{w}\boxplus\mathbf{v}}{\|\boxminus\mathbf{w}\boxplus\mathbf{v}\|}
\end{aligned}\tag{54}$$

where we use the notation $\boxminus\mathbf{u} = -\mathbf{u}$, etc.

For pedagogical purposes students are encouraged to corroborate numerically the surprising result,

$$\alpha' + \beta' + \gamma' = \pi \tag{55}$$

for any hyperbolic triangle $\Delta\mathbf{uvw}$ in the Einstein gyrovector space $(\mathbb{R}^3_c, \oplus, \otimes)$. Identity (55) captures a most elegant analogy with Euclidean geometry, according to which the hyperbolic triangle coangle sum is always π. Interested readers may find the mathematical proof of (55) in [5]. Students who experience the validity of (55) by corroborating it numerically are likely to be impressed by the power and elegance that the relativistic effect of the Thomas precession stores.

IX. THE EUCLIDEAN AND THE HYPERBOLIC MIDPOINT – NEWTONIAN AND EINSTEINIAN MECHANICAL INTERPRETATION

The Euclidean midpoint in Newtonian velocity spaces has the well-known classical mechanical interpretation as the center of momentum of two moving particles with equal masses [35], as we explain below. Similarly, we will show that the hyperbolic midpoint in Einsteinian velocity spaces has the relativistic mechanical interpretation as the center of relativistic momentum of two moving particles with equal relativistic rest masses, thus uncovering new analogies.

Let us consider two particles with equal masses m moving with Newtonian velocities $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$. Their respective momenta are $m\mathbf{u}$ and $m\mathbf{v}$ so that their classical center of momentum is

$$\mathbf{c}_{\mathbf{uv}}^{newtonian} = \frac{m\mathbf{u} + m\mathbf{v}}{m+m} = \frac{\mathbf{u}+\mathbf{v}}{2} \tag{56}$$

It turns out that the Newtonian center of momentum (56) coincides with the Euclidean midpoint

$$\mathbf{m}_{\mathbf{uv}}^{euclidean} = \frac{\mathbf{u}+\mathbf{v}}{2} \tag{57}$$

of $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$. The Euclidean midpoint (57) thus has the Newtonian classical mechanical interpretation (56) as a center of momentum,

$$\mathbf{m}_{\mathbf{uv}}^{euclidean} = \mathbf{c}_{\mathbf{uv}}^{newtonian} \tag{58}$$

By analogy with (56) and (57) let us consider two particles with equal rest masses m moving with Einsteinian velocities $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3_c$. Their respective momenta must be

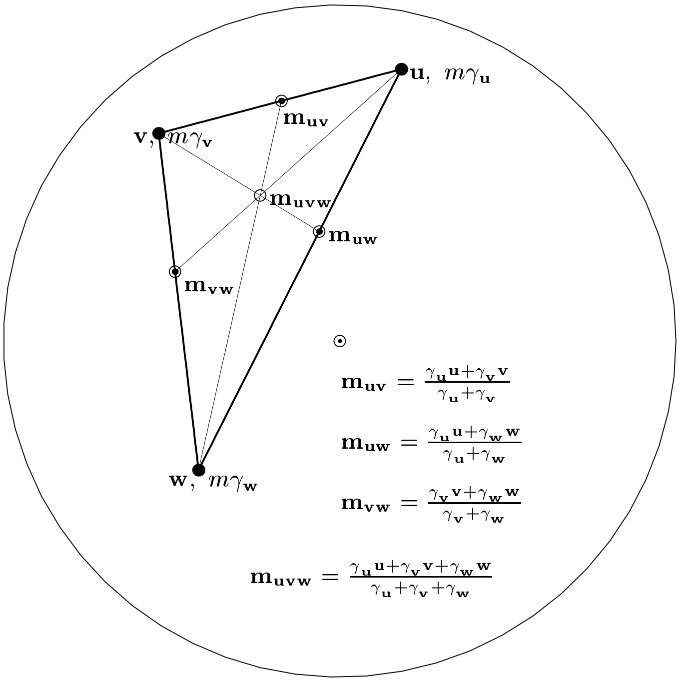


FIG. 4: The hyperbolic centroid $\mathbf{m}_{\mathbf{uvw}}^{hyperbolic} = \mathbf{m_{uvw}}$ of a hyperbolic triangle $\Delta\mathbf{uvw}$ in the Einsteinian velocity gyrovector space $(\mathbb{R}_c^2, \oplus, \otimes)$ coincides with the relativistic center of momentum of three massive objects with equal rest masses situated at the vertices $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$, of the triangle $\Delta\mathbf{uvw}$, as shown in (64). The analogous Newtonian counterpart is well known, and is obtained in the Newtonian limit, $c \to \infty$.

relativistically corrected so that they are, respectively, $m\gamma_{\mathbf{u}}\mathbf{u}$ and $m\gamma_{\mathbf{v}}\mathbf{v}$. Their center of momentum is, accordingly,

$$\begin{aligned} \mathbf{c}_{\mathbf{uv}}^{einsteinian} &= \frac{m\gamma_{\mathbf{u}} + m\gamma_{\mathbf{v}}\mathbf{v}}{m\gamma_{\mathbf{u}} + m\gamma_{\mathbf{v}}} \\ &= \frac{\gamma_{\mathbf{u}}\mathbf{u} + \gamma_{\mathbf{v}}\mathbf{v}}{\gamma_{\mathbf{u}} + \gamma_{\mathbf{v}}} \end{aligned} \tag{59}$$

It turns out that the Einsteinian center of momentum (59) coincides with the hyperbolic midpoint (49), that is,

$$\mathbf{m}_{\mathbf{uv}}^{hyperbolic} = \frac{\gamma_{\mathbf{u}}\mathbf{u} + \gamma_{\mathbf{v}}\mathbf{v}}{\gamma_{\mathbf{u}} + \gamma_{\mathbf{v}}} \tag{60}$$

of $\mathbf{u}, \mathbf{v} \in \mathbb{R}_c^3$. The hyperbolic midpoint (60) thus has the Einsteinian relativistic mechanical interpretation (59) as a relativistic center of momentum,

$$\mathbf{m}_{\mathbf{uv}}^{hyperbolic} = \mathbf{c}_{\mathbf{uv}}^{einsteinian} \tag{61}$$

in full analogy with the Newtonian classical mechanical interpretation (56) of the Euclidean midpoint (57), presented in (58). The problem of understanding the relativistic center of momentum has long, checkered history described in Ref. [36].

In the analogy (58) $\leftrightarrow$ (61) between the pairs (1) we have employed the hyperbolic geometry of Bolyai and Lobachevski to improve our understanding of both the hyperbolic midpoint of the Beltrami model and the relativistic center of momentum of two particles in special relativity in terms of analogies they share with their respective classical counterparts.

In Fig. 4 we extend the observations made in Fig. 1 from two relativistically moving particles with equal rest masses to three. The three vertices of the hyperbolic triangle $\Delta\mathbf{uvw}$ in Fig. 4 represent three particles with equal rest masses m moving with relativistic velocities $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}_c^3$ relative to some rest frame $\Sigma_{\mathbf{0}}$. Relativistically corrected, the relativistic momenta of the three particles are, respectively, $m\gamma_{\mathbf{u}}\mathbf{u}$, $m\gamma_{\mathbf{v}}\mathbf{v}$, and $m\gamma_{\mathbf{w}}\mathbf{w}$, so that their relativistic center of momentum is

$$\begin{aligned} \mathbf{c}_{\mathbf{uvw}}^{einsteinian} &= \frac{m\gamma_{\mathbf{u}}\mathbf{u} + m\gamma_{\mathbf{v}}\mathbf{v} + m\gamma_{\mathbf{w}}\mathbf{w}}{m\gamma_{\mathbf{u}} + m\gamma_{\mathbf{v}} + m\gamma_{\mathbf{w}}} \\ &= \frac{\gamma_{\mathbf{u}}\mathbf{u} + \gamma_{\mathbf{v}}\mathbf{v} + \gamma_{\mathbf{w}}\mathbf{w}}{\gamma_{\mathbf{u}} + \gamma_{\mathbf{v}} + \gamma_{\mathbf{w}}} \end{aligned} \tag{62}$$

This will turn out below to be the hyperbolic centroid $\mathbf{m}_{\mathbf{uvw}}^{hyperbolic}$ of the hyperbolic triangle $\Delta\mathbf{uvw}$, Fig. 4, as well.

The geodesic segment in triangle $\Delta\mathbf{uvw}$, Fig. 4, joining a midpoint of a side with its opposite vertex is called a *median*. As in Euclidean geometry, the three hyperbolic triangle medians are concurrent, the point of concurrency being the triangle centroid [37]. Fig. 4 shows that, as expected, the three medians of the hyperbolic triangle $\Delta\mathbf{uvw}$ are concurrent. The point of concurrency, $\mathbf{m_{uvw}} = \mathbf{m}_{\mathbf{uvw}}^{hyperbolic}$, is called the hyperbolic centroid of the hyperbolic triangle $\Delta\mathbf{uvw}$. It can readily be calculated by elementary linear algebra, obtaining

$$\mathbf{m}_{\mathbf{uvw}}^{hyperbolic} = \frac{\gamma_{\mathbf{u}}\mathbf{u} + \gamma_{\mathbf{v}}\mathbf{v} + \gamma_{\mathbf{w}}\mathbf{w}}{\gamma_{\mathbf{u}} + \gamma_{\mathbf{v}} + \gamma_{\mathbf{w}}} \tag{63}$$

The task of verifying (63) from results shown in Fig. 4 can be given to students as a project in elementary linear algebra. Indeed, it is interesting to realize that geodesics in the Beltrami model are Euclidean straight lines, and midpoints and centroids in that model, like the ones in (60) and (63), introduce linear structure into hyperbolic geometry. This linear structure, in turn, allows some problems in hyperbolic geometry to be solved by methods of linear algebra.

Following (62) and (63) we have

$$\mathbf{m}_{\mathbf{uvw}}^{hyperbolic} = \mathbf{c}_{\mathbf{uvw}}^{einsteinian} \tag{64}$$

Eq. (64) extends the identity in (61) from a system of two particles with velocities $\mathbf{u}$ and $\mathbf{v}$ to a system of three particles with velocities $\mathbf{u}, \mathbf{v}$ and $\mathbf{w}$, demonstrating that the hyperbolic triangle centroid can be interpreted as a relativistic center of momentum of three moving particles with equal rest masses.

The Euclidean analog of (63) is clearly the Euclidean centroid

$$\mathbf{m}_{\mathbf{uvw}}^{euclidean} = \frac{\mathbf{u} + \mathbf{v} + \mathbf{w}}{3} \tag{65}$$

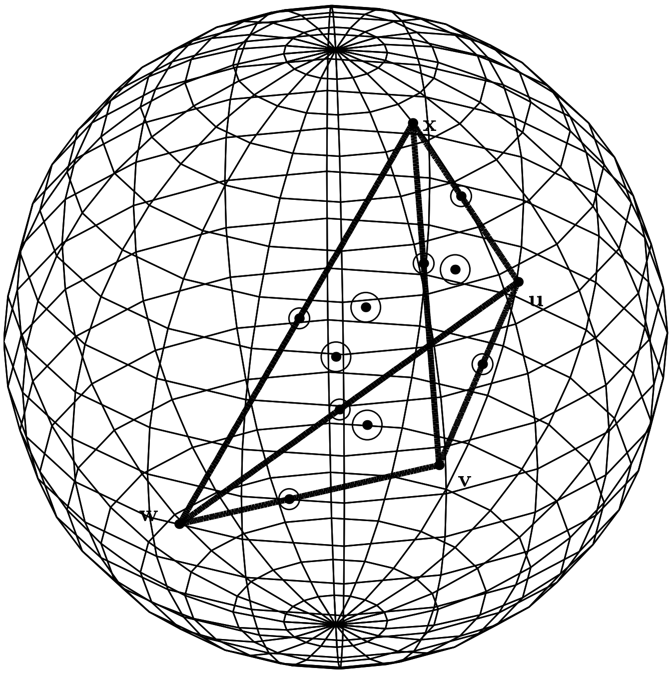


FIG. 5: The hyperbolic tetrahedron $\Delta\mathbf{uvwx}$, called a gyrotetrahedron, is shown in the Einstein gyrovector space
$$\mathbb{R}^3_c = (\mathbb{R}^3_c, \oplus, \otimes),$$
underlying the Beltrami ball model of hyperbolic geometry, with vertices $\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{x} \in \mathbb{R}^3_c$. The gyrotetrahedron $\Delta\mathbf{uvwx}$ is shown inside the c-ball $\mathbb{R}^3_c$ of the Euclidean 3-space $\mathbb{R}^3$ where it lives. The faces of the gyrotetrahedron are gyrotriangles.

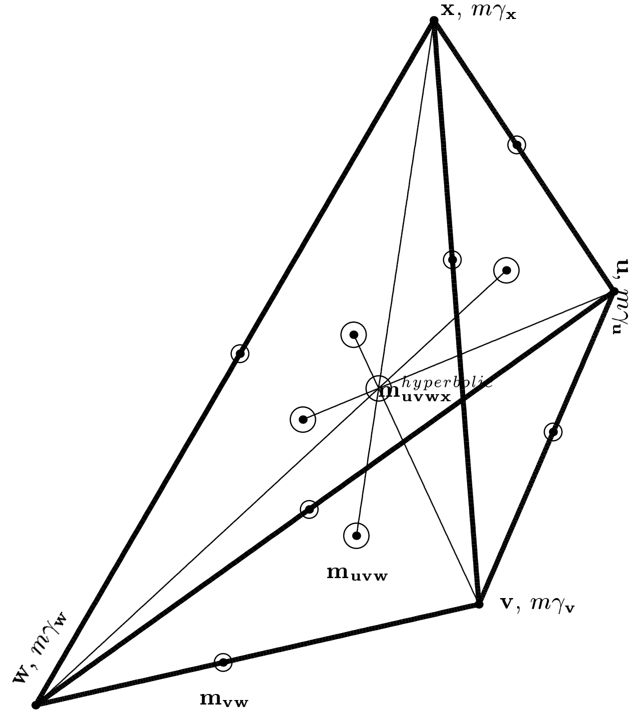


FIG. 6: Shown are also the midpoints of the 6 sides and the centroids of the 4 faces. The gyroline joining a vertex of a gyrotetrahedron and the hyperbolic centroid, gyrocentroid, of the opposite face is called a gyrotetrahedron median. The four medians of the gyrotetrahedron $\Delta\mathbf{uvwx}$, determined by the four indicated centroids, are concurrent. The point of concurrency is the gyrotetrahedron centroid $\mathbf{m}_{\mathbf{uvwx}}$ given by (68).

of the Euclidean triangle $\Delta\mathbf{uvw}$.

Moreover, the Newtonian-classical analog of the Einsteinian-relativistic center of momentum (62) is the Newtonian center of momentum of three moving particles with equal masses,

$$\mathbf{c}^{newtonian}_{\mathbf{uvw}} = \frac{\mathbf{u} + \mathbf{v} + \mathbf{w}}{3} \tag{66}$$

Hence, by (65) and (66), we have

$$\mathbf{m}^{euclidean}_{\mathbf{uvw}} = \mathbf{c}^{newtonian}_{\mathbf{uvw}} \tag{67}$$

thus obtaining the classical counterpart of (64). Eq. (67) demonstrates that the Euclidean triangle centroid can be interpreted as a classical center of momentum. The analogy (67) $\leftrightarrow$ (64) extends the analogy (58) $\leftrightarrow$ (61) between the pairs in (1) from a system of two particles to a system of three particles. Further extension to four particles is shown in Figs. 5 and 6. In these figures we see four particles with equal rest masses m, represented by the points $\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{x}$ of the Einstein gyrovector space $(\mathbb{R}^3_c, \oplus, \otimes)$ of Einsteinian velocities. The four particles, accordingly, have the respective relativistic velocities $\mathbf{u}, \mathbf{v}, \mathbf{w}$, and $\mathbf{x}$. The relativistic center of momentum is given by the equation

$$\begin{aligned}\mathbf{c}^{einsteinian}_{\mathbf{uvwx}} &= \frac{m\gamma_{\mathbf{u}}\mathbf{u} + m\gamma_{\mathbf{v}}\mathbf{v} + m\gamma_{\mathbf{w}}\mathbf{w} + m\gamma_{\mathbf{x}}\mathbf{x}}{m\gamma_{\mathbf{u}} + m\gamma_{\mathbf{v}} + m\gamma_{\mathbf{w}} + m\gamma_{\mathbf{x}}} \\ &= \frac{\gamma_{\mathbf{u}}\mathbf{u} + \gamma_{\mathbf{v}}\mathbf{v} + \gamma_{\mathbf{w}}\mathbf{w} + \gamma_{\mathbf{x}}\mathbf{x}}{\gamma_{\mathbf{u}} + \gamma_{\mathbf{v}} + \gamma_{\mathbf{w}} + \gamma_{\mathbf{x}}}\end{aligned} \tag{68}$$

and the hyperbolic centroid of the hyperbolic tetrahedron $\Delta\mathbf{uvwx}$ is

$$\mathbf{m}^{hyperbolic}_{\mathbf{uvwx}} = \frac{\gamma_{\mathbf{u}}\mathbf{u} + \gamma_{\mathbf{v}}\mathbf{v} + \gamma_{\mathbf{w}}\mathbf{w} + \gamma_{\mathbf{x}}\mathbf{x}}{\gamma_{\mathbf{u}} + \gamma_{\mathbf{v}} + \gamma_{\mathbf{w}} + \gamma_{\mathbf{x}}} \tag{69}$$

so that, Fig. 6,

$$\mathbf{m}^{hyperbolic}_{\mathbf{uvwx}} = \mathbf{c}^{einsteinian}_{\mathbf{uvwx}} \tag{70}$$

Understanding the geometry that underlies (69) enables us to improve our understanding of the physics of the relativistic center of momentum (68). As an example illustrating the physical understanding that we gain from geometry, we note that it follows from the geometric significance of the hyperbolic triangle centroid that it remains invariant under the hyperbolic rigid motions, which are left gyrotranslations and rotations of the ball $\mathbb{R}^3_c$. Let us therefore left gyrotranslate the points in (69) by $\ominus\mathbf{y}\in\mathbb{R}^3_c$, obtaining

$$\ominus\mathbf{y}\oplus\mathbf{m}_{\mathbf{uvwx}}^{hyperbolic} = \frac{\gamma_{\ominus\mathbf{y}\oplus\mathbf{u}}(\ominus\mathbf{y}\oplus\mathbf{u}) + \gamma_{\ominus\mathbf{y}\oplus\mathbf{v}}(\ominus\mathbf{y}\oplus\mathbf{v}) + \gamma_{\ominus\mathbf{y}\oplus\mathbf{w}}(\ominus\mathbf{y}\oplus\mathbf{w}) + \gamma_{\ominus\mathbf{y}\oplus\mathbf{x}}(\ominus\mathbf{y}\oplus\mathbf{x})}{\gamma_{\ominus\mathbf{y}\oplus\mathbf{u}} + \gamma_{\ominus\mathbf{y}\oplus\mathbf{v}} + \gamma_{\ominus\mathbf{y}\oplus\mathbf{w}} + \gamma_{\ominus\mathbf{y}\oplus\mathbf{x}}} \tag{71}$$

Hence, by means of the equality (70) between hyperbolic triangle centroids and the relativistic center of momentum we have from (71),

$$\ominus\mathbf{y}\oplus\mathbf{c}_{\mathbf{uvwx}}^{hyperbolic} = \frac{\gamma_{\ominus\mathbf{y}\oplus\mathbf{u}}(\ominus\mathbf{y}\oplus\mathbf{u}) + \gamma_{\ominus\mathbf{y}\oplus\mathbf{v}}(\ominus\mathbf{y}\oplus\mathbf{v}) + \gamma_{\ominus\mathbf{y}\oplus\mathbf{w}}(\ominus\mathbf{y}\oplus\mathbf{w}) + \gamma_{\ominus\mathbf{y}\oplus\mathbf{x}}(\ominus\mathbf{y}\oplus\mathbf{x})}{\gamma_{\ominus\mathbf{y}\oplus\mathbf{u}} + \gamma_{\ominus\mathbf{y}\oplus\mathbf{v}} + \gamma_{\ominus\mathbf{y}\oplus\mathbf{w}} + \gamma_{\ominus\mathbf{y}\oplus\mathbf{x}}} \tag{72}$$

Clearly, (68) gives the velocity of the relativistic center of momentum frame as measured by an observer who is at rest relative to the rest frame $\Sigma_{\mathbf{0}}$ and, similarly, (72) gives the velocity of the same relativistic center of momentum frame, but as seen by an observer who is at rest relative to the inertial frame $\Sigma_{\mathbf{y}}$. The relativistic center of momentum, as a result, is observer independent. The relativistic center of momentum is thus a useful example illustrating the improved understanding of relativity physics gained by hyperbolic geometry considerations.

The extension of the notion of the relativistic center of momentum to systems of any number of particles with rest masses that need not be equal is presented in Ref. [36]. Having the relativistic center of momentum and its harmonious interplay with hyperbolic geometry in hand, one can show that within the context of hyperbolic geometry the relativistic center of momentum is observer independent in the same way that the hyperbolic centroid of the hyperbolic tetrahedron (the gyrotetrahedron in Figs. 5 and 6) is invariant under the hyperbolic rigid motions (which are left gyrotranslations and rotations) of the hyperbolic tetrahedron.

Further study that uncovers the gyro-structure in some Lie groups and algebras, that could be useful in general relativity, is found in [38].

X. THE LORENTZ TRANSFORMATION AND ITS GYRO-ALGEBRA

The gyro-algebra of Thomas gyration captures analogies that the Lorentz transformation shares with its Galilean counterpart as well. These analogies can help students to extend their intuitive understanding of the Galilei transformation group to intuitive understanding of the Lorentz transformation group. It will be found in this section that in the context of hyperbolic geometry the Lorentz transformation appears as natural and intuitive as its Galilean counterpart appears in the context of Euclidean geometry.

Let $(t,\mathbf{x})^t$, where exponent t denotes transposition, be a spacetime event, $\mathbf{x}\in\mathbb{R}^3$, $t\in\mathbb{R}^+$, $\mathbb{R}^+$ being the positive ray of the real line, $\mathbb{R}^+ = \{r\in\mathbb{R} : r\geq 0\}$. A Lorentz transformation of spacetime coordinates is a coordinate transformation that leaves the norm

$$\left\|\begin{pmatrix} t \\ \mathbf{x}\end{pmatrix}\right\| = \sqrt{t^2 - \frac{\|\mathbf{x}\|^2}{c^2}} \tag{73}$$

of a spacetime event $(t,\mathbf{x})^t$ invariant. A Lorentz transformation without rotation is known in the jargon as a *boost*. Let $L(\mathbf{u})$ be a Lorentz boost parametrized by a relativistically admissible velocity $\mathbf{u}\in\mathbb{R}^3_c$. Its application to spacetime coordinates $(t,\mathbf{x})^t = (t,\mathbf{v}t)^t$ is given by the equation

$$\begin{aligned} L(\mathbf{u})\begin{pmatrix} t \\ \mathbf{x}\end{pmatrix} &= \begin{pmatrix} \gamma_{\mathbf{u}}(t+\frac{1}{c^2}\mathbf{u}\cdot\mathbf{x}) \\ (\mathbf{u}\oplus\mathbf{v})\gamma_{\mathbf{u}}(1+\frac{1}{c^2}\mathbf{u}\cdot\mathbf{v})t \end{pmatrix} \\ &= \begin{pmatrix} \gamma_{\mathbf{u}}(t+\frac{1}{c^2}\mathbf{u}\cdot\mathbf{x}) \\ \gamma_{\mathbf{u}}\mathbf{u}t + \mathbf{x} + \frac{1}{c^2}\frac{\gamma_{\mathbf{u}}^2}{1+\gamma_{\mathbf{u}}}(\mathbf{u}\cdot\mathbf{x})\mathbf{u} \end{pmatrix} \\ &= \begin{pmatrix} t' \\ \mathbf{x}'\end{pmatrix} \end{aligned} \tag{74}$$

leaving the spacetime norm (73) invariant,

$$\left\|\begin{pmatrix} t \\ \mathbf{x}\end{pmatrix}\right\| = \left\|\begin{pmatrix} t' \\ \mathbf{x}'\end{pmatrix}\right\| \tag{75}$$

The boost is a linear transformation of spacetime coordinates that has the matrix representation $L_m(\mathbf{v})$ given by the equation [17, 39, 40],

$$L_m(\mathbf{v}) = \begin{pmatrix} \gamma_{\mathbf{v}} & c^{-2}\gamma_{\mathbf{v}}v_1 & c^{-2}\gamma_{\mathbf{v}}v_2 & c^{-2}\gamma_{\mathbf{v}}v_3 \\ \gamma_{\mathbf{v}}v_1 & 1+c^{-2}\frac{\gamma_{\mathbf{v}}^2}{\gamma_{\mathbf{v}}+1}v_1^2 & c^{-2}\frac{\gamma_{\mathbf{v}}^2}{\gamma_{\mathbf{v}}+1}v_1v_2 & c^{-2}\frac{\gamma_{\mathbf{v}}^2}{\gamma_{\mathbf{v}}+1}v_1v_3 \\ \gamma_{\mathbf{v}}v_2 & c^{-2}\frac{\gamma_{\mathbf{v}}^2}{\gamma_{\mathbf{v}}+1}v_1v_2 & 1+c^{-2}\frac{\gamma_{\mathbf{v}}^2}{\gamma_{\mathbf{v}}+1}v_2^2 & c^{-2}\frac{\gamma_{\mathbf{v}}^2}{\gamma_{\mathbf{v}}+1}v_2v_3 \\ \gamma_{\mathbf{v}}v_3 & c^{-2}\frac{\gamma_{\mathbf{v}}^2}{\gamma_{\mathbf{v}}+1}v_1v_3 & c^{-2}\frac{\gamma_{\mathbf{v}}^2}{\gamma_{\mathbf{v}}+1}v_2v_3 & 1+c^{-2}\frac{\gamma_{\mathbf{v}}^2}{\gamma_{\mathbf{v}}+1}v_3^2 \end{pmatrix} \tag{76}$$

so that

$$L(\mathbf{v})\begin{pmatrix} t \\ \mathbf{x} \end{pmatrix} = L_m(\mathbf{v})\begin{pmatrix} t \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} t' \\ x_1' \\ x_2' \\ x_3' \end{pmatrix} = \begin{pmatrix} t' \\ \mathbf{x}_1' \end{pmatrix} \tag{77}$$

Here $\mathbf{v} = (v_1, v_2, v_3)^t \in \mathbb{R}_c^3$, $\mathbf{x} = (x_1, x_2, x_3)^t \in \mathbb{R}^3$, $\mathbf{x}' = (x_1', x_2', x_3')^t \in \mathbb{R}^3$, and $t, t' \in \mathbb{R}^+$.

In the Newtonian limit of large vacuum speed of light, $c \to \infty$, the Lorentz boost $L(\mathbf{v})$, (77), reduces to the Galilei boost $G(\mathbf{v})$, $\mathbf{v} = (v_1, v_2, v_3) \in \mathbb{R}^3$,

$$\begin{aligned} G(\mathbf{v})\begin{pmatrix} t \\ \mathbf{x} \end{pmatrix} &= \lim_{c\to\infty} L(\mathbf{v})\begin{pmatrix} t \\ \mathbf{x} \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ v_1 & 1 & 0 & 0 \\ v_2 & 0 & 1 & 0 \\ v_3 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} t \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} \\ &= \begin{pmatrix} t \\ x_1 + v_1 t \\ x_2 + v_2 t \\ x_3 + v_3 t \end{pmatrix} \\ &= \begin{pmatrix} t \\ \mathbf{x} + \mathbf{v}t \end{pmatrix} \end{aligned} \tag{78}$$

where $\mathbf{x} = (x_1, x_2, x_3)^t \in \mathbb{R}^3$ and $t \in \mathbb{R}^+$.

The composition of two Galilei boosts is equivalent to a single Galilei boost according to the equation

$$G(\mathbf{u})G(\mathbf{v}) = G(\mathbf{u}+\mathbf{v}) \tag{79}$$

as it is clear from (78).

The composition of Lorentz boosts is more complicated than that of Galilei boosts since, in general, the composition of two Lorentz boosts is not a boost but, rather, a boost preceded (or followed) by a space rotation that Thomas precession generates.

Let $V \in SO(3)$ be a space rotation, and let $E(V)$ be the space rotation of a spacetime event, extended from V by the equation

$$E(V)\begin{pmatrix} t \\ \mathbf{x} \end{pmatrix} = \begin{pmatrix} t \\ V\mathbf{x} \end{pmatrix} \tag{80}$$

$t \in \mathbb{R}^+$, $\mathbf{x} \in \mathbb{R}^3$. Similarly, let $\mathrm{Gyr}[\mathbf{u}, \mathbf{v}]$ be the space gyration of a spacetime event, extended from $\mathrm{gyr}[\mathbf{u}, \mathbf{v}]$ by the equation

$$\mathrm{Gyr}[\mathbf{u}, \mathbf{v}] = E(\mathrm{gyr}[\mathbf{u}, \mathbf{v}]) \tag{81}$$

Then, the relativistic analog of (79) is [5],

$$\begin{aligned} L(\mathbf{u})L(\mathbf{v}) &= L(\mathbf{u}\oplus\mathbf{v})\mathrm{Gyr}[\mathbf{u}, \mathbf{v}] \\ &= \mathrm{Gyr}[\mathbf{u}, \mathbf{v}]L(\mathbf{v}\oplus\mathbf{u}) \end{aligned} \tag{82}$$

Hence, the composition of two successive Lorentz boosts, $L(\mathbf{v})$ followed by $L(\mathbf{u})$,

(1) is equivalent to a single boost, $L(\mathbf{u}\oplus\mathbf{v})$, preceded by the Thomas gyration generated by $\mathbf{u}$ and $\mathbf{v}$; and equally well, it

(2) is equivalent to a single boost, $L(\mathbf{v}\oplus\mathbf{u})$, followed by the Thomas gyration generated by $\mathbf{v}$ and $\mathbf{u}$.

The Lorentz transformation $L(\mathbf{v}, V)$ of spacetime events $(t, \mathbf{x})^t \in \mathbb{R}^+ \times \mathbb{R}^3$ is a boost preceded by a space rotation,

$$L(\mathbf{v}, V)\begin{pmatrix} t \\ \mathbf{x} \end{pmatrix} = L(\mathbf{v})E(V)\begin{pmatrix} t \\ \mathbf{x} \end{pmatrix} \tag{83}$$

Hence, explicitly, the Lorentz transformation $L(\mathbf{v}, V)$ takes the form

$$L(\mathbf{v}, V)\begin{pmatrix} t \\ \mathbf{x} \end{pmatrix} = \begin{pmatrix} \gamma_{\mathbf{v}}(t + \frac{1}{c^2}\mathbf{v}\cdot V\mathbf{x}) \\ \gamma_{\mathbf{v}}\mathbf{v}t + V\mathbf{x} + \frac{1}{c^2}\frac{\gamma_{\mathbf{v}}^2}{1+\gamma_{\mathbf{v}}}(\mathbf{v}\cdot V\mathbf{x})\mathbf{v} \end{pmatrix} \tag{84}$$

$\mathbf{v}\in\mathbb{R}_c^3$, $V\in SO(3)$, $\mathbf{x}\in\mathbb{R}^3$, $t\in\mathbb{R}^+$, as we see from (74).

In the Newtonian limit of large vacuum speed of light, $c \to \infty$, the Lorentz transformation $L(\mathbf{v}, V)$ in (84) reduces to its Galilean counterpart $G(\mathbf{v}, V)$,

$$G(\mathbf{v}, V)\begin{pmatrix} t \\ \mathbf{x} \end{pmatrix} = \begin{pmatrix} t \\ \mathbf{v}t + V\mathbf{x} \end{pmatrix} \tag{85}$$

$\mathbf{v} \in \mathbb{R}^3$, $V \in SO(3)$, $\mathbf{x} \in \mathbb{R}^3$, $t \in \mathbb{R}$.

Clearly, the Lorentz transformation $L(\mathbf{v}, I)$ with no rotation, I being the identity element of $SO(3)$, is a Lorentz boost,

$$L(\mathbf{v}, I) = L(\mathbf{v}) \tag{86}$$

for all $\mathbf{v} \in \mathbb{R}_c^3$, shown in (77). Similarly, for all $\mathbf{v} \in \mathbb{R}^3$

$$G(\mathbf{v}, I) = G(\mathbf{v}) \tag{87}$$

is a Galilei boost, shown in (78).

The composition of two Galilei transformations is given by parameters composition according to the equation

$$G(\mathbf{u}, U)G(\mathbf{v}, V) = G(\mathbf{u} + U\mathbf{v}, UV) \tag{88}$$

where $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$, and $U, V \in SO(n)$.

In full analogy, the composition of two Lorentz transformations is given by parameters composition according to the equation [5],

$$L(\mathbf{u}, U)L(\mathbf{v}, V) = L(\mathbf{u} \oplus U\mathbf{v}, \mathrm{gyr}[\mathbf{u}, U\mathbf{v}]UV) \tag{89}$$

where $\mathbf{u}, \mathbf{v} \in \mathbb{R}_c^3$, and $U, V \in SO(3)$.

The Galilei transformation composition law (88) is obvious and well known in the literature. In contrast, the Lorentz transformation composition law (89) is a new result that follows from gyrogroup-theoretic considerations [5, 40–42]. Its usefulness stems from the analogies that it shares with its Galilean counterpart (88).

The Lorentz transformations (84) form a subgroup of the so called full Lorentz group. This subgroup is known as the *homogeneous, proper, orthochronous* Lorentz group $SO(3, 1)$. It is:

(1) *homogeneous*, since each of its elements is a Lorentz transformation that takes the origin of spacetime coordinates into an origin of spacetime coordinates; it is

(2) *proper*, since each of its elements is a Lorentz transformation that is continuously connected to the identity transformation of spacetime; and it is

(3) *orthochronous*, since each of its elements is a Lorentz transformation that preserves the sign of time, that is, it takes positive (negative) time into positive (negative) time.

Identity (89) is thus the group operation of the homogeneous, proper, orthochronous Lorentz group $SO(3, 1)$.

Let

$$\begin{pmatrix} t \\ \mathbf{u}t \end{pmatrix} \qquad \textit{and} \qquad \begin{pmatrix} t' \\ \mathbf{v}t' \end{pmatrix} \tag{90}$$

be two Galilean spacetime events, $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$, and $t, t' \in \mathbb{R}$, with equal time components,

$$t = t' \tag{91}$$

Then, the unique Galilei transformation without rotation (boost) that links the two spacetime events in (90) – (91) has the velocity parameter $\mathbf{v} - \mathbf{u}$, satisfying

$$G(\mathbf{v} - \mathbf{u}) \begin{pmatrix} t \\ \mathbf{u}t \end{pmatrix} = \begin{pmatrix} t' \\ \mathbf{v}t' \end{pmatrix} \tag{92}$$

as one can readily see from (78). We call (92) a *Galilei link*.

In the analogous relativistic link, the Galilean negative velocity addition, $-$, in (92) becomes the Einstein negative coaddition, $\boxminus$, in (32), as we will see in (95) below. Let

$$\begin{pmatrix} t \\ \mathbf{u}t \end{pmatrix} \qquad \textit{and} \qquad \begin{pmatrix} t' \\ \mathbf{v}t' \end{pmatrix} \tag{93}$$

be two relativistic spacetime events, $\mathbf{u}, \mathbf{v} \in \mathbb{R}_c^3$, and $t, t' \in \mathbb{R}^+$, with equal spacetime norms,

$$t'\sqrt{1 - \frac{\|\mathbf{v}\|^2}{c^2}} = t\sqrt{1 - \frac{\|\mathbf{u}\|^2}{c^2}} \tag{94}$$

Then, the unique Lorentz transformation without rotation (boost) that links the two spacetime events in (93) – (94) has the velocity parameter $\mathbf{v} \boxminus \mathbf{u}$, satisfying

$$L(\mathbf{v} \boxminus \mathbf{u}) \begin{pmatrix} t \\ \mathbf{u}t \end{pmatrix} = \begin{pmatrix} t' \\ \mathbf{v}t' \end{pmatrix} \tag{95}$$

as we see from pp. 343 – 348 in Ref. [5].

We call (95) a *Lorentz link*. The analogies that the Lorentz link (95) shares with the Galilean link (92) are obvious. Remarkably, the analogies that the Lorentz and the Galilei links share are uncovered in terms of Einstein coaddition rather than Einstein addition. This observation demonstrates that in order to capture analogies that Einsteinian velocities share with Newtonian velocities the two mutually dual Einstein additions, $\oplus$ and $\boxplus$, must be invoked.

Furthermore, the most general Galilei transformation that links the simultaneous spacetime events in (90) – (91) is $G(\mathbf{v} - R\mathbf{u}, R)$, satisfying

$$G(\mathbf{v} - R\mathbf{u}, R) \begin{pmatrix} t \\ \mathbf{u}t \end{pmatrix} = \begin{pmatrix} t' \\ \mathbf{v}t' \end{pmatrix} \tag{96}$$

for all $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$, and $t, t' \in \mathbb{R}$, where $R \in SO(3)$ is a free rotation parameter. Identity (96) reduces to (92) when the rotation R vanishes, that is, when R reduces to the identity map I.

In full analogy, the most general homogeneous, proper, orthochronous Lorentz transformation that links the equinorm spacetime events in (93) – (94) is $L(\mathbf{v} \boxminus R\mathbf{u}, R)$ satisfying [5, 42, 43],

$$L(\mathbf{v} \boxminus R\mathbf{u}, R) \begin{pmatrix} t \\ \mathbf{u}t \end{pmatrix} = \begin{pmatrix} t' \\ \mathbf{v}t' \end{pmatrix} \tag{97}$$

for all $\mathbf{u}, \mathbf{v} \in \mathbb{R}_c^3$, and $t, t' \in \mathbb{R}^+$, where $R \in SO(3)$ is a free rotation parameter.

Historically, until recently [44], the problem of determining Lorentz links between given spacetime events was only partially solved by several explorers, as evidenced from Ref. [45].

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