A Trip
Beyond the Einstein Addition Law

Presentation at the Fall Reception
University of Wisconsin – Superior

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Beyond the Einstein Addition Law and its Gyroscopic Thomas Precession: The Theory of Gyrogroups and Gyrovector Spaces

Kluwer, Acad. 2001

http://www.wkap.nl/book.htm/0-7923-6909-2

This book, by Abraham A. Ungar, is available at the Library of University of Wisconsin – Superior

The Einstein addition $\oplus_{\mathrm{E}}$

in $(\mathbb{V}_c, \oplus_{\mathrm{E}})$

$$\boxed{\mathbf{u} \oplus_{\mathrm{E}} \mathbf{v} = \frac{1}{1 + \frac{\mathbf{u} \cdot \mathbf{v}}{c^2}} \left\{ \mathbf{u} + \frac{1}{\gamma_{\mathbf{u}}} \mathbf{v} + \frac{1}{c^2} \frac{\gamma_{\mathbf{u}}}{1 + \gamma_{\mathbf{u}}} (\mathbf{u} \cdot \mathbf{v}) \mathbf{u} \right\}} \tag{1}$$

for all $\mathbf{u}$ and $\mathbf{v}$ in the c-ball $\mathbb{V}_c$

$$\mathbb{V}_c = \{ \mathbf{v} \in \mathbb{V} : \|\mathbf{v}\| < c \} \tag{2}$$

of any real inner product space $\mathbb{V}$, $\gamma_{\mathbf{u}}$ being the Lorentz factor

$$\gamma_{\mathbf{u}} = \frac{1}{\sqrt{1 - \frac{\|\mathbf{u}\|^2}{c^2}}} \tag{3}$$

Einstein's addition is noncommutative. In general

$$\boxed{\mathbf{u}\oplus\mathbf{v} \neq \mathbf{v}\oplus\mathbf{u}} \tag{4}$$

$\mathbf{u}, \mathbf{v} \in \mathbb{V}_c$. Moreover, Einstein's addition is also nonassociative. In general

$$\boxed{(\mathbf{u}\oplus\mathbf{v})\oplus\mathbf{w} \neq \mathbf{u}\oplus(\mathbf{v}\oplus\mathbf{w})} \tag{5}$$

$\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{V}_c$. Before the advent of gyrogroup theory in 1988 physicists have difficulty explaining why the Einstein velocity addition fell into such despair.

One might suppose that there is a price to pay in mathematical regularity when replacing ordinary vector addition with Einstein's addition. But we will now see that this is not the case.

$$a \oplus b = \mathrm{gyr}[a, b](b \oplus a) \qquad \text{Gyrocommutative Law}$$

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Coincidentally, the gyration that repairs the breakdown of commutativity in the Möbius addition repairs the breakdown of associativity as well, giving rise to identities that capture analogies,

$$a \oplus (b \oplus c) = (a \oplus b) \oplus \mathrm{gyr}[a, b]c \qquad \text{Left Gyroass. Law}$$
$$(a \oplus b) \oplus c = a \oplus (b \oplus \mathrm{gyr}[b, a]c) \qquad \text{Right Gyroa. Law}$$
$$\mathrm{gyr}[a, b] = \mathrm{gyr}[a \oplus b, b] \qquad \text{Left Loop Property}$$
$$\mathrm{gyr}[a, b] = \mathrm{gyr}[a, b \oplus a] \qquad \text{Right Loop Prop.}$$

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The emerging coincidences expose an algebraic structure that merits extension by abstraction, leading to the concept of the grouplike structure called a gyrogroup.

Gyrogroups are generalized groups that share remarkable analogies with groups. In full analogy with groups

(1) gyrogroups are classified into gyrocommutative gyrogroups and non-gyrocommutative gyrogroups; and

(2) some gyrocommutative gyrogroups admit scalar multiplication, turning them into gyrovector spaces.

(3) Gyrovector spaces, in turn, provide the setting for hyperbolic geometry just as vector spaces provide the setting for Euclidean geometry, thus enabling the two geometries to be unified.

Definition 1 (Gyrogroups). *The groupoid* $(G, \oplus)$ *is a gyrogroup if its binary operation satisfies the following axioms. In* G *there is at least one element,* 0*, called a left identity, satisfying for all* $a, b, z \in G$

$0 \oplus a = a$	*Left Identity*
$\ominus a \oplus a = 0$	*Left Inverse*
$a \oplus (b \oplus z) = (a \oplus b) \oplus \mathrm{gyr}[a, b]z$	*Left Gyroassociative*
$\mathrm{gyr}[a, b] \in \mathrm{Aut}(G, \oplus)$	*Gyroautomorphism*
$\mathrm{gyr}[a, b] = \mathrm{gyr}[a \oplus b, b]$	*Left Loop Property*

Definition 2 (Gyrocommutative Gyrogroups). *The gyrogroup* $(G, \oplus)$ *is gyrocommutative if for all* $a, b \in G$

$a \oplus b = \mathrm{gyr}[a, b](b \oplus a)$	*Gyrocommutative Law*

The Gyration
Expressed in Terms of
Einstein Addition

Solving the Left Gyroassociative Law

$$a \oplus (b \oplus z) = (a \oplus b) \oplus \mathrm{gyr}[a, b]z$$

we have

$$\boxed{\mathrm{gyr}[a, b]z = \ominus(a \oplus b) \oplus \{a \oplus (b \oplus z)\}}$$

and

$$\boxed{\|\mathrm{gyr}[a, b]z\| = \|z\|}$$

calling $\mathrm{gyr}[a, b]$ the gyration generated by a and b.

Thus, the application of $\mathrm{gyr}[a, b]$ to z gives a rotation of z, so that gyrations are, in fact, rotations.

The Möbius addition $\oplus_{\mathrm{M}}$

in $(\mathbb{V}_c, \oplus_{\mathrm{M}})$

$$\boxed{\mathbf{u} \oplus_{\mathrm{M}} \mathbf{v} = \frac{(1 + \frac{2}{c^2}\mathbf{u}\cdot\mathbf{v} + \frac{1}{c^2}\|\mathbf{v}\|^2)\mathbf{u} + (1 - \frac{1}{c^2}\|\mathbf{u}\|^2)\mathbf{v}}{1 + \frac{2}{c^2}\mathbf{u}\cdot\mathbf{v} + \frac{1}{c^4}\|\mathbf{u}\|^2\|\mathbf{v}\|^2}} \tag{6}$$

for all $\mathbf{u}$ and $\mathbf{v}$ in the c-ball $\mathbb{V}_c$

$$\mathbb{V}_c = \{\mathbf{v} \in \mathbb{V} : \|\mathbf{v}\| < c\} \tag{7}$$

of any real inner product space $\mathbb{V}$, $\gamma_{\mathbf{u}}$ being the Lorentz factor

$$\gamma_{\mathbf{u}} = \frac{1}{\sqrt{1 - \dfrac{\|\mathbf{u}\|^2}{c^2}}} \tag{8}$$

Motivating the Möbius Vector Addition

Identifying vectors in $\mathbb{R}^2$ with complex numbers in the usual way we have

$$\mathbf{u} = (u_1, u_2) = u_1 + iu_2 = u \tag{9}$$

The inner product and the norm in $\mathbb{R}^2$ then become the real numbers

$$\begin{aligned} \mathbf{u}\cdot\mathbf{v} &= Re(\bar{u}v) = \frac{\bar{u}v + u\bar{v}}{2} \\ \|\mathbf{u}\| &= |u| \end{aligned} \tag{10}$$

where $\bar{u}$ is the complex conjugate of u.

Under the translation (10) of elements of the disc $\mathbb{R}^2_{c=1}$ of $\mathbb{R}^2$ into elements of the complex unit disc $\mathbb{D}$, the Möbius addition (6), with $c = 1$ for simplicity, takes the form

$$
\begin{aligned}
\mathbf{u}\oplus_{\mathrm{M}}\mathbf{v} &= \frac{(1+2\mathbf{u}\cdot\mathbf{v}+\|\mathbf{v}\|^2)\mathbf{u}+(1-\|\mathbf{u}\|^2)\mathbf{v}}{1+2\mathbf{u}\cdot\mathbf{v}+\|\mathbf{u}\|^2\|\mathbf{v}\|^2} \\
&= \frac{(1+\bar{u}v+u\bar{v}+|v|^2)u+(1-|u|^2)v}{1+\bar{u}v+u\bar{v}+|u|^2|v|^2} \\
&= \frac{(1+u\bar{v})(u+v)}{(1+\bar{u}v)(1+u\bar{v})} \\
&= \frac{u+v}{1+\bar{u}v} \\
&= u\oplus_{\mathrm{M}} v
\end{aligned}
\tag{11}
$$

for all $\mathbf{u},\mathbf{v}\in\mathbb{R}^2_{c=1}$, and all $u,v\in\mathbb{D}$.

Both Einstein's addition and Möbius' addition are important in modern physics. Gyro-theoretic techniques uncover the reason:

They are gyro-isomorphic to each other.

The Isomorphism Between
The Einstein Addition $\oplus_{\mathrm{E}}$
and
The Möbius Addition $\oplus_{\mathrm{M}}$

$$\boxed{\mathbf{u}\oplus_{\mathrm{M}}\mathbf{v}=\frac{1}{2}\otimes(2\otimes\mathbf{u}\oplus_{\mathrm{E}}2\otimes\mathbf{v})} \tag{12}$$

$$\boxed{\mathbf{u}\oplus_{\mathrm{E}}\mathbf{v}=2\otimes(\frac{1}{2}\otimes\mathbf{u}\oplus_{\mathrm{M}}\frac{1}{2}\otimes\mathbf{v})} \tag{13}$$

Scalar Multiplication $\otimes$

for Möbius and Einstein's gyrogroups $(\mathbb{V}_c, \oplus)$

$$\boxed{\begin{aligned} r\otimes\mathbf{v} &= c\frac{(1+\|\mathbf{v}\|/c)^r-(1-\|\mathbf{v}\|/c)^r}{(1+\|\mathbf{v}\|/c)^r+(1-\|\mathbf{v}\|/c)^r}\,\frac{\mathbf{v}}{\|\mathbf{v}\|} \\ &= c\,tanh\left(r\,tanh^{-1}\frac{\|\mathbf{v}\|}{c^2}\right)\frac{\mathbf{v}}{\|\mathbf{v}\|} \end{aligned}} \tag{14}$$

where $r \in \mathbb{R}$, $\mathbf{v} \in \mathbb{V}_c$, $\mathbf{v} \neq \mathbf{0}$; and $r\otimes\mathbf{0} = \mathbf{0}$.

The scalar multiplication possesses the following properties. For any positive integer n and for all $r, r_1, r_2 \in \mathbb{R}$ and $\mathbf{v} \in \mathbb{V}_c$,

$$n\otimes\mathbf{v} = \mathbf{v}\oplus \ldots \oplus\mathbf{v} \qquad \text{n terms}$$

$$(r_1 + r_2)\otimes\mathbf{v} = r_1\otimes\mathbf{v}\oplus r_2\otimes\mathbf{v}$$

$$(r_1 r_2)\otimes\mathbf{v} = r_1\otimes(r_2\otimes\mathbf{v})$$

$$r\otimes(r_1\otimes\mathbf{v}\oplus r_2\otimes\mathbf{v}) = r\otimes(r_1\otimes\mathbf{v})\oplus r\otimes(r_2\otimes\mathbf{v})$$

$$\|r\otimes\mathbf{v}\| = |r|\otimes\|\mathbf{v}\|$$

$$\frac{|r|\otimes\mathbf{v}}{\|r\otimes\mathbf{v}\|} = \frac{\mathbf{v}}{\|\mathbf{v}\|}$$

$$\|\mathbf{u}\oplus\mathbf{v}\| \leq \|\mathbf{u}\|\oplus\|\mathbf{v}\|$$

$$\mathrm{gyr}[a, b](r\otimes\mathbf{v}) = r\otimes\mathrm{gyr}[a, b]\mathbf{v}$$

$$\mathrm{gyr}[r_1\otimes\mathbf{v}, r_2\otimes\mathbf{v}] = I \qquad (15)$$

Remarkably, we have a *monodistributive law* but no distributive law.

Solving the Equation

$$\mathbf{a}\oplus\mathbf{x} = \mathbf{b} \tag{16}$$

for x in a gyrogroup $(G, \oplus)$. If $\mathbf{x}$ is a solution, then by the right gyroassociative law and the identity $\mathrm{gyr}[\mathbf{a}, -\mathbf{a}] = id$ we have

$$\begin{aligned} \mathbf{x} &= \mathbf{0}\oplus\mathbf{x} \\ &= (-\mathbf{a}\oplus\mathbf{a})\oplus\mathbf{x} \\ &= -\mathbf{a}\oplus(\mathbf{a}\oplus\mathrm{gyr}[\mathbf{a}, -\mathbf{a}]\mathbf{x}) \\ &= -\mathbf{a}\oplus(\mathbf{a}\oplus\mathbf{x}) \\ &= -\mathbf{a}\oplus\mathbf{b} \end{aligned} \tag{17}$$

Thus if a solution exists, it must be given uniquely by

$$\mathbf{x} = -\mathbf{a}\oplus\mathbf{b} \tag{18}$$

Solving the Equation

$$\mathbf{x} \oplus \mathbf{a} = \mathbf{b} \tag{19}$$

for x in a gyrogroup $(G, \oplus)$. If $\mathbf{x}$ is a solution then, by the left gyroassociative law and the left loop property we have

$$\begin{aligned}
\mathbf{x} &= \mathbf{x} \oplus \mathbf{0} \\
&= \mathbf{x} \oplus (\mathbf{a} \oplus (\ominus \mathbf{a})) \\
&= (\mathbf{x} \oplus \mathbf{a}) \oplus \mathrm{gyr}[\mathbf{x}, \mathbf{a}](\ominus \mathbf{a}) \\
&= (\mathbf{x} \oplus \mathbf{a}) \oplus (\ominus \mathrm{gyr}[\mathbf{x}, \mathbf{a}]\mathbf{a}) \\
&= (\mathbf{x} \oplus \mathbf{a}) \ominus \mathrm{gyr}[\mathbf{x}, \mathbf{a}]\mathbf{a} \\
&= \mathbf{b} \ominus \mathrm{gyr}[\mathbf{x}, \mathbf{a}]\mathbf{a} \\
&= \mathbf{b} \ominus \mathrm{gyr}[\mathbf{x} \oplus \mathbf{a}, \mathbf{a}]\mathbf{a} \\
&= \mathbf{b} \ominus \mathrm{gyr}[\mathbf{b}, \mathbf{a}]\mathbf{a} \\
&= \mathbf{b} \boxminus \mathbf{a}
\end{aligned} \tag{20}$$

where we use the obvious notation:

$$\mathbf{a} \ominus \mathbf{b} = \mathbf{a} \oplus (\ominus \mathbf{b}).$$

Definition 3 (Gyrogroup Dual Operation). *Let $(G, \oplus)$ be a gyrogroup. A secondary binary operation $\boxplus$ in G is defined by the equation*

$$a \boxplus b = a \oplus \mathrm{gyr}[a, \ominus b]b \qquad \textit{Secondary Operation}$$

Theorem 4. *Let $(G, \oplus)$ be a gyrogroup and let $\boxplus$ be its dual binary operation given by Definition (3),*

$$\boxed{a \boxplus b = a \oplus \mathrm{gyr}[a, \ominus b]b} \tag{21}$$

Then

$$\boxed{a \oplus b = a \boxplus \mathrm{gyr}[a, b]b} \tag{22}$$

and

$$\mathrm{Aut}(G, \boxplus) = \mathrm{Aut}(G, \oplus) \tag{23}$$

Theorem 5 (Gyrogroup Canellation Laws). *Let $(G, \oplus)$ be a gyrogroup, and let $\boxplus$ be its dual operation. Then, for all $a, b, c \in G$,*

$$a \oplus (\ominus a \oplus b) = b \qquad \textit{Left Cancellation Law}$$
$$(b \boxminus a) \oplus a = b \qquad \textit{Right Cancellation Law}$$
$$(b \ominus a) \boxplus a = b \qquad \textit{Dual Right Cancellation Law}$$

Euclidean Geodesics

$$L_p = \mathbf{a} + (-\mathbf{a} + \mathbf{b})t$$
$$L_s = (\mathbf{b} - \mathbf{a})t + \mathbf{a} \tag{24}$$

Newtonian Mechanics: $\boxed{\mathbf{v}_0 + \mathbf{a}t \quad \text{or} \quad \mathbf{a}t + \mathbf{v}_0}$

Relative to the Euclidean Metric

$$d(\mathbf{a}, \mathbf{b}) = \|\mathbf{a} - \mathbf{b}\|$$
$$d(\mathbf{a}, \mathbf{b}) = \|\mathbf{a} - \mathbf{b}\| \tag{25}$$

Hyperbolic Geodesics

$$L_p = \mathbf{a} \oplus (\ominus \mathbf{a} \oplus \mathbf{b}) \otimes t$$
$$L_s = (\mathbf{b} \boxminus \mathbf{a}) \otimes t \oplus \mathbf{a} \tag{26}$$

Einsteinian Mechanics: $\boxed{\mathbf{v}_0 \oplus \mathbf{a} \otimes t \quad \text{or} \quad \mathbf{a} \otimes t \oplus \mathbf{v}_0}$

Relative to the Hyperbolic Dual Metrics

$$d_{\oplus}(\mathbf{a}, \mathbf{b}) = \|\mathbf{a} \ominus \mathbf{b}\|$$
$$d_{\boxplus}(\mathbf{a}, \mathbf{b}) = \|\mathbf{a} \boxminus \mathbf{b}\| \tag{27}$$

Einstein gyrovector spaces
form the setting
for
the Beltrami ball model
of
hyperbolic geometry

just as

vector spaces
form the setting
for
the common model
of
Euclidean geometry.

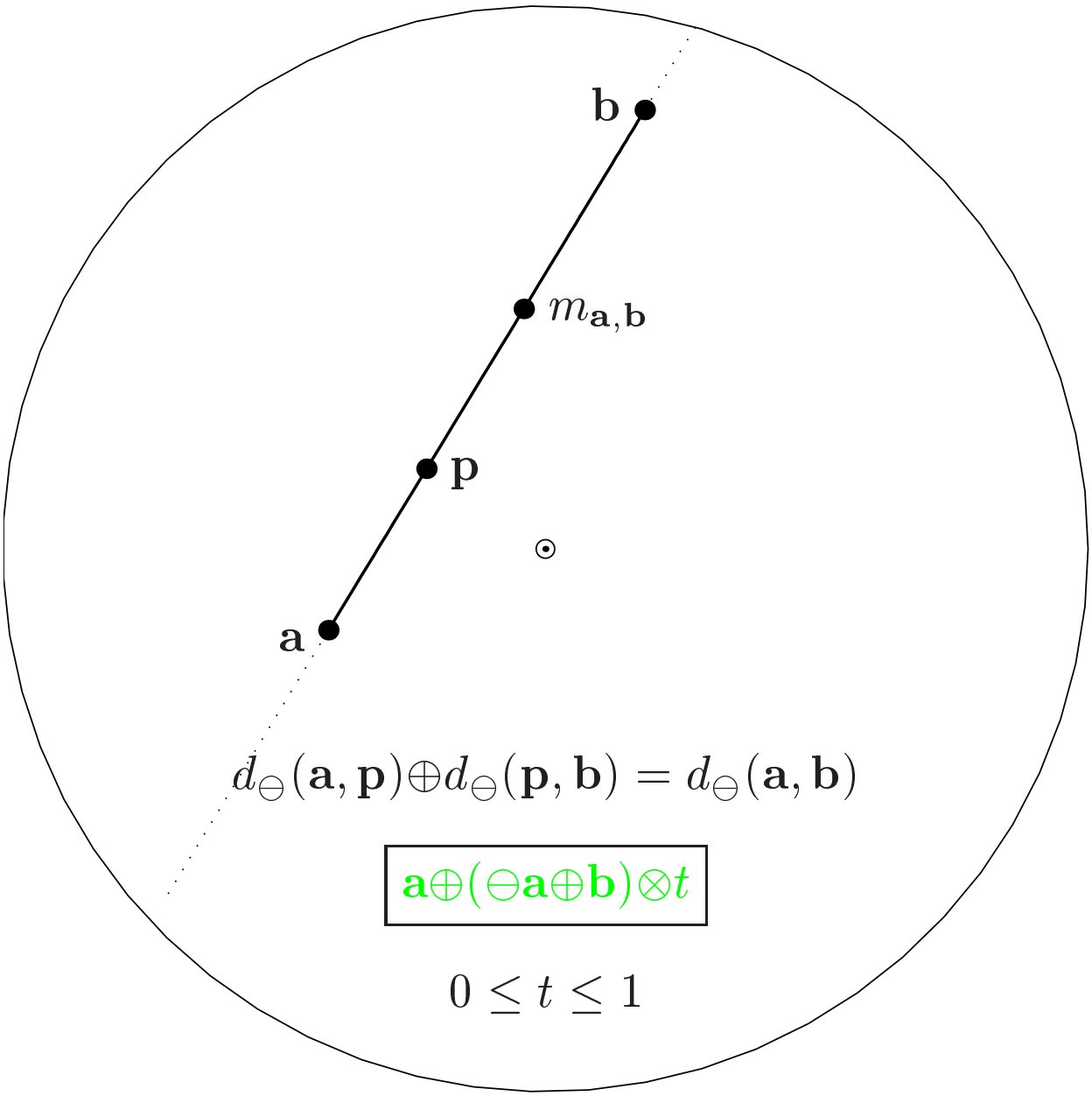

Figure 1: $\oplus = \oplus_{\mathrm{E}}$. The gyroline segment linking the two points $\mathbf{a}$ and $\mathbf{b}$ in an Einstein gyrovector space $(\mathbb{V}_c, \oplus_{\mathrm{E}}, \otimes)$. $\mathbf{p}$ is a generaic point between $\mathbf{a}$ and $\mathbf{b}$ and $m_{\mathbf{a},\mathbf{b}}$ is the midpoint of $\mathbf{a}$ and $\mathbf{b}$.

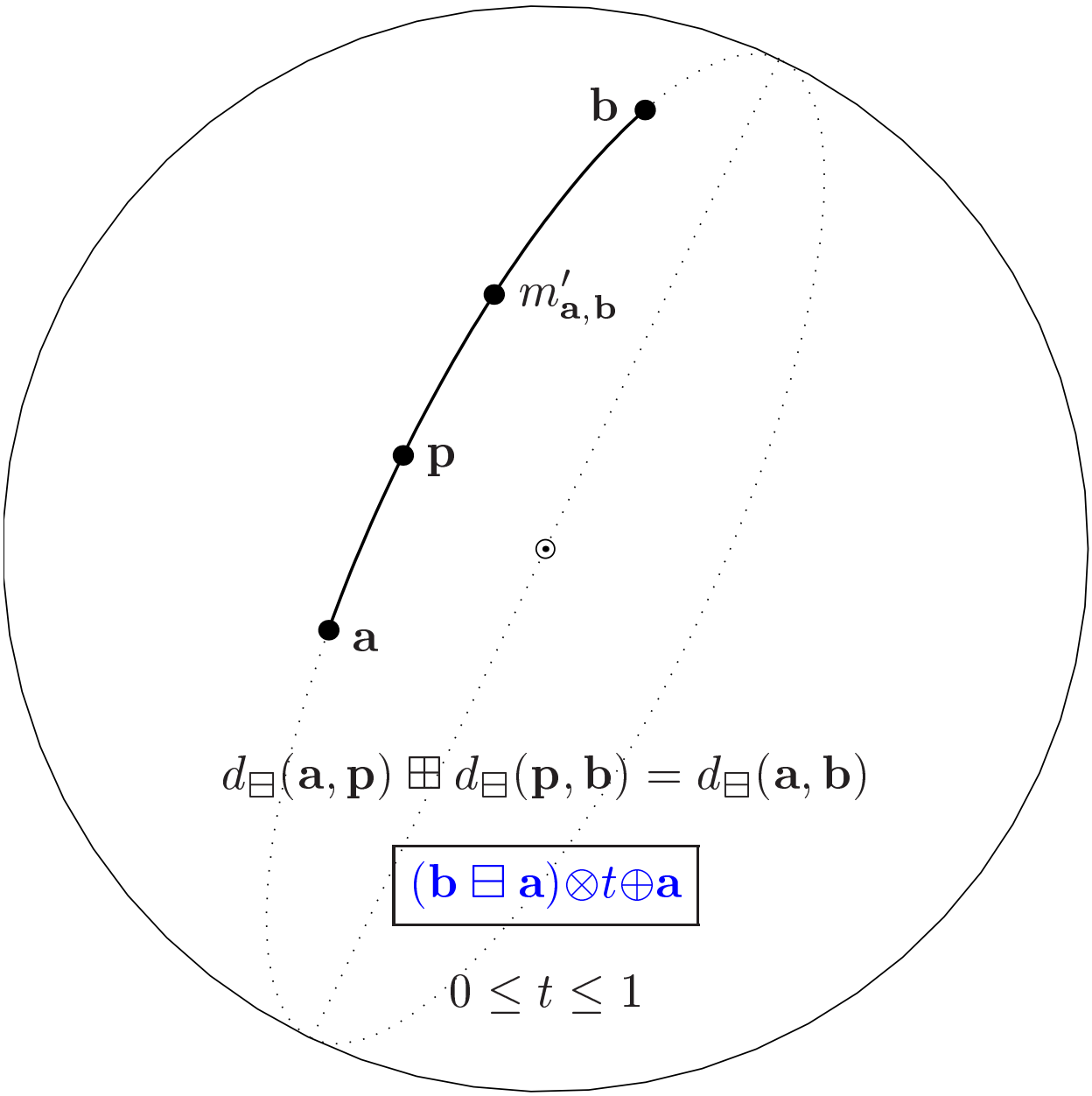

Figure 2: $\oplus = \oplus_{\mathrm{E}}$. The cogyroline segment linking the two points $\mathbf{a}$ and $\mathbf{b}$ in an Einstein gyrovector space $(\mathbb{V}_c, \oplus_{\mathrm{E}}, \otimes)$. $\mathbf{p}$ is a generaic point cobetween $\mathbf{a}$ and $\mathbf{b}$ and $m'_{\mathbf{a},\mathbf{b}}$ is the comidpoint of $\mathbf{a}$ and $\mathbf{b}$.

Möbius gyrovector spaces
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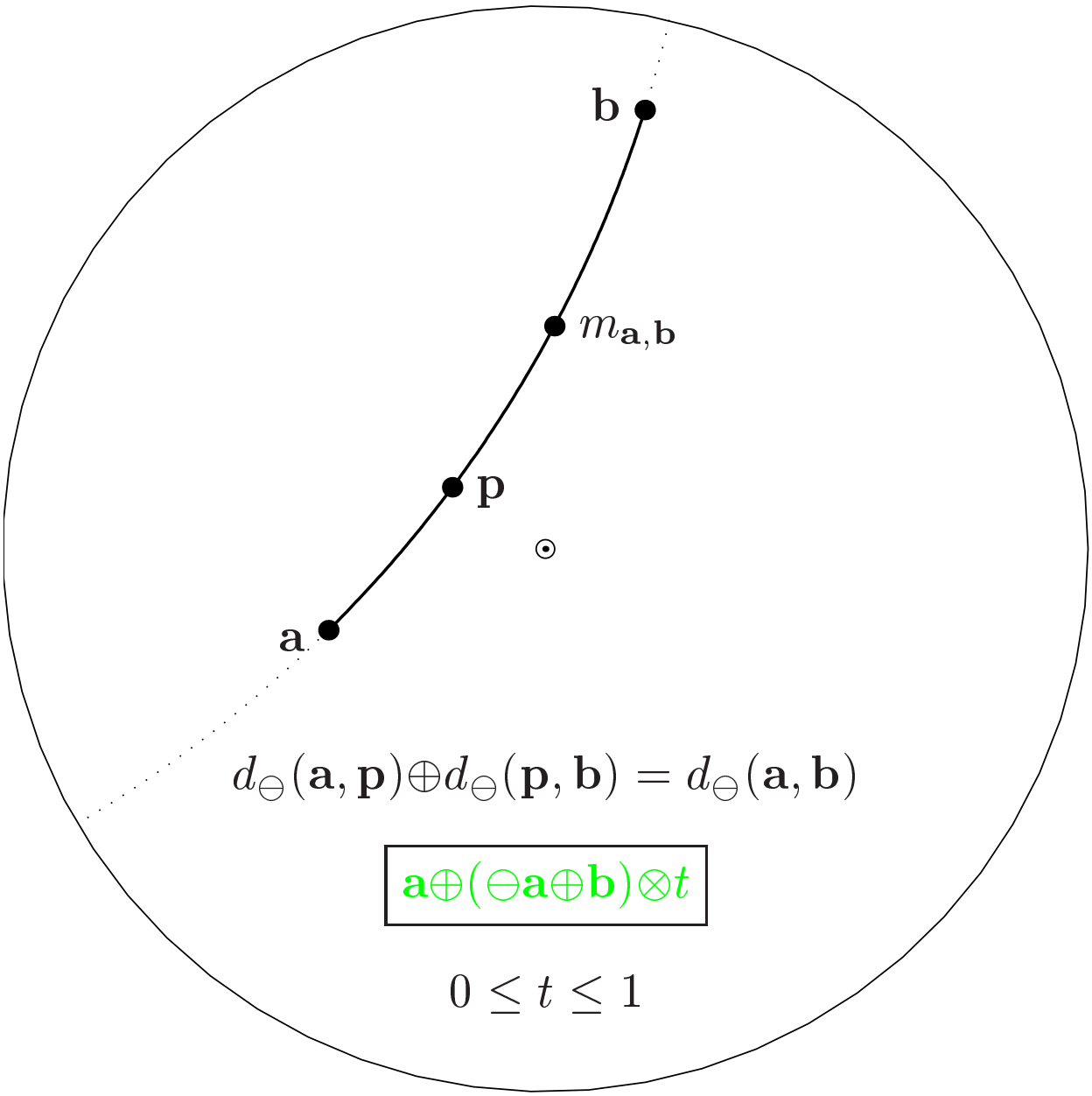


Figure 3: $\oplus = \oplus_{\mathrm{M}}$. The gyroline segment linking the two points $\mathbf{a}$ and $\mathbf{b}$ in a Möbius gyrovector space $(\mathbb{V}_c, \oplus_{\mathrm{M}}, \otimes)$. $\mathbf{p}$ is a generic point between $\mathbf{a}$ and $\mathbf{b}$ and $m_{\mathbf{a},\mathbf{b}}$ is the midpoint of $\mathbf{a}$ and $\mathbf{b}$.

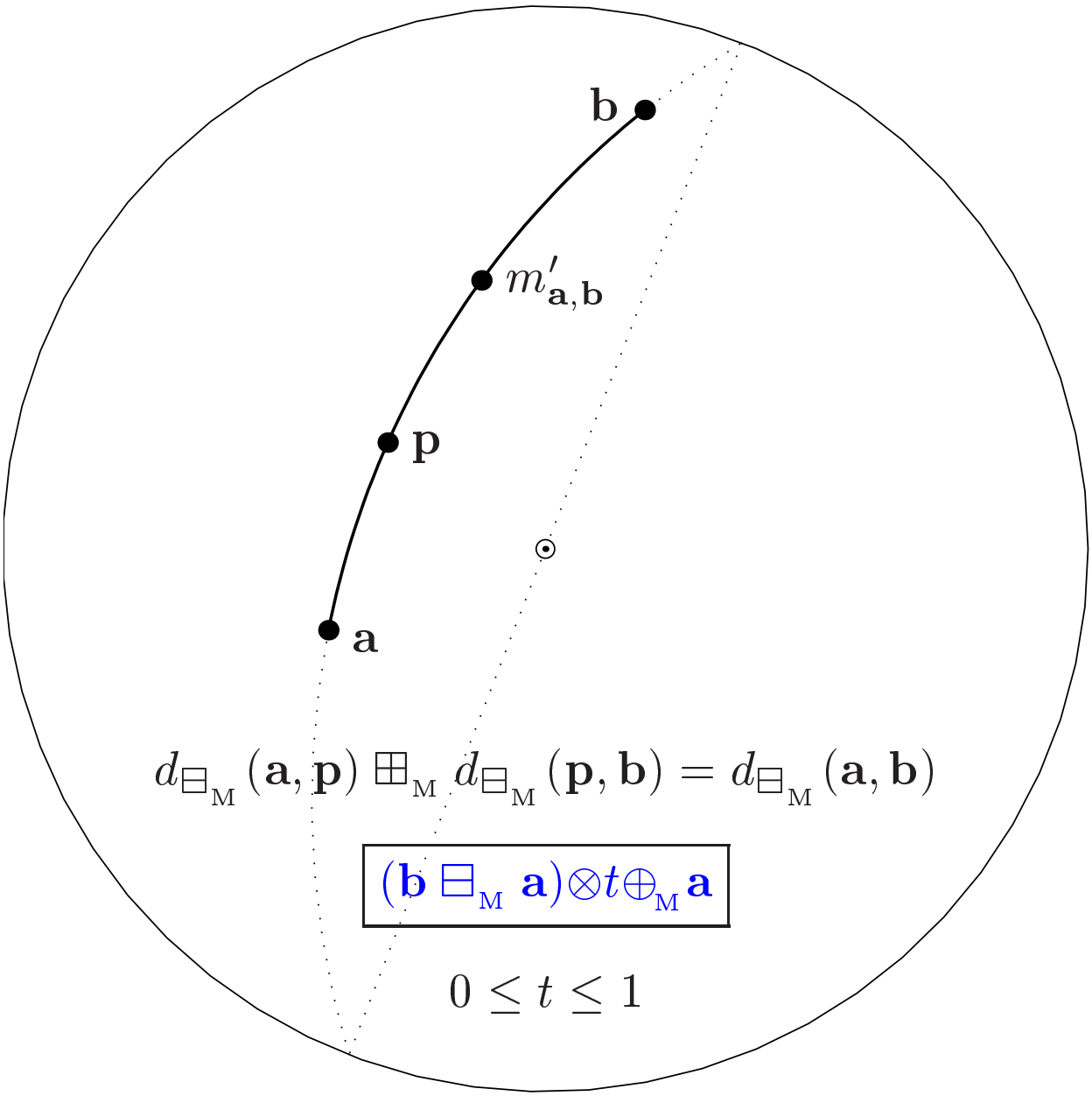


Figure 4: $\oplus = \oplus_{\mathrm{M}}$. The cogyroline segment linking the two points $\mathbf{a}$ and $\mathbf{b}$ in a Möbius gyrovector space $(\mathbb{V}_c, \oplus_{\mathrm{M}}, \otimes)$. $\mathbf{p}$ is a generic point cobetween $\mathbf{a}$ and $\mathbf{b}$ and $m'_{\mathbf{a},\mathbf{b}}$ is the comidpoint of $\mathbf{a}$ and $\mathbf{b}$.

The dual gyrotriangle inequality

$$\|\mathbf{u} \boxminus \mathbf{v}\| \boxplus \|\mathbf{v} \boxminus \mathbf{w}\| \geq \|\mathbf{u} \boxminus \mathrm{gyr}[\mathbf{u} \boxminus \mathbf{v}, \mathbf{v} \boxminus \mathbf{w}]\mathbf{w}\| \tag{28}$$

Theorem 6 (Gyration Gyrotransitive Laws). *Let L_p and L_s be dual geodesics,* (26)*, in a gyrovector space* $(\mathbb{V}_c, \oplus, \otimes)$.

Let $\{\mathbf{a}_1, \cdots, \mathbf{a}_n\}$ *be a set of n points lying on* L_p *(in any order), and*

let $\{\mathbf{b}_1, \cdots, \mathbf{b}_n\}$ *be a set of n points lying on* L_s *(in any order). Then we have the gyration gyrotransitive law for geodesics,*

$$\mathrm{gyr}[\mathbf{a}_1, \ominus\mathbf{a}_2]\mathrm{gyr}[\mathbf{a}_2, \ominus\mathbf{a}_3]\cdots\mathrm{gyr}[\mathbf{a}_{n-1}, \ominus\mathbf{a}_n] \tag{29}$$

$$= \mathrm{gyr}[\mathbf{a}_1, \ominus\mathbf{a}_n] \tag{30}$$

and the dual gyration gyrotransitive law for dual geodesics,

$$\mathrm{gyr}[\mathbf{b}_1, \mathbf{b}_2]\mathrm{gyr}[\mathbf{b}_2, \mathbf{b}_3]\cdots\mathrm{gyr}[\mathbf{b}_{n-1}, \mathbf{b}_n] \tag{31}$$

$$= \mathrm{gyr}[\mathbf{b}_1, \mathbf{b}_n] \tag{32}$$

The Hyperbolic Angle
and
The Hyperbolic Dual Angle

$$\boxed{\cos\alpha = \frac{\ominus\mathbf{a}\oplus\mathbf{b}}{\|\ominus\mathbf{a}\oplus\mathbf{b}\|}\cdot\frac{\ominus\mathbf{a}\oplus\mathbf{c}}{\|\ominus\mathbf{a}\oplus\mathbf{c}\|}} \tag{33}$$

Definition 7 (The Hyperbolic Dual Angle). *The measure of the hyperbolic dual angle* α *between two geometric dual gyrovectors* $\mathbf{b}\boxminus\mathbf{a}$ *and* $\mathbf{d}\boxminus\mathbf{c}$ *is given by the equation*

$$\boxed{\cos\alpha = \frac{\boxminus\mathbf{a}\boxplus\mathbf{b}}{\|\boxminus\mathbf{a}\boxplus\mathbf{b}\|}\cdot\frac{\boxminus\mathbf{c}\boxplus\mathbf{d}}{\|\boxminus\mathbf{c}\boxplus\mathbf{d}\|}} \tag{34}$$

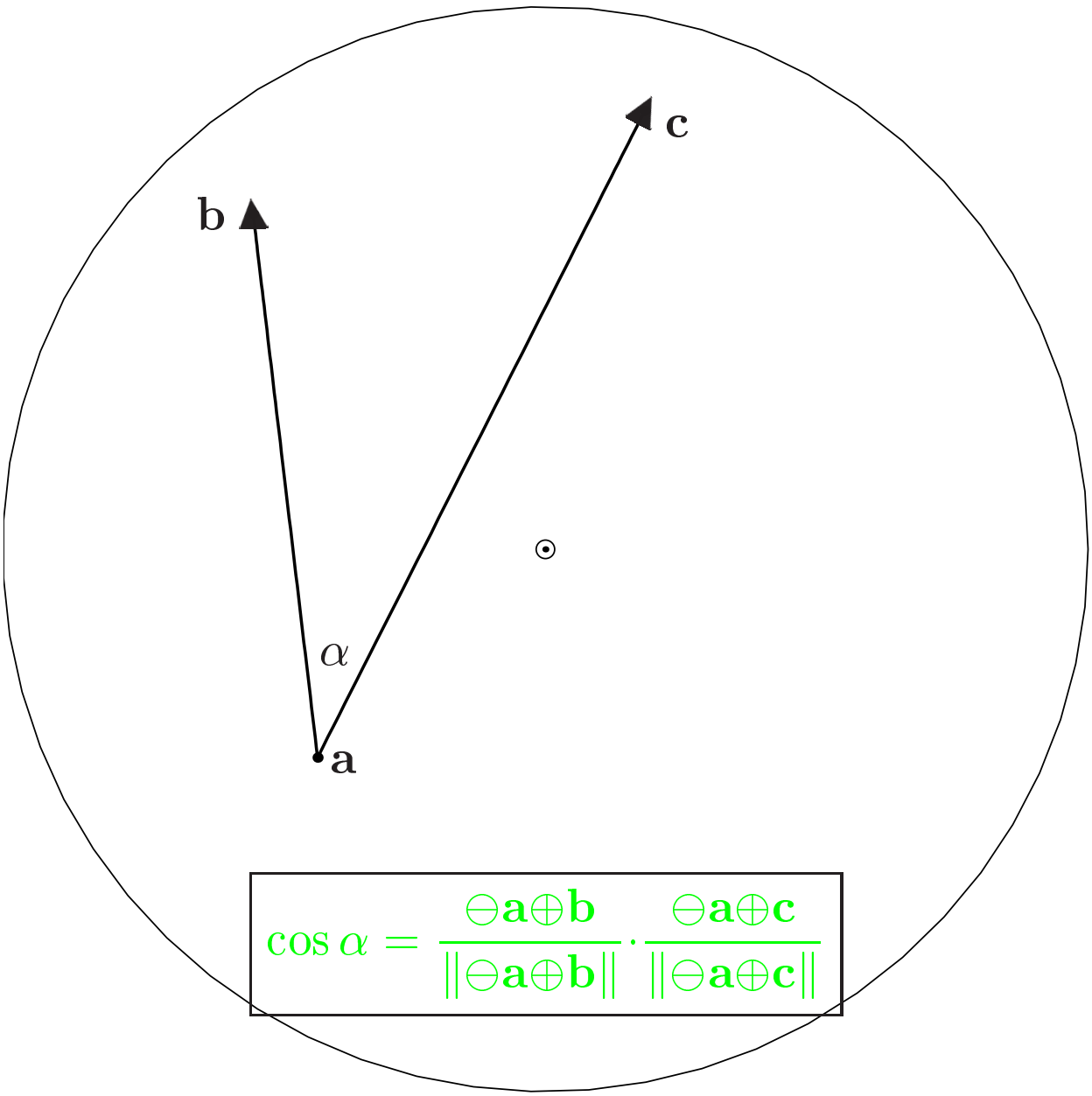


Figure 5: $\oplus = \oplus_{\mathrm{E}}$. The Hyperbolic Angle in the Einstein gyrovector plane = The Beltrami disc model of hyperbolic geometry.

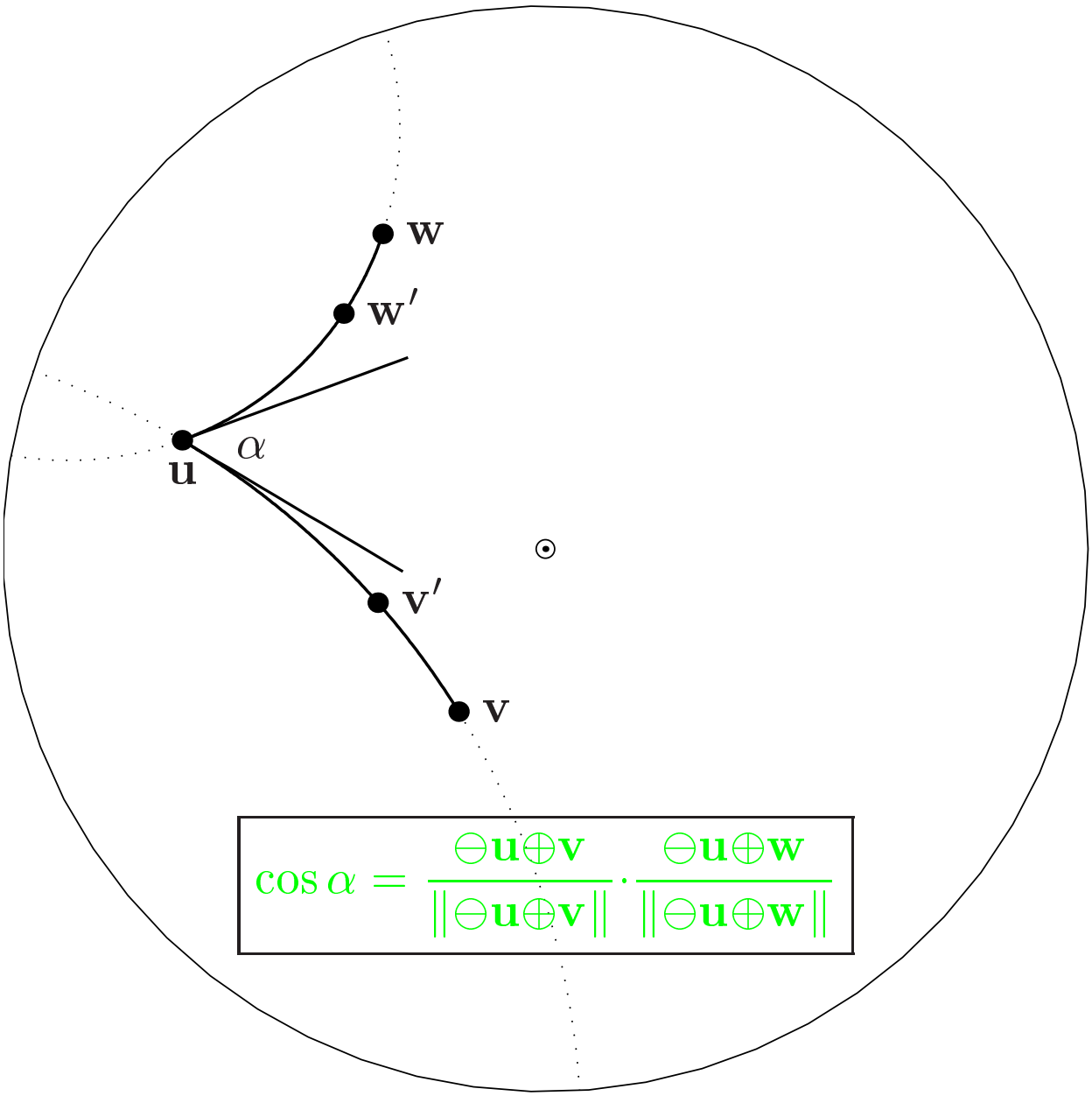


Figure 6: $\oplus = \oplus_{\mathrm{M}}$. A Möbius angle α generated by the two intersecting Möbius geodesic rays. It is coincident with the hyperbolic angle of the Poincaré disc model of hyperbolic geometry.

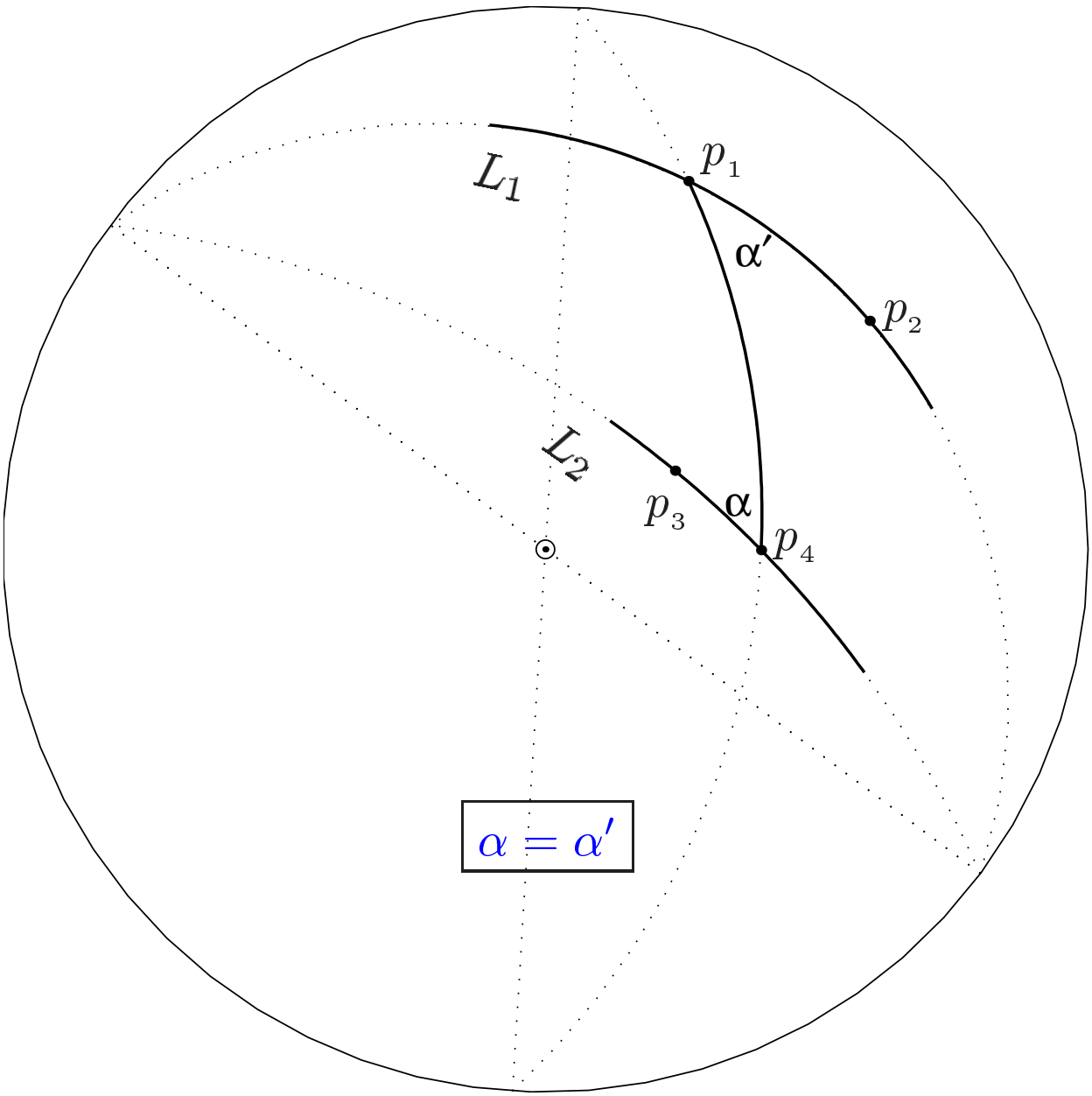


Figure 7: Hyperbolic Alternate Interior Coangles in a Möbius gyrovector space.

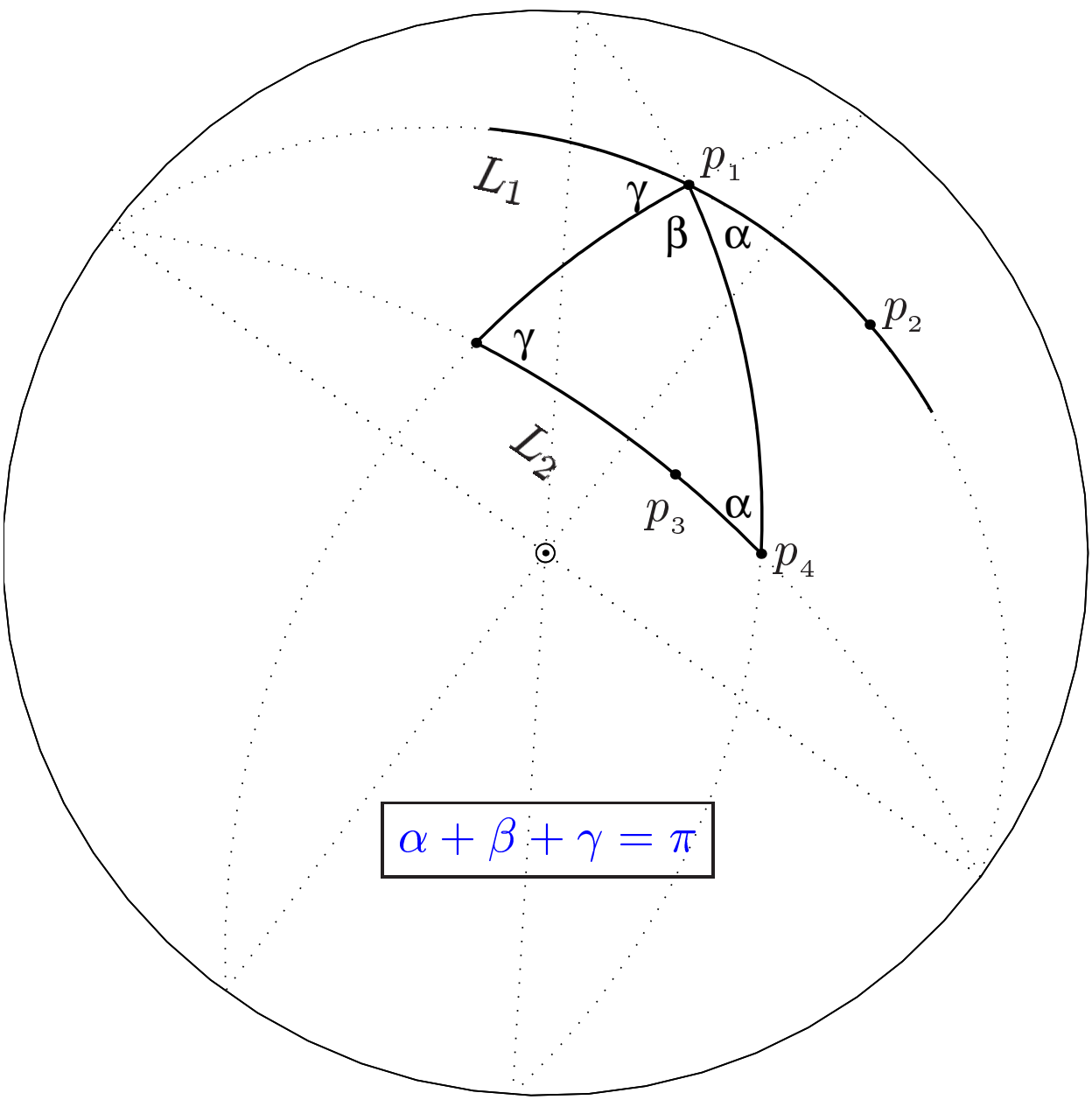


Figure 8: The Hyperbolic π Theorem in the Poincaré Disc Model: The cotriangle coangle sum is π.

Theorem 8 (The Hyperbolic π Theorem). *Let* $\Delta\mathbf{abc}$ *be the hyperbolic dual triangle in an Einstein gyrovector space* $(\mathbb{V}_c, \oplus, \otimes)$ *whose three vertices are the points* $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{V}_c$. *Then, its three hyperbolic dual angles* α, β *and* γ, *given by*

$$\begin{aligned}
\cos\alpha &= \frac{\mathbf{b} \boxminus \mathbf{a}}{\|\mathbf{b} \boxminus \mathbf{a}\|} \cdot \frac{\mathbf{c} \boxminus \mathbf{a}}{\|\mathbf{c} \boxminus \mathbf{a}\|} \\
\cos\beta &= \frac{\mathbf{a} \boxminus \mathbf{b}}{\|\mathbf{a} \boxminus \mathbf{b}\|} \cdot \frac{\mathbf{c} \boxminus \mathbf{b}}{\|\mathbf{c} \boxminus \mathbf{b}\|} \\
\cos\gamma &= \frac{\mathbf{a} \boxminus \mathbf{c}}{\|\mathbf{a} \boxminus \mathbf{c}\|} \cdot \frac{\mathbf{b} \boxminus \mathbf{c}}{\|\mathbf{b} \boxminus \mathbf{c}\|}
\end{aligned} \tag{35}$$

satisfy the identity

$$\boxed{\alpha + \beta + \gamma = \pi} \tag{36}$$

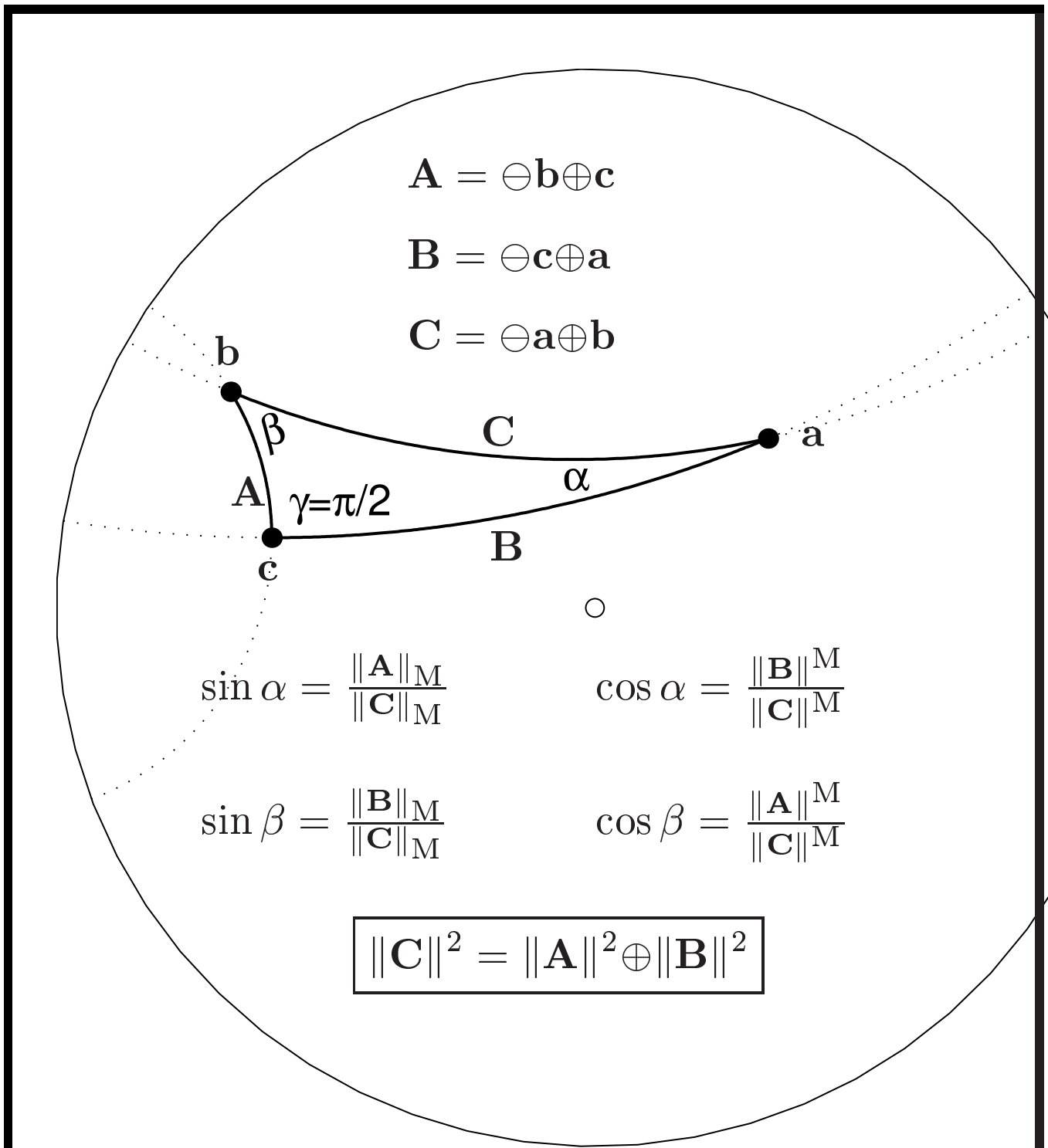


Figure 9: Hyperbolic trigonometry in Möbius gyrovector spaces.

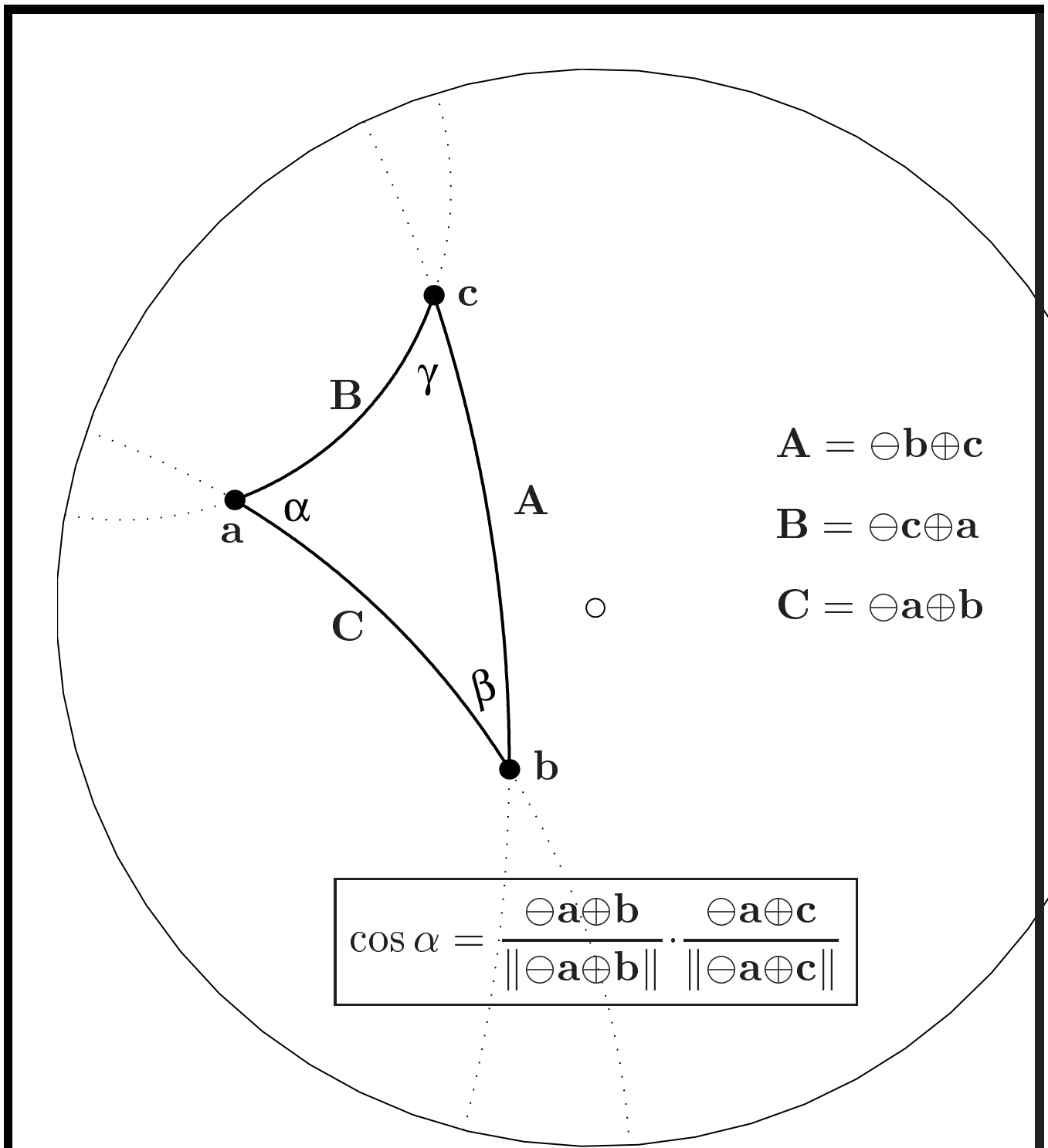


Figure 10: $\oplus = \oplus_{\mathrm{M}}$. A Möbius triangle $\mathbf{\Delta abc}$ in the Möbius gyrovector plane $\mathbb{D} = (\mathbb{R}^2_c, \oplus_{\mathrm{M}}, \otimes)$.

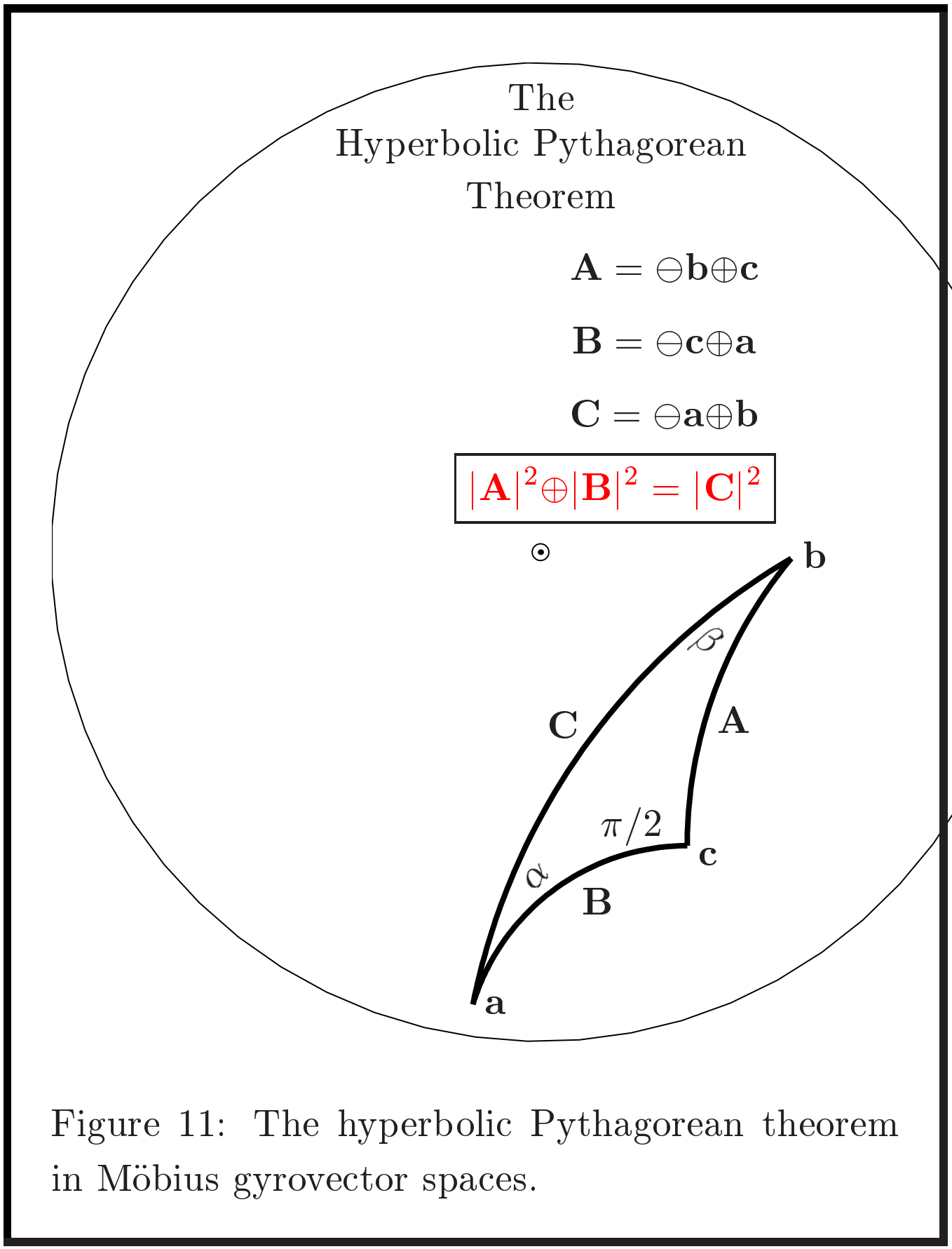


Figure 11: The hyperbolic Pythagorean theorem in Möbius gyrovector spaces.

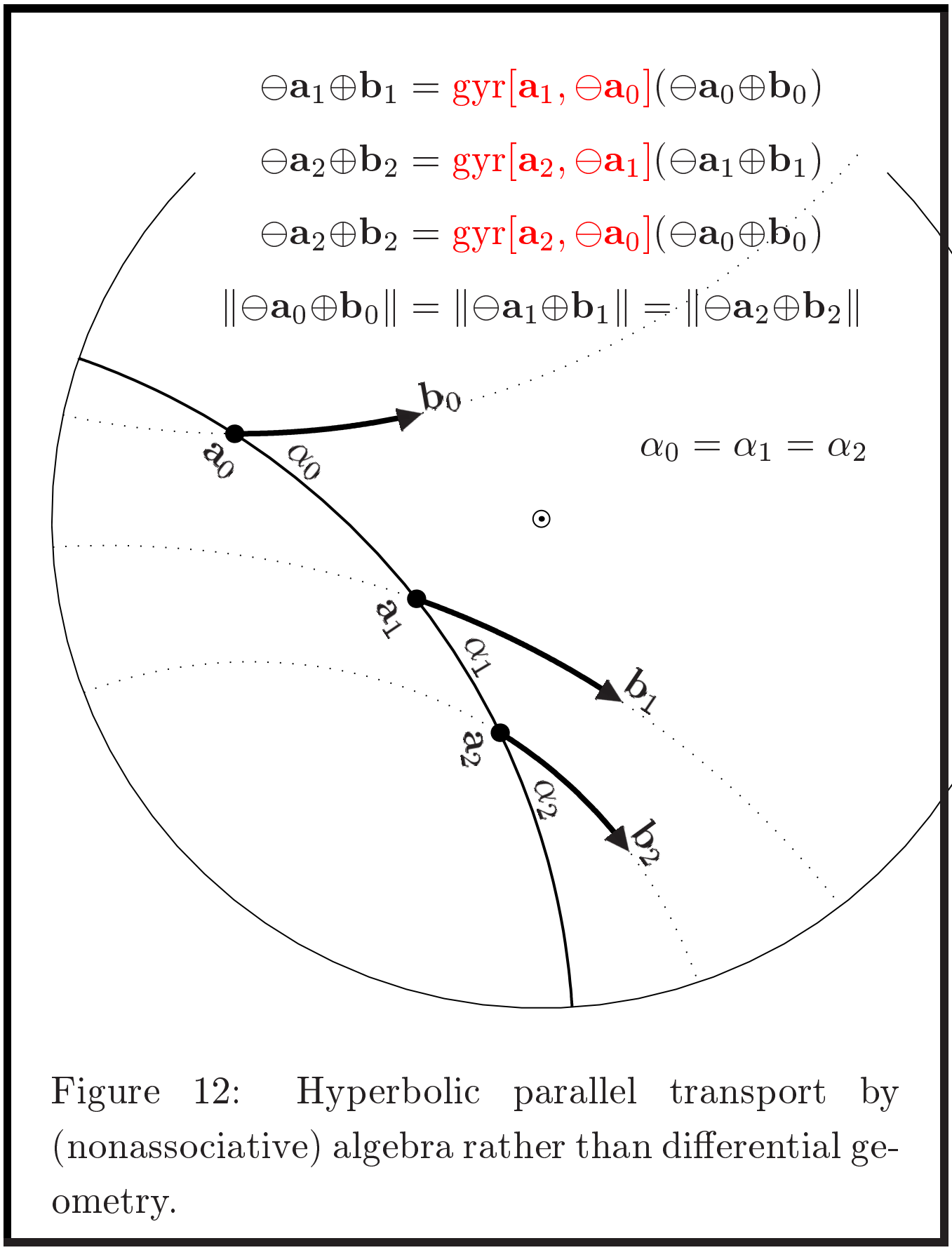


Figure 12: Hyperbolic parallel transport by (nonassociative) algebra rather than differential geometry.

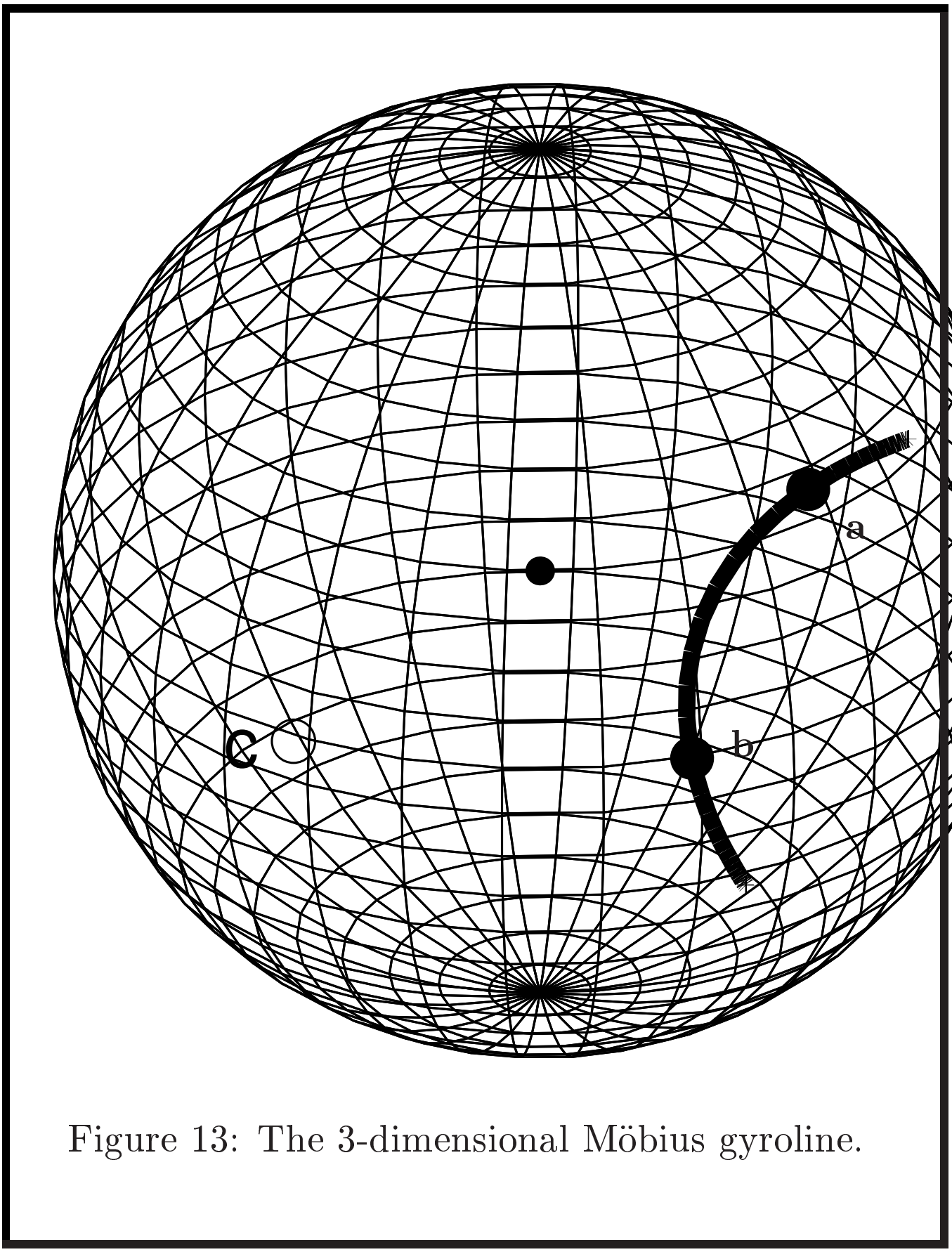


Figure 13: The 3-dimensional Möbius gyroline.

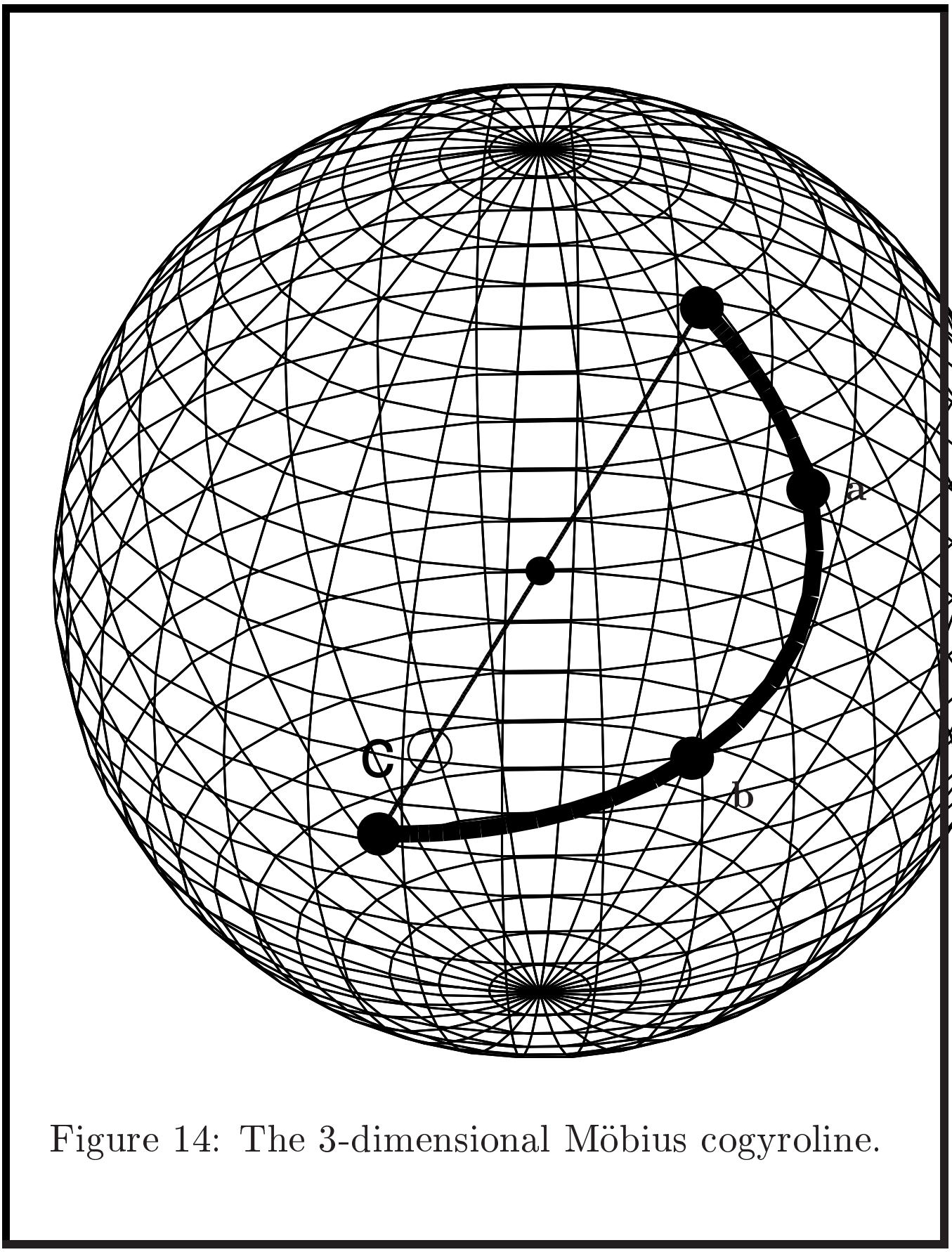


Figure 14: The 3-dimensional Möbius cogyroline.

Gyrovector spaces
and
Riemannian Geometry

The Riemannian line element in a gyrovector space $(\mathbb{V}_c, \oplus, \otimes)$ is

$$\boxed{ds^2 = \| - \mathbf{x} \oplus (\mathbf{x} + d\mathbf{x}) \|^2} \tag{37}$$

Example I: When $\oplus = \oplus_{\mathrm{M}}$, ds^2 turns out to be the Riemannian line element

$$\boxed{\begin{aligned} ds^2 &= \| - \mathbf{x} \oplus_{\mathrm{M}} (\mathbf{x} + d\mathbf{x}) \|^2 \\ &= \frac{\sum_{i=1}^n dx_i^2}{\left(1 + \frac{1}{4} K \sum_{i=1}^n x_i^2\right)^2} \end{aligned}} \tag{38}$$

with curvature $K = -4/c^2$. For the special case when $n = 2$ this line element turns out to be the one of the Poincaré ball model of hyperbolic geometry.

Example II: When $\oplus = \oplus_{\mathrm{E}}$, ds^2 turns out to be the Riemannian line element

$$\boxed{ds^2 = \| - \mathbf{x} \oplus_{\mathrm{E}} (\mathbf{x} + d\mathbf{x}) \|^2 = \quad c^2 \frac{\sum_{i=1}^{n} \{c^2 - (r^2 - x_i^2)\} dx_i^2 + 2 \sum_{\substack{i,j=1 \\ i<j}}^{n} x_i x_j dx_i dx_j}{(c^2 - r^2)^2}} \tag{39}$$

where $r^2 = \sum_{i=1}^{n} x_i^2$. For the special case when $n = 2$ this line element turns out to be the one of the Beltrami (or Klein) ball model of hyperbolic geometry.

Gyrogroup-theoretic techniques capture analogies between the Galilei and the Lorentz transformation groups as well:

$$\boxed{L(\mathbf{v},V)\begin{pmatrix} t \\ \mathbf{x} \end{pmatrix} = \begin{pmatrix} \gamma_{\mathbf{v}}(t+\frac{1}{c^2}\mathbf{v}\cdot V\mathbf{x}) \\ \\ \gamma_{\mathbf{v}}\mathbf{v}t + V\mathbf{x} + \frac{1}{c^2}\frac{\gamma_{\mathbf{v}}^2}{1+\gamma_{\mathbf{v}}}(\mathbf{v}\cdot V\mathbf{x})\mathbf{v} \end{pmatrix}} \tag{40}$$

$\mathbf{v} \in \mathbb{R}_c^n$, $V \in SO(n)$, $\mathbf{x} \in \mathbb{R}^n$, $t \in \mathbb{R}^+$.

In the limit of large vacuum speed of light, $c \to \infty$, the Lorentz transformation $L(\mathbf{v},V)$ in (40) reduces to its Galilean counterpart $G(\mathbf{v},V)$,

$$\boxed{G(\mathbf{v},V)\begin{pmatrix} t \\ \mathbf{x} \end{pmatrix} = \begin{pmatrix} t \\ \\ \mathbf{v}t + V\mathbf{x} \end{pmatrix}} \tag{41}$$

$\mathbf{v} \in \mathbb{R}^n$, $V \in SO(n)$, $\mathbf{x} \in \mathbb{R}^n$, $t \in \mathbb{R}$.

The Galilei Transformation Composition Law Is Given By Parameter Composition:

$$\boxed{G(\mathbf{u}, U)G(\mathbf{v}, V) = G(\mathbf{u} + U\mathbf{v}, UV)} \tag{42}$$

where $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$, and $U, V \in SO(n)$. The Lorentz

Transformation Composition Law Is Given By Parameter Composition:

$$\boxed{L(\mathbf{u}, U)L(\mathbf{v}, V) = L(\mathbf{u} \oplus U\mathbf{v}, \text{gyr}[\mathbf{u}, U\mathbf{v}]UV)} \tag{43}$$

where $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n_c$, and $U, V \in SO(n)$.

The set of all Galilei transformations that link any two given "equitime" spacetime events is well known:

$$\boxed{G(\mathbf{v}-R\mathbf{u},R)\begin{pmatrix} t \\ \mathbf{u}t \end{pmatrix} = \begin{pmatrix} t' \\ \mathbf{v}t' \end{pmatrix}} \qquad (44)$$

for all $\mathbf{u},\mathbf{v}\in\mathbb{R}^n$, and $t,t'\in\mathbb{R}$, where $R\in SO(n)$ is a free rotation parameter.

In full analogy, the set of all Lorentz transformations that link any two given "equinorm" spacetime events is, surprisingly:

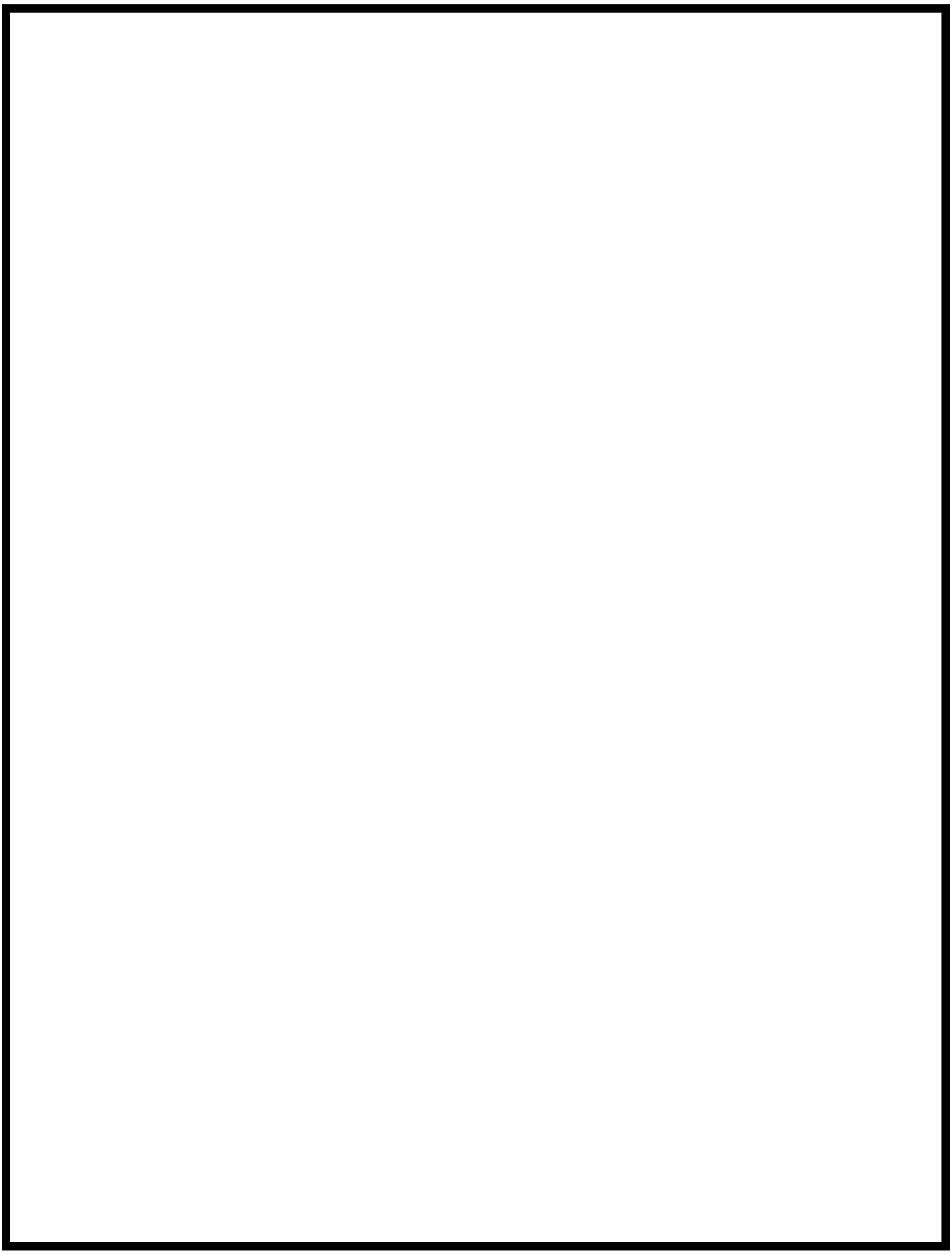
$$\boxed{L(\mathbf{v}\boxminus R\mathbf{u},R)\begin{pmatrix} t \\ \mathbf{u}t \end{pmatrix} = \begin{pmatrix} t' \\ \mathbf{v}t' \end{pmatrix}} \qquad (45)$$

for all $\mathbf{u},\mathbf{v}\in\mathbb{R}_c^n$, and $t,t'\in\mathbb{R}^+$, where $R\in SO(n)$ is a free rotation parameter.

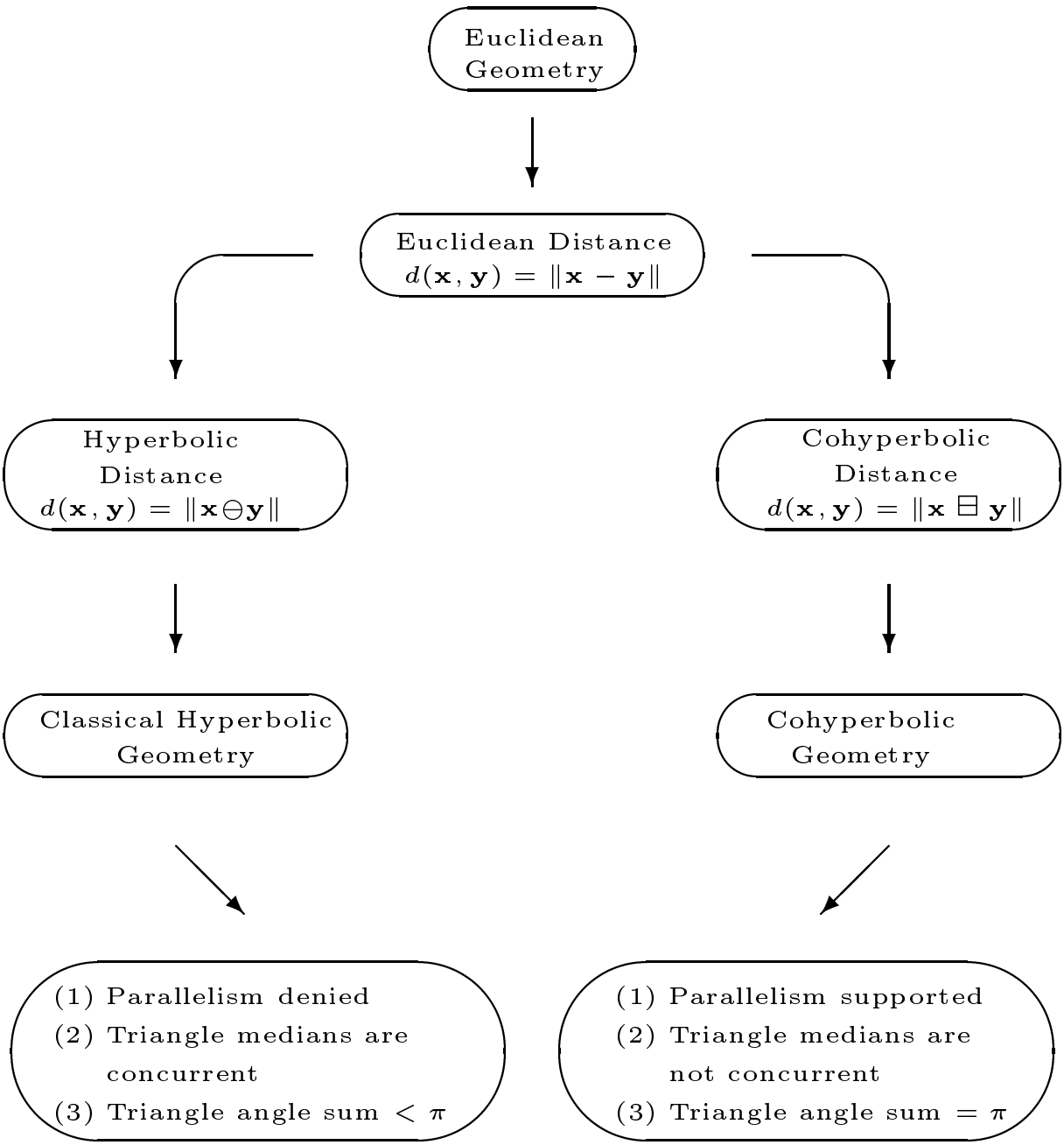
The gyrogroup dual operation $\boxplus$ thus plays centrally in the ability of gyrogroup-theoretic techniques to capture analogies.

	Composition Law	binary operation
Galilei	$\boxed{G(\mathbf{u},U)G(\mathbf{v},V)=G(\mathbf{u}+U\mathbf{v},UV)}$ $\mathbf{u},\mathbf{v}\in\mathbb{R}^n,\quad U,V\in SO(n)$ Structure: Semidirect Product Group	$+$
Lorentz	$\boxed{L(\mathbf{u},U)L(\mathbf{v},V)=L(\mathbf{u}\oplus U\mathbf{v},\mathrm{gyr}[\mathbf{u},U\mathbf{v}]UV)}$ $\mathbf{u},\mathbf{v}\in\mathbb{R}^n_c,\quad U,V\in SO(n)$ Structure: Gyrosemidirect Product Group	$\oplus$
	Link	
Galilei	$\boxed{G(\mathbf{v}-R\mathbf{u},R)\begin{pmatrix}t\\ \mathbf{u}t\end{pmatrix}=\begin{pmatrix}t'\\ \mathbf{v}t'\end{pmatrix}}$ $t'=t$ $R\in SO(n)\quad free\ orientation\ parameter$	$-$
Lorentz	$\boxed{L(\mathbf{v}\boxminus R\mathbf{u},R)\begin{pmatrix}t\\ \mathbf{u}t\end{pmatrix}=\begin{pmatrix}t'\\ \mathbf{v}t'\end{pmatrix}}$ $t'\sqrt{1-\|\mathbf{v}\|^2/c^2}=t\sqrt{1-\|\mathbf{u}\|^2/c^2}$ $R\in SO(n)\quad free\ orientation\ parameter$	$\boxminus$

Table 1: Galilei and Lorentz transformation analogies in their composition law and links. Remarkably, in order to capture analogies the two dual Einstein additions, $\oplus$ and $\boxplus$, are employed.



The Hyperbolic Bifurcation



Those who want to know more about gyrogroups and gyrovector spaces will find it rewarding to read the new book

Beyond The Einstein Addition Law And Its Gyroscopic Thomas Precession The Theory Of Gyrogroups And Gyrovector Spaces

by A.A. Ungar (Kluwer Acad. 2001)

See URL:
www.wkap.nl/book.htm/0-7923-6909-2